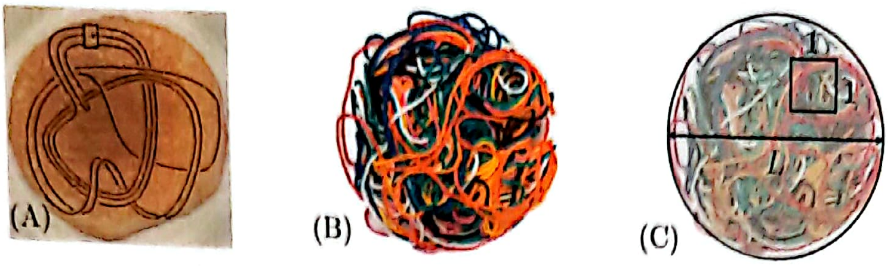

make 1×1 square smaller.

FIGURE 1. (A) A planar diagram of a knot in a pancake (knot diagram by Piccirillo [Pic20]). (B) A yarn ball. (C) Measurements on a yarn ball knot.

very flat and wide subsets of $\mathbb{R}^3$, as in Figure 1 (A). This pancake description of a knot is somewhat artificial and does not realistically describe knots that occur in nature. For example, knotted DNA is shaped much more like the ball of yarn in Figure 1 (B), than the pancake knot in (A). Since we aim to have a 3D understanding of knot invariants, we study knots in the shape of the yarn ball as opposed to the pancake. We define a *yarn ball knot* to be a knotted tube of uniform width 1 that is tightly packed in a ball, as in Figure 1 (B). Equivalently, a yarn ball knot is a knot embedded in a grid, which we discuss in depth in Section 2. Let L be the diameter of the yarn ball knot, then the volume of the knot is $V \sim L^3$. The volume also measures the length of the yarn, or how much yarn was used to make the knot.

By projecting the yarn ball knot in a generic direction onto a disk, we attain a planar knot diagram. The crossing number of this projection can be estimated by subdividing the disk into 1×1 squares, as in Figure 1 (C). Most such 1×1 squares will have L layers of strands above them, and one can expect these strands to cross around $\sim \binom{L}{2} \sim L^2$ times in that square. Since there are $\sim L^2$ squares, the total crossing number of this projection is around $n \sim L^2 L^2 = L^4 = V^{4/3}$.

We see that to describe a yarn ball knot of volume (or length) V as a planar diagram, we would need $\sim V^{4/3}$ crossings. For sufficiently large V, $V^{4/3} \gg V$ and it requires many more bits to describe a yarn ball knot via its projection rather than directly as a yarn ball.

If ζ is a knot invariant, we denote by $C_\zeta(2D, n)$ the worst-case complexity of computing ζ on a knot given by a planar diagram with n crossings, and by $C_\zeta(3D, V)$ the worst-case complexity of computing ζ on a knot given as a yarn ball of volume V. Given a yarn ball of volume V we can always compute ζ by first projecting to the plane[2], obtaining a planar diagram with $\sim V^{4/3}$ crossings, and then computing ζ using our best 2D techniques. Hence always, $C_\zeta(3D, V) \leq C_\zeta(2D, V^{4/3})$. It is interesting to know when 3D techniques can do even better. Our first main result is to show that for the link invariant the *linking number*, there is a 3D computation technique whose worst-case complexity is faster than every worst-case 2D technique.

Theorem 2.1. (Proof in Section 2) *Let lk denote the linking number of a 2-component link. Then $C_{lk}(2D, n) \sim n$ while $C_{lk}(3D, V) \sim V$.*

The linking number of a link is an example of a *finite type invariant*. Finite type invariants underlie many of the classical knot invariants, for instance they give the coefficients of the Jones, Alexander, and more generally HOMPFLY-PT polynomials [BL93, BN95a]. We prove

[2]The projection itself can be computed quickly, in time $\sim V^{4/3}$, and for all interesting ζ, this extra work is negligible

the following computational bounds for all finite type invariants.

Theorem 3.2. (Proof in Section 3) *If ζ is a finite type invariant of type d then $C_\zeta(2D, n)$ is at most $\sim n^d$.*

Theorem 3.3. (Proof in Section 3) *If ζ is a finite type invariant of type d then $C_\zeta(3D, V)$ is at most $\sim V^d$.*

The actual complexities $C_\zeta(2D, n)$ for some specific though "special" finite type ζ's, such as the coefficients of the Alexander polynomial, are known to be much smaller. However, for generic ζ's, these theorems suggest that 3D techniques will be more computationally efficient than 2D ones. We suspect the complexities in Theorem 3.2 and 3.3 can be improved by closer consideration of the counting arguments used the proofs. However, we view these theorems as a significant starting point that we hope to improve upon in the future and encourage our readers do to the same.

1.1. **Discussion.** The opinion we present in this paper is that knots are three-dimensional and the best way to understand a knot should be three-dimensional~~ly~~. We propose some new language to aid the knot theory community in discussion surrounding current understanding of an invariant.

Conversation Starter 1. *A knot invariant ζ is said to be* computationally 3D, *or C3D, if*

$$C_\zeta(3D, V) \ll C_\zeta(2D, V^{4/3}).$$

In other words, ζ is C3D if substantial savings can be made to the computation of ζ on a yarn ball knot, relative to the complexity of computing ζ by first projecting the yarn ball to the plane.

This is not a formal definition! The notion of an invariant being computationally 3D is dependent on the current knowledge of the invariant. As our understanding grows and our computational techniques get better, an invariant might become *C3D*, or lose its *C3D* status. However, the question whether an invariant is *C3D*, as we understand it at a given time, still has merit as it measures our understanding, as a community, of knot theory as a 3D subject. With this new terminology, Theorem 2.1 could be restated as

Theorem 2.1. (restated) *The linking number of a 2-component link is C3D.*

The results of Theorems 3.2 and 3.3 naively suggest that finite type invariants are also C3D, but the theorems only give one sided bounds. Yet we believe that our naive conclusion remains valid, at least in the form "most finite type invariants are C3D".

The opinion in this paper is that in general knot invariants should be *C3D*. Unfortunately, as of the time this paper is written, very few knot invariants are known to be *C3D*. Are the Alexander, Jones, or HOMFLY-PT polynomials *C3D*? Why or why not? Are the Reshetikhin-Turaev invariants *C3D*? Are knot homologies *C3D*? While we seem to have a weak understanding of these fundamental invariants from a 3D perspective, this is cause for optimism; there is still much work to be done.

Instead of using computational complexity to compare 2D and 3D understandings of invariants, we can also use the notion of the maximal value of other quantities relating to the size of the knot, which motivates the next conversation starter.

Conversation Starter 2. *If η is a stingy quantity (i.e. we expect it to be small for small knots), we say η has savings in 3D, or has 'S3D' if*

$$M_\eta(3D, V) \ll M_\eta(2D, V^{4/3}),$$

where $M_\eta(kD, s)$ is the maximum value of η on all knots described k-dimensionally and of size s.

For example, the hyperbolic volume is a stingy quantity–the more complicated a knot is, the more complicated its complement in S^3 will be, which makes the question of putting a hyperbolic structure on it harder. We expect hyperbolic volume to have savings in 3 dimensions.

Conjecture (Bar-Natan, van der Veen) Hyperbolic volume has *S3D.*

The genus of a knot is another example of a stingy quantity, but we do not know if the genus of a knot has *S3D*, or not. If a knot is given in 3-dimensions, is the best way to find the genus truly to compute the Seifert surface from a projection to 2D, at a great cost? The genus is by all means a 3D property of a knot, and it seems as though it *should* be best computed in a 3D manner.

We hope that these conversation starters will encourage our readers to think about more 3D computational methods. The remaining two sections of this paper are dedicated to proving Theorems 2.1 and 3.3.

Acknowledgements. This work was partially supported by NSERC grant RGPIN-2018-04350. And And by The Chu Family Foundation (NYC).

2. Grid Knots and Linking Number

To emphasize the 3D nature of knots, we think of them as yarn ball knots, instead of as pancake knots. An equivalent notion is "grid knots" (see also [BL12]). A *grid knot (or link) of size L* is a labeled parametrized knot or link embedded as a subset of a grid with side length L. The arcs of a grid knot are enumerated in order of the parametrization of the knot along the unit grid segments of the grid lines. For a grid link, the link components are enumerated, and each arc of the link is labeled by a pair (c, p), where c is the arc's component enumeration, and p is the arc's parametrization enumeration. Some examples of grid knots are shown in Figure 2, and Figure 3 (A) and (B) shows an example of a $4 \times 4 \times 4$ grid from different perspectives.

The process of converting an oriented yarn ball knot of length/volume V to a grid knot is as follows. Replace the yarn by an approximation along grid lines with grid spacing say $\frac{1}{10}$'th the unit width of the yarn. Rescale so that the grid squares are unit length again. Starting at any corner of the grid knot, label the arcs of the knot in order according to the orientation. The resulting knot is bounded in a box of size $\sim 10^3 V$, and this process takes $\sim V$ computation steps. To convert a grid knot to a yarn ball knot, scale the grid so the distance between neighbouring grid points is say 3 or 5 units. Replace the arcs of the knot with yarn of width 1 and round out the corners. This process takes time proportional to the length of the knot. When computing an invariant of a yarn ball knot, first converting the knot into a grid knot adds a negligible amount of computation time.

For the remainder of this paper, we conventionally view grids with the slightly askew top-down view as in Figure 3 (B). From this perspective, all of the crossings of a grid knot occur in triangular *crossing fields* of the grid– highlighted in Figure 3 (C). The grid lines in the x, y

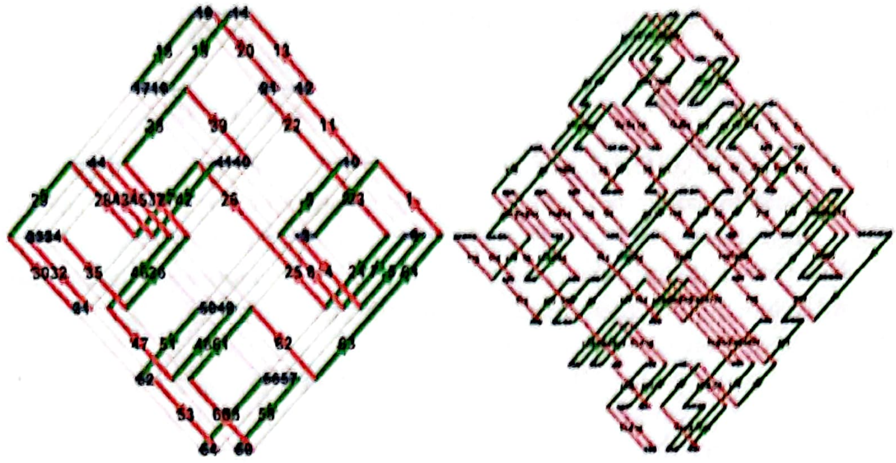

FIGURE 2. Two examples of grid knots. The left grid knot has $L = 3$ and 64 labeled arcs, and the right has $L = 5$ and 216 labeled arcs.

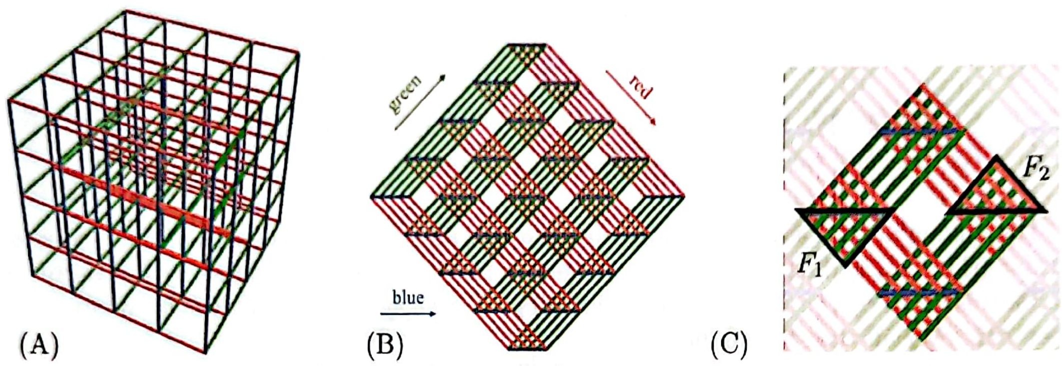

FIGURE 3. The grid in (B) shows a slightly askew top down view of the grid from (A). (C) highlights two crossing fields, F_1 and F_2, of a grid.

directions are colored in green and red, and the grid lines in the vertical direction are colored blue. From the askew top down perspective as in Figure 3 (B), we keep the convention that "/" grid lines are called "green", "\" grid lines are called "red", and horizontal grid lines are called "blue". All of the crossings in a crossing field occur between green and red grid lines. The vertical blue grid lines never participate in a crossing.

There are $2L^2$ triangular crossing fields; 2 at each $(L-1)^2$ interior corners, and one along each exterior corners except two corners, which is $2L + 2(L-1)$, for a total of $2(L-1)^2 + 2L + 2(L-1) = 2L^2$.

2.1. **Linking Number.** For a two-component link $\mathcal{L}$, the *linking number* of $\mathcal{L}$, denoted $lk(\mathcal{L})$, is a classical link invariant that measures how the two components are linked. From a planar projection of $\mathcal{L}$, $lk(\mathcal{L})$ can be computed as follows: Only counting "mixed" crossings that involve both components (the over strand is from one component and the under strand is from the other component), $lk(\mathcal{L})$ is one half (the number of positive crossings minus the number of negative crossings). Using this 2D method for a planar diagram with n crossings, computing lk requires $\sim n$ steps– one for every crossing[1] which shows that $C_{lk}(2D, n) \sim n$. Using grid ugly parenthetization

Foot 1: Note that in the worst case, and presumably also typically, the number of mixed crossings is $\sim n$.