Pensieve header: Do solutions of R4, unitarity and cap satisfy the new $\bar{V}^{21}=S_1S_2V^{-1}$?.

In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
<< FreeLie.m;
<< AwCalculus.m;
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
α = LS[{x, y}, αs]; β = LS[{x, y}, βs]; γ = CWS[{x, y}, γs];
V = Es[⟨x → α, y → β⟩, γ];
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
R4Eqn = V ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V ;
UnitarityEqn = (V ** (V // dA[x] // dA[y]) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]]);
CapEqn = ((V ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap * (Cap // dσ[x, y]) // dc[x] // dc[y]));
βs[x] = 1 / 2; βs[y] = 0;
SeriesSolve[{α, β, γ, κ}, (ħ^-1 R4Eqn) && UnitarityEqn && CapEqn];
{V, κ}
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw,
  CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
  Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
  Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
  t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ħ, ⟌, ⟍}.

FreeLie` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA,
  hm, hS, hΔ, hη, hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150909.

SeriesSolve: In degree 1 arbitrarily setting {κs[x] → 0}.

SeriesSolve: In degree 3 arbitrarily setting {αs[x, y, y] → 0}.
```

Out[•]=

$$\left\{\text{Es}\left[\left\langle \text{x}\to \text{LS}\left[0,-\frac{\overline{\text{x y}}}{24},0,\ldots\right],\ \text{y}\to \text{LS}\left[\frac{\overline{\text{x}}}{2},-\frac{\overline{\text{x y}}}{12},0,\ldots\right]\right\rangle,\ \text{CWS}\left[0,-\frac{\widehat{\text{xy}}}{48},0,\ldots\right]\right],\right.$$
$$\left.\text{CWS}\left[0,-\frac{\widehat{\text{xx}}}{96},0,\ldots\right]\right\}$$

In[•]:=
```
V@{6} // Timing
```

SeriesSolve: In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

Out[•]=

$$\Big\{0.8125,\ \text{Es}\Big[\Big\langle x\to \text{LS}\Big[0,\ -\frac{\overline{xy}}{24},\ 0,\ \frac{7\,\overline{x\,x\,\overline{xy}}}{5760}-\frac{7\,\overline{x\,\overline{xy}\,y}}{5760}+\frac{\overline{\overline{xy}\,y\,y}}{1440},\ 0,$$
$$-\frac{31\,\overline{x\,x\,x\,\overline{x\,\overline{xy}}}}{967\,680}+\frac{31\,\overline{x\,x\,x\,\overline{xy}\,y}}{483\,840}-\frac{83\,\overline{x\,x\,\overline{xy}\,y\,y}}{967\,680}-\frac{31\,\overline{x\,\overline{xy}\,\overline{xy}\,y}}{725\,760}-\frac{31\,\overline{x\,\overline{x\,\overline{xy}}\,\overline{xy}}}{645\,120}+$$
$$\frac{13\,\overline{x\,\overline{xy}\,y\,yy}}{241\,920}+\frac{101\,\overline{xy}\,\overline{\overline{xy}\,y\,y}}{1\,451\,520}+\frac{527\,\overline{x\,\overline{xy}\,y}\,\overline{xy}}{5\,806\,080}-\frac{\overline{\overline{xy}\,y\,yyy}}{60\,480},\ \ldots\Big],$$
$$y\to \text{LS}\Big[\frac{\overline{x}}{2},\ -\frac{\overline{xy}}{12},\ 0,\ \frac{\overline{x\,x\,\overline{xy}}}{5760}-\frac{1}{720}\,\overline{x\,\overline{xy}\,y}+\frac{1}{720}\,\overline{\overline{xy}\,y\,y},\ -\frac{\overline{x\,x\,x\,\overline{xy}}}{7680}+\frac{\overline{x\,x\,\overline{xy}\,y}}{3840}-\frac{\overline{x\,\overline{xy}\,\overline{xy}}}{6912},$$
$$-\frac{\overline{x\,x\,x\,\overline{x\,\overline{xy}}}}{645\,120}+\frac{23\,\overline{x\,x\,x\,\overline{xy}\,y}}{483\,840}-\frac{13\,\overline{x\,x\,\overline{xy}\,y\,y}}{161\,280}-\frac{\overline{x\,\overline{xy}\,\overline{xy}\,y}}{22\,680}-\frac{41\,\overline{x\,\overline{x\,\overline{xy}}\,\overline{xy}}}{580\,608}+$$
$$\frac{\overline{x\,\overline{xy}\,y\,yy}}{15\,120}+\frac{\overline{xy}\,\overline{\overline{xy}\,y\,y}}{12\,096}+\frac{71\,\overline{x\,\overline{xy}\,y}\,\overline{xy}}{483\,840}-\frac{\overline{\overline{xy}\,y\,yyy}}{30\,240},\ \ldots\Big]\Big\rangle,$$
$$\text{CWS}\Big[0,\ -\frac{\widehat{xy}}{48},\ 0,\ \frac{\widehat{xxxy}}{2880}+\frac{\widehat{xxyy}}{2880}+\frac{\widehat{xyxy}}{5760}+\frac{\widehat{xyyy}}{2880},\ 0,\ -\frac{\widehat{xxxxxy}}{120\,960}-\frac{\widehat{xxxxyy}}{120\,960}-\frac{\widehat{xxxyxy}}{120\,960}-$$
$$\frac{\widehat{xxxyyy}}{120\,960}-\frac{\widehat{xxyxxy}}{241\,920}-\frac{\widehat{xxyxyy}}{120\,960}-\frac{\widehat{xxyyxy}}{120\,960}-\frac{\widehat{xxyyyy}}{120\,960}-\frac{\widehat{xyxyxy}}{362\,880}-\frac{\widehat{xyxyyy}}{120\,960}-\frac{\widehat{xyyxyy}}{241\,920}-\frac{\widehat{xyyyyy}}{120\,960},\ \ldots\Big]\Big]\Big\}$$

Checking $\overline{V}^{21} = S_1\, S_2\, V^{-1}$.

In[•]:=
```
V // dA[x] // dA[y]
```

Out[•]=

$$\text{Es}\Big[\Big\langle x\to \text{LS}\Big[0,\ \frac{\overline{xy}}{24},\ -\frac{1}{48}\,\overline{x\,\overline{xy}},\ \ldots\Big],\ y\to \text{LS}\Big[-\frac{\overline{x}}{2},\ \frac{\overline{xy}}{12},\ -\frac{1}{48}\,\overline{x\,\overline{xy}},\ \ldots\Big]\Big\rangle,\ \text{CWS}\Big[0,\ \frac{\widehat{xy}}{48},\ 0,\ \ldots\Big]\Big]$$

In[•]:=
```
V // dA[x] // dA[y] // dσ[{x, y} → {y, x}]
```

Out[•]=

$$\text{Es}\Big[\Big\langle x\to \text{LS}\Big[-\frac{\overline{y}}{2},\ -\frac{\overline{xy}}{12},\ -\frac{1}{48}\,\overline{\overline{xy}\,y},\ \ldots\Big],\ y\to \text{LS}\Big[0,\ -\frac{\overline{xy}}{24},\ -\frac{1}{48}\,\overline{\overline{xy}\,y},\ \ldots\Big]\Big\rangle,$$
$$\text{CWS}\Big[0,\ \frac{\widehat{xy}}{48},\ 0,\ \ldots\Big]\Big]$$

In[•]:=
```
V^-1
```

Out[•]=

$$\text{Es}\left[\left\langle x \to \text{LS}\left[0, \frac{\overline{xy}}{24}, -\frac{1}{48}\,\overline{x\overline{xy}}, \ldots\right], y \to \text{LS}\left[-\frac{\overline{x}}{2}, \frac{\overline{xy}}{12}, -\frac{1}{48}\,\overline{x\overline{xy}}, \ldots\right]\right\rangle, \text{CWS}\left[0, \frac{\widehat{xy}}{48}, 0, \ldots\right]\right]$$

In[•]:=
```
V^-1 // dS[x] // dS[y]
```

Out[•]=

$$\text{Es}\left[\left\langle x \to \text{LS}\left[0, -\frac{\overline{xy}}{24}, 0, \ldots\right], y \to \text{LS}\left[-\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, 0, \ldots\right]\right\rangle, \text{CWS}\left[0, -\frac{\widehat{xy}}{48}, 0, \ldots\right]\right]$$

In[•]:=
```
(V // dA[x] // dA[y] // dσ[{x, y} → {y, x}]) ≡ (V^-1 // dS[x] // dS[y])
```

Out[•]=

$$\text{BS}\left[\text{True}, -\frac{\overline{y}}{2} == 0 \,\&\&\, 0 == -\frac{\overline{x}}{2},\right.$$
$$-\frac{\overline{y}}{2} == 0 \,\&\&\, -\frac{\overline{xy}}{12} == -\frac{\overline{xy}}{24} \,\&\&\, 0 == -\frac{\overline{x}}{2} \,\&\&\, -\frac{\overline{xy}}{24} == -\frac{\overline{xy}}{12} \,\&\&\, \frac{\widehat{xy}}{48} == -\frac{\widehat{xy}}{48}, -\frac{\overline{y}}{2} == 0 \,\&\&\, -\frac{\overline{xy}}{12} == -\frac{\overline{xy}}{24} \,\&\&$$
$$\left.-\frac{1}{48}\,\overline{\overline{xy}\,y} == 0 \,\&\&\, 0 == -\frac{\overline{x}}{2} \,\&\&\, -\frac{\overline{xy}}{24} == -\frac{\overline{xy}}{12} \,\&\&\, -\frac{1}{48}\,\overline{\overline{xy}\,y} == 0 \,\&\&\, \frac{\widehat{xy}}{48} == -\frac{\widehat{xy}}{48}, \ldots\right]$$