Pensieve header: Figuring out triality(s).

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
<< WKO4.m
```

FreeLie` implements / extends

$\{*, +, **, \$SeriesShowDegree, \langle\rangle, \int, \equiv, ad, Ad, adSeries, AllCyclicWords,$ AllLyndonWords,

 AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, CW,

 CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp,

 Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,

 Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,

 $t, tb, TopBracketForm, tr, UndeterminedCoefficients, \alpha Map, \Gamma, \iota, \Lambda, \sigma, \hbar, \frown, \smile\}.$

FreeLie` is in the public domain. Dror Bar-Natan

 is committed to support it within reason until July 15, 2022.

AwCalculus` implements / extends $\{*, **, \equiv, dA, dc, deg, dm, dS,$

 $d\Delta, d\eta, d\sigma, El, Es, hA, hm, hS, h\Delta, h\eta, h\sigma, tA, tha, tm, tS, t\Delta, t\eta, t\sigma, \Gamma, \Lambda\}.$

AwCalculus` is in the public domain. Dror Bar-Natan

 is committed to support it within reason until July 15, 2022.

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting $\{\kappa s[x] \to 0\}$.

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting $\{\alpha s[x, y, y] \to 0\}$.

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting $\{\alpha s[x, x, x, y, y] \to 0\}$.

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

```
SeedRandom[0];
ζe = Es[⟨x → RandomLieSeries[{x, y}], y → RandomLieSeries[{x, y}]⟩,
    RandomCWSeries[{x, y}]];
ζf = Es[⟨x → RandomLieSeries[{x, y}], y → RandomLieSeries[{x, y}]⟩,
   RandomCWSeries[{x, y}]]
```

$\text{Es}\Big[\Big\langle x \to \text{LS}\Big[-\overline{x} - 2\,\overline{y},\ 2\,\overline{xy},\ -\frac{4}{3}\,\overline{x\,\overline{xy}} - \frac{7}{6}\,\overline{\overline{xy}\,y},\ \frac{1}{24}\,\overline{x\,\overline{x\,\overline{xy}}} - \frac{3}{4}\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{5}{6}\,\overline{\overline{\overline{xy}\,y}\,y},\ \ldots\Big],$

$\qquad y \to \text{LS}\Big[-\overline{x} - 2\,\overline{y},\ \frac{3\,\overline{xy}}{2},\ -\overline{x\,\overline{xy}} - \frac{3}{2}\,\overline{\overline{xy}\,y},\ \frac{3}{4}\,\overline{x\,\overline{x\,\overline{xy}}} + \frac{37}{24}\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{1}{12}\,\overline{\overline{\overline{xy}\,y}\,y},\ \ldots\Big]\Big\rangle,$

$\qquad \text{CWS}\Big[-\widehat{x},\ \overline{xx} + \frac{\overline{xy}}{2} + \frac{\overline{yy}}{2},\ -\frac{11\,\overline{xxx}}{6} - \frac{5\,\overline{xxy}}{3} - \frac{11\,\overline{xyy}}{6} - \frac{3\,\overline{yyy}}{2},$

$\qquad \frac{\overline{xxxx}}{2} - \frac{47\,\overline{xxxy}}{24} - \frac{29\,\overline{xxyy}}{24} - \frac{5\,\overline{xyxy}}{24} - \frac{17\,\overline{xyyy}}{24} + \frac{23\,\overline{yyyy}}{12},\ \ldots\Big]\Big]$

```
ρρ2[V_] := Rs[x, y] ** (V // (-1)^deg);
(ζe ≡ (ζe // ρρ2 // ρρ2))@{8}
BS[9 True, ...]
```

$\rho_3[\mathcal{E}\_\text{Es}] := \mathcal{E}$ // dS[y] // dΔ[y, y, z] // dm[x, z, x] // dσ[{x, y} → {y, x}];

{V₀@{2}, (V₀ // ρ_3)@{2}, (V₀ // ρ_3 // ρ_3 // ρ_3)@{2}}

$$\left\{ \text{Es}\left[\left\langle x \to \text{LS}\left[0, -\frac{\overline{x\,y}}{24}, \ldots \right], y \to \text{LS}\left[\frac{\overline{x}}{2}, -\frac{\overline{x\,y}}{12}, \ldots \right] \right\rangle, \text{CWS}\left[0, -\frac{\overline{x\,y}}{48}, \ldots \right] \right], \right.$$

$$\text{Es}\left[\left\langle x \to \text{LS}\left[-\frac{\overline{y}}{2}, \frac{\overline{x\,y}}{12}, \ldots \right], y \to \text{LS}\left[-\frac{\overline{y}}{2}, \frac{\overline{x\,y}}{24}, \ldots \right] \right\rangle, \text{CWS}\left[-\frac{\overline{y}}{2}, \frac{\overline{x\,y}}{48} + \frac{\overline{y\,y}}{48}, \ldots \right] \right],$$

$$\left. \text{Es}\left[\left\langle x \to \text{LS}\left[0, -\frac{\overline{x\,y}}{24}, \ldots \right], y \to \text{LS}\left[\frac{\overline{x}}{2}, -\frac{\overline{x\,y}}{12}, \ldots \right] \right\rangle, \text{CWS}\left[0, -\frac{\overline{x\,y}}{48}, \ldots \right] \right] \right\}$$

(V₀ ≡ (V₀ // ρ_3 // ρ_3 // ρ_3))@{6}

BS[7 True, ...]

(ξ_e ≡ (ξ_e // ρ_3 // ρ_3 // ρ_3))@{6}

BS[7 True, ...]

$\rho_{3'}[\mathcal{E}\_\text{Es}] := \mathcal{E}$ // dS[y] // dΔ[y, y, z] // dm[z, x, x] // dσ[{x, y} → {y, x}];

(ξ_e ≡ (ξ_e // $\rho_{3'}$ // $\rho_{3'}$ // $\rho_{3'}$))@{6}

BS[7 True, ...]

$\rho_3[\xi_e] \equiv \rho_{3'}[\xi_e]$

$\text{BS}\Big[\text{True}, \ -4\,\widehat{x} - 3\,\widehat{y} == -4\,\widehat{x} - 2\,\widehat{y}, \ -\dfrac{3\,\overline{x\,y}}{2} == -\dfrac{7\,\overline{x\,y}}{2} \ \&\& \ -2\,\overline{x\,y} == -3\,\overline{x\,y} \ \&\&$

$-4\,\widehat{x} - 3\,\widehat{y} == -4\,\widehat{x} - 2\,\widehat{y} \ \&\& \ \widehat{xx} + 4\,\overline{xy} + \dfrac{9\,\overline{yy}}{2} == \widehat{xx} + 6\,\overline{xy} + \dfrac{11\,\overline{yy}}{2},$

$-\dfrac{3\,\overline{x\,y}}{2} == -\dfrac{7\,\overline{x\,y}}{2} \ \&\& \ -\dfrac{11}{6}\,\overline{\overline{x\,y}\,y} == -4\,\overline{x\,\overline{x\,y}} - \dfrac{7}{3}\,\overline{\overline{x\,y}\,y} \ \&\& \ -2\,\overline{x\,y} == -3\,\overline{x\,y} \ \&\&$

$-\dfrac{19}{12}\,\overline{x\,\overline{x\,y}} - \dfrac{31}{12}\,\overline{\overline{x\,y}\,y} == -\dfrac{43}{12}\,\overline{x\,\overline{x\,y}} - \dfrac{49}{12}\,\overline{\overline{x\,y}\,y} \ \&\&$

$-4\,\widehat{x} - 3\,\widehat{y} == -4\,\widehat{x} - 2\,\widehat{y} \ \&\& \ \widehat{xx} + 4\,\overline{xy} + \dfrac{9\,\overline{yy}}{2} == \widehat{xx} + 6\,\overline{xy} + \dfrac{11\,\overline{yy}}{2} \ \&\&$

$-\dfrac{11\,\widehat{xxx}}{6} - \dfrac{29\,\widehat{xxy}}{6} - \dfrac{13\,\widehat{xyy}}{3} + \dfrac{\widehat{yyy}}{6} == -\dfrac{11\,\widehat{xxx}}{6} - \dfrac{19\,\widehat{xxy}}{6} - \dfrac{37\,\widehat{xyy}}{12} - \dfrac{5\,\widehat{yyy}}{12},$

$-\dfrac{3\,\overline{x\,y}}{2} == -\dfrac{7\,\overline{x\,y}}{2} \ \&\& \ -\dfrac{11}{6}\,\overline{\overline{x\,y}\,y} == -4\,\overline{x\,\overline{x\,y}} - \dfrac{7}{3}\,\overline{\overline{x\,y}\,y} \ \&\&$

$\dfrac{3}{2}\,\overline{x\,\overline{x\,y}} - \dfrac{91}{24}\,\overline{x\,\overline{\overline{x\,y}\,y}} + \dfrac{59}{24}\,\overline{\overline{\overline{x\,y}\,y}\,y} == -\dfrac{11}{6}\,\overline{x\,\overline{x\,y}} - \dfrac{199}{24}\,\overline{x\,\overline{\overline{x\,y}\,y}} + \dfrac{101}{24}\,\overline{\overline{\overline{x\,y}\,y}\,y} \ \&\&$

$-2\,\overline{x\,y} == -3\,\overline{x\,y} \ \&\& \ -\dfrac{19}{12}\,\overline{x\,\overline{x\,y}} - \dfrac{31}{12}\,\overline{\overline{x\,y}\,y} == -\dfrac{43}{12}\,\overline{x\,\overline{x\,y}} - \dfrac{49}{12}\,\overline{\overline{x\,y}\,y} \ \&\&$

$\dfrac{35}{8}\,\overline{x\,\overline{x\,y}} - \dfrac{139}{24}\,\overline{x\,\overline{\overline{x\,y}\,y}} + \dfrac{9}{8}\,\overline{\overline{\overline{x\,y}\,y}\,y} == \dfrac{65}{24}\,\overline{x\,\overline{x\,y}} - \dfrac{97}{8}\,\overline{x\,\overline{\overline{x\,y}\,y}} + \dfrac{23}{24}\,\overline{\overline{\overline{x\,y}\,y}\,y} \ \&\&$

$-4\,\widehat{x} - 3\,\widehat{y} == -4\,\widehat{x} - 2\,\widehat{y} \ \&\& \ \widehat{xx} + 4\,\overline{xy} + \dfrac{9\,\overline{yy}}{2} == \widehat{xx} + 6\,\overline{xy} + \dfrac{11\,\overline{yy}}{2} \ \&\&$

$-\dfrac{11\,\widehat{xxx}}{6} - \dfrac{29\,\widehat{xxy}}{6} - \dfrac{13\,\widehat{xyy}}{3} + \dfrac{\widehat{yyy}}{6} == -\dfrac{11\,\widehat{xxx}}{6} - \dfrac{19\,\widehat{xxy}}{6} - \dfrac{37\,\widehat{xyy}}{12} - \dfrac{5\,\widehat{yyy}}{12} \ \&\&$

$-\dfrac{11\,\widehat{xxxx}}{6} - \dfrac{85\,\widehat{xxxy}}{12} - \dfrac{127\,\widehat{xxyy}}{24} - \dfrac{19\,\widehat{xyxy}}{12} - \dfrac{11\,\widehat{xyyy}}{8} + \dfrac{\widehat{yyyy}}{3} ==$

$-\dfrac{11\,\widehat{xxxx}}{6} - \dfrac{181\,\widehat{xxxy}}{24} + \dfrac{43\,\widehat{xxyy}}{24} - \dfrac{197\,\widehat{xyxy}}{12} - \dfrac{59\,\widehat{xyyy}}{6} - \dfrac{41\,\widehat{yyyy}}{12}, \ \ldots\Big]$

$\big(\big(\rho_3[\xi_e] \; // \; dS\big) \equiv \rho_{3'}[\xi_e \; // \; dS]\big) @ \{6\}$

$\text{BS}[7\,\text{True}, \ \ldots]$

$\rho_3[\rho_3[\xi_e]] \equiv \rho_{3'}[\xi_e]$

$\mathtt{BS}\big[\mathtt{True},\ \overline{x} + 2\,\overline{y} == -2\,\overline{x} - 2\,\overline{y}\ \&\&\ 3\,\overline{x} + 2\,\overline{y} == -\overline{x} + \overline{y}\ \&\&\ 4\,\widehat{x} + \widehat{y} == -4\,\widehat{x} - 2\,\widehat{y},$

$\overline{x} + 2\,\overline{y} == -2\,\overline{x} - 2\,\overline{y}\ \&\&\ 2\,\overline{xy} == -\dfrac{7\,\overline{xy}}{2}\ \&\&\ 3\,\overline{x} + 2\,\overline{y} == -\overline{x} + \overline{y}\ \&\&\ -\dfrac{3\,\overline{xy}}{2} == -3\,\overline{xy}\ \&\&$

$\quad 4\,\widehat{x} + \widehat{y} == -4\,\widehat{x} - 2\,\widehat{y}\ \&\&\ \dfrac{9\,\widehat{xx}}{2} + 5\,\widehat{xy} + \dfrac{3\,\widehat{yy}}{2} == \widehat{xx} + 6\,\widehat{xy} + \dfrac{11\,\widehat{yy}}{2},$

$\overline{x} + 2\,\overline{y} == -2\,\overline{x} - 2\,\overline{y}\ \&\&\ 2\,\overline{xy} == -\dfrac{7\,\overline{xy}}{2}\ \&\&\ \dfrac{31}{12}\,\overline{x\,\overline{xy}} - \dfrac{25}{6}\,\overline{\overline{xy}\,y} == -4\,\overline{x\,\overline{xy}} - \dfrac{7}{3}\,\overline{\overline{xy}\,y}\ \&\&$

$\quad 3\,\overline{x} + 2\,\overline{y} == -\overline{x} + \overline{y}\ \&\&\ -\dfrac{3\,\overline{xy}}{2} == -3\,\overline{xy}\ \&\&\ -\dfrac{5}{3}\,\overline{x\,\overline{xy}} - \dfrac{1}{6}\,\overline{\overline{xy}\,y} == -\dfrac{43}{12}\,\overline{x\,\overline{xy}} - \dfrac{49}{12}\,\overline{\overline{xy}\,y}\ \&\&$

$\quad 4\,\widehat{x} + \widehat{y} == -4\,\widehat{x} - 2\,\widehat{y}\ \&\&\ \dfrac{9\,\widehat{xx}}{2} + 5\,\widehat{xy} + \dfrac{3\,\widehat{yy}}{2} == \widehat{xx} + 6\,\widehat{xy} + \dfrac{11\,\widehat{yy}}{2}\ \&\&$

$\quad -\dfrac{\widehat{xxx}}{6} - \dfrac{29\,\widehat{xxy}}{6} - \dfrac{13\,\widehat{xyy}}{3} - \dfrac{3\,\widehat{yyy}}{2} == -\dfrac{11\,\widehat{xxx}}{6} - \dfrac{19\,\widehat{xxy}}{6} - \dfrac{37\,\widehat{xyy}}{12} - \dfrac{5\,\widehat{yyy}}{12},$

$\overline{x} + 2\,\overline{y} == -2\,\overline{x} - 2\,\overline{y}\ \&\&\ 2\,\overline{xy} == -\dfrac{7\,\overline{xy}}{2}\ \&\&\ \dfrac{31}{12}\,\overline{x\,\overline{xy}} - \dfrac{25}{6}\,\overline{\overline{xy}\,y} == -4\,\overline{x\,\overline{xy}} - \dfrac{7}{3}\,\overline{\overline{xy}\,y}\ \&\&$

$\quad -\dfrac{9}{8}\,\overline{x\,\overline{x\,\overline{xy}}} - \dfrac{85}{24}\,\overline{x\,\overline{\overline{xy}\,y}} + \dfrac{7}{24}\,\overline{\overline{\overline{xy}\,y}\,y} == -\dfrac{11}{6}\,\overline{x\,\overline{x\,\overline{xy}}} - \dfrac{199}{24}\,\overline{x\,\overline{\overline{xy}\,y}} + \dfrac{101}{24}\,\overline{\overline{\overline{xy}\,y}\,y}\ \&\&$

$\quad 3\,\overline{x} + 2\,\overline{y} == -\overline{x} + \overline{y}\ \&\&\ -\dfrac{3\,\overline{xy}}{2} == -3\,\overline{xy}\ \&\&\ -\dfrac{5}{3}\,\overline{x\,\overline{xy}} - \dfrac{1}{6}\,\overline{\overline{xy}\,y} == -\dfrac{43}{12}\,\overline{x\,\overline{xy}} - \dfrac{49}{12}\,\overline{\overline{xy}\,y}\ \&\&$

$\quad -\dfrac{11}{8}\,\overline{x\,\overline{x\,\overline{xy}}} + \dfrac{9}{4}\,\overline{x\,\overline{\overline{xy}\,y}} - \dfrac{15}{8}\,\overline{\overline{\overline{xy}\,y}\,y} == \dfrac{65}{24}\,\overline{x\,\overline{x\,\overline{xy}}} - \dfrac{97}{8}\,\overline{x\,\overline{\overline{xy}\,y}} + \dfrac{23}{24}\,\overline{\overline{\overline{xy}\,y}\,y}\ \&\&$

$\quad 4\,\widehat{x} + \widehat{y} == -4\,\widehat{x} - 2\,\widehat{y}\ \&\&\ \dfrac{9\,\widehat{xx}}{2} + 5\,\widehat{xy} + \dfrac{3\,\widehat{yy}}{2} == \widehat{xx} + 6\,\widehat{xy} + \dfrac{11\,\widehat{yy}}{2}\ \&\&$

$\quad -\dfrac{\widehat{xxx}}{6} - \dfrac{29\,\widehat{xxy}}{6} - \dfrac{13\,\widehat{xyy}}{3} - \dfrac{3\,\widehat{yyy}}{2} == -\dfrac{11\,\widehat{xxx}}{6} - \dfrac{19\,\widehat{xxy}}{6} - \dfrac{37\,\widehat{xyy}}{12} - \dfrac{5\,\widehat{yyy}}{12}\ \&\&$

$\quad \dfrac{\widehat{xxxx}}{3} + \dfrac{65\,\widehat{xxxy}}{24} - \dfrac{29\,\widehat{xxyy}}{24} + \dfrac{11\,\widehat{xyxy}}{24} - \dfrac{29\,\widehat{xyyy}}{24} + \dfrac{\widehat{yyyy}}{12} ==$

$\qquad -\dfrac{11\,\widehat{xxxx}}{6} - \dfrac{181\,\widehat{xxxy}}{24} + \dfrac{43\,\widehat{xxyy}}{24} - \dfrac{197\,\widehat{xyxy}}{12} - \dfrac{59\,\widehat{xyyy}}{6} - \dfrac{41\,\widehat{yyyy}}{12},\ \ldots\big]$

ξ_e

$\mathtt{Es}\Big[\Big\langle x \to \mathtt{LS}\Big[2\,\overline{x} - \overline{y},\ \dfrac{3\,\overline{xy}}{2},\ \dfrac{1}{6}\,\overline{x\,\overline{xy}} - \dfrac{11}{6}\,\overline{\overline{xy}\,y},\ \dfrac{15}{8}\,\overline{x\,\overline{x\,\overline{xy}}} - \dfrac{3}{2}\,\overline{x\,\overline{\overline{xy}\,y}} + \overline{\overline{\overline{xy}\,y}\,y},\ \ldots\Big],$

$\qquad y \to \mathtt{LS}\Big[-2\,\overline{y},\ \dfrac{3\,\overline{xy}}{2},\ -\dfrac{11}{6}\,\overline{x\,\overline{xy}},\ -\dfrac{1}{6}\,\overline{x\,\overline{x\,\overline{xy}}} - \dfrac{19}{24}\,\overline{x\,\overline{\overline{xy}\,y}} - \dfrac{3}{2}\,\overline{\overline{\overline{xy}\,y}\,y},\ \ldots\Big]\Big\rangle,$

$\qquad \mathtt{CWS}\Big[\widehat{x} + 2\,\widehat{y},\ \dfrac{3\,\widehat{xx}}{2} - 2\,\widehat{xy} + \widehat{yy},\ \dfrac{3\,\widehat{xxx}}{2} + \dfrac{\widehat{xxy}}{6} + \dfrac{2\,\widehat{xyy}}{3} + \dfrac{11\,\widehat{yyy}}{6},$

$\qquad \dfrac{\widehat{xxxx}}{12} + \dfrac{37\,\widehat{xxxy}}{24} + \dfrac{37\,\widehat{xxyy}}{24} + \dfrac{11\,\widehat{xyxy}}{6} - \dfrac{\widehat{xyyy}}{4} - \dfrac{11\,\widehat{yyyy}}{6},\ \ldots\Big]\Big]\Big]$

ζₑ // ρ₃, // ρ₃ // ρ₃

$$\text{Es}\Big[\Big\langle x \to \text{LS}\Big[2\,\overline{x} - \overline{y},\ \frac{5\,\overline{x\,y}}{2},\ -\frac{17}{6}\,\overline{x\,\overline{x}\,\overline{y}} + \frac{1}{6}\,\overline{\overline{x}\,\overline{y}\,y},\ \frac{119}{24}\,\overline{x\,\overline{x}\,\overline{x}\,\overline{y}} - \frac{27}{4}\,\overline{x\,\overline{x}\,\overline{y}\,y} + \frac{8}{3}\,\overline{\overline{x}\,\overline{y}\,y\,y},\ \ldots\Big],$$

$$y \to \text{LS}\Big[-2\,\overline{y},\ \frac{7\,\overline{x\,y}}{2},\ -\frac{19}{3}\,\overline{x\,\overline{x}\,\overline{y}} + 4\,\overline{\overline{x}\,\overline{y}\,y},\ \frac{71}{12}\,\overline{x\,\overline{x}\,\overline{x}\,\overline{y}} - \frac{287}{24}\,\overline{x\,\overline{x}\,\overline{y}\,y} + \frac{11}{6}\,\overline{\overline{x}\,\overline{y}\,y\,y},\ \ldots\Big]\Big\rangle,$$

$$\text{CWS}\Big[2\,\widehat{x} + 2\,\widehat{y},\ \frac{\widehat{xx}}{2} - 4\,\widehat{xy} + \widehat{yy},\ \frac{4\,\widehat{xxx}}{3} + \frac{9\,\widehat{xxy}}{4} + \frac{7\,\widehat{xyy}}{3} + \frac{11\,\widehat{yyy}}{6},$$

$$-\frac{5\,\widehat{xxxx}}{2} - \frac{33\,\widehat{xxxy}}{8} + \frac{229\,\widehat{xxyy}}{24} - \frac{301\,\widehat{xyxy}}{24} + \frac{5\,\widehat{xyyy}}{24} - \frac{11\,\widehat{yyyy}}{6},\ \ldots\Big]\Big]$$

ζₑ // dA

$$\text{Es}\Big[\Big\langle x \to \text{LS}\Big[-2\,\overline{x} + \overline{y},\ \frac{\overline{x\,y}}{2},\ -\frac{7}{6}\,\overline{x\,\overline{x}\,\overline{y}} + \frac{5}{6}\,\overline{\overline{x}\,\overline{y}\,y},\ -\frac{31}{8}\,\overline{x\,\overline{x}\,\overline{x}\,\overline{y}} + \frac{15}{2}\,\overline{x\,\overline{x}\,\overline{y}\,y} - \frac{13}{4}\,\overline{\overline{x}\,\overline{y}\,y\,y},\ \ldots\Big],$$

$$y \to \text{LS}\Big[2\,\overline{y},\ -\frac{3\,\overline{x\,y}}{2},\ \frac{11}{6}\,\overline{x\,\overline{x}\,\overline{y}} - \frac{3}{2}\,\overline{\overline{x}\,\overline{y}\,y},\ \frac{1}{6}\,\overline{x\,\overline{x}\,\overline{x}\,\overline{y}} + \frac{33}{8}\,\overline{x\,\overline{x}\,\overline{y}\,y} + \frac{3}{4}\,\overline{\overline{x}\,\overline{y}\,y\,y},\ \ldots\Big]\Big\rangle,$$

$$\text{CWS}\Big[-\widehat{x} + 4\,\widehat{y},\ \frac{3\,\widehat{xx}}{2} - 3\,\widehat{xy} + \widehat{yy},\ \frac{3\,\widehat{xxx}}{2} + 3\,\widehat{xxy} + \frac{35\,\widehat{xyy}}{12} + \frac{11\,\widehat{yyy}}{6},$$

$$\frac{\widehat{xxxx}}{12} + \frac{53\,\widehat{xxxy}}{12} + \frac{49\,\widehat{xxyy}}{12} + \frac{7\,\widehat{xyxy}}{4} + \frac{37\,\widehat{xyyy}}{12} - \frac{11\,\widehat{yyyy}}{6},\ \ldots\Big]\Big]$$

```
θ[x_, s_] := Module[{y}, Θs[x, y, -s] // dS[y] // dm[x, y, x]];
θ[1, 1] @ {6}
```

$$\text{Es}\Big[\Big\langle 1 \to \text{LS}\Big[2\,\overline{1},\ 0,\ 0,\ 0,\ 0,\ 0,\ \ldots\Big]\Big\rangle,\ \text{CWS}\Big[\widehat{1},\ 0,\ 0,\ 0,\ 0,\ 0,\ \ldots\Big]\Big]$$

```
V₁ = V₀ ** Θs[x, y, -1/4] **
    Es[⟨x → LS@0, y → LS@0⟩, CWS[CW[x]/12 - CW[y]/12] - (2 Cap⟦2⟧ // tΔ[x, x, y])];
(V₁ ≡ ρ₃[V₁]) @ {6}
```

```
BS[7 True, ...]
```

```
(V₁ ≡ ρ₃[V₁]) @ {7}
```

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {αs[x, x, x, x, y, y] → 0}.

```
BS[8 True, ...]
```

```
w1 = Es[⟨x → LS@0, y → LS@0⟩, CWS[CW[x]/12 - CW[y]/12] - (2 Cap⟦2⟧ // tΔ[x, x, y])]
```

$$\text{Es}\Big[\Big\langle x \to \text{LS}[0, 0, 0, 0, \ldots],\ y \to \text{LS}[0, 0, 0, 0, \ldots]\Big\rangle,$$

$$\text{CWS}\Big[\frac{\widehat{x}}{12} - \frac{\widehat{y}}{12},\ \frac{\widehat{xx}}{48} + \frac{\widehat{xy}}{24} + \frac{\widehat{yy}}{48},\ 0,\ -\frac{\widehat{xxxx}}{5760} - \frac{\widehat{xxxy}}{1440} - \frac{\widehat{xxyy}}{1440} - \frac{\widehat{xyxy}}{2880} - \frac{\widehat{xyyy}}{1440} - \frac{\widehat{yyyy}}{5760},\ \ldots\Big]\Big]$$

```
V₂ = V₀ ** Θs[x, y, -1/4] ** w1;
(V₂ ≡ ρ₃[V₂]) @ {6}
```

```
BS[7 True, ...]
```

```
(V₀ ** Θs[x, y, -1/4] ** w1 ≡ ρ₃[V₀ ** Θs[x, y, -1/4] ** w1]) @ {6}
```

```
BS[7 True, ...]
```

Q = V₀ ∗∗ ρ₃[V₀]⁻¹

$$\text{Es}\Big[\Big\langle x \to \text{LS}\Big[\frac{\overline{y}}{2}, -\frac{\overline{x\,y}}{8}, \frac{1}{32}\overline{\overline{x\,y}\,y}, \frac{1}{384}\overline{x\,\overline{x\,\overline{x\,y}}} - \frac{1}{128}\overline{x\,\overline{\overline{x\,y}\,y}}, \dots\Big],$$

$$y \to \text{LS}\Big[\frac{\overline{x}}{2} + \frac{\overline{y}}{2}, -\frac{\overline{x\,y}}{4}, \frac{1}{24}\overline{x\,\overline{x\,y}} + \frac{1}{24}\overline{\overline{x\,y}\,y}, -\frac{1}{48}\overline{x\,\overline{\overline{x\,y}\,y}}, \dots\Big]\Big\rangle,$$

$$\text{CWS}\Big[\frac{\overline{y}}{2}, -\frac{\overline{x\,y}}{24} - \frac{\overline{y\,y}}{48}, 0, \frac{\overline{x\,x\,x\,y}}{1440} - \frac{\overline{x\,x\,y\,y}}{360} + \frac{11\,\overline{x\,y\,x\,y}}{2880} + \frac{\overline{x\,y\,y\,y}}{1440} + \frac{\overline{y\,y\,y\,y}}{5760}, \dots\Big]\Big]\Big]$$

Q@{6}

$$\text{Es}\Big[\Big\langle x \to \text{LS}\Big[\frac{\overline{y}}{2}, -\frac{\overline{x\,y}}{8}, \frac{1}{32}\overline{\overline{x\,y}\,y}, \frac{1}{384}\overline{x\,\overline{x\,\overline{x\,y}}} - \frac{1}{128}\overline{x\,\overline{\overline{x\,y}\,y}},$$

$$-\frac{\overline{x\,x\,\overline{\overline{x\,y}\,y}}}{1536} + \frac{1}{512}\overline{x\,\overline{\overline{\overline{x\,y}\,y}\,y}} + \frac{1}{384}\overline{\overline{x\,y}\,\overline{\overline{x\,y}\,y}} + \frac{\overline{x\,\overline{x\,y}\,\overline{x\,y}}}{1536} - \frac{\overline{\overline{\overline{x\,y}\,y}\,y\,y}}{1536},$$

$$-\frac{\overline{x\,x\,x\,\overline{x\,\overline{x\,y}}}}{15\,360} + \frac{\overline{x\,x\,x\,\overline{\overline{x\,y}\,y}}}{3072} - \frac{\overline{x\,x\,\overline{\overline{\overline{x\,y}\,y}\,y}}}{2048} - \frac{\overline{x\,\overline{x\,y}\,\overline{\overline{x\,y}\,y}}}{1536} - \frac{\overline{x\,\overline{x\,\overline{x\,y}}\,\overline{x\,y}}}{6144} + \frac{\overline{x\,\overline{\overline{\overline{x\,y}\,y}\,y\,y}}}{6144}, \dots\Big],$$

$$y \to \text{LS}\Big[\frac{\overline{x}}{2} + \frac{\overline{y}}{2}, -\frac{\overline{x\,y}}{4}, \frac{1}{24}\overline{x\,\overline{x\,y}} + \frac{1}{24}\overline{\overline{x\,y}\,y}, -\frac{1}{48}\overline{x\,\overline{\overline{x\,y}\,y}},$$

$$-\frac{\overline{x\,x\,\overline{x\,\overline{x\,y}}}}{1440} + \frac{1}{360}\overline{x\,x\,\overline{\overline{x\,y}\,y}} + \frac{1}{360}\overline{x\,\overline{\overline{\overline{x\,y}\,y}\,y}} + \frac{1}{240}\overline{\overline{x\,y}\,\overline{\overline{x\,y}\,y}} + \frac{1}{720}\overline{x\,\overline{x\,y}\,\overline{x\,y}} - \frac{\overline{\overline{\overline{x\,y}\,y}\,y\,y}}{1440},$$

$$\frac{\overline{x\,x\,x\,\overline{\overline{x\,y}\,y}}}{2880} - \frac{1}{720}\overline{x\,x\,\overline{\overline{\overline{x\,y}\,y}\,y}} - \frac{1}{480}\overline{x\,\overline{x\,y}\,\overline{\overline{x\,y}\,y}} - \frac{\overline{x\,\overline{x\,\overline{x\,y}}\,\overline{x\,y}}}{1440} + \frac{\overline{x\,\overline{\overline{\overline{x\,y}\,y}\,y\,y}}}{2880}, \dots\Big]\Big\rangle,$$

$$\text{CWS}\Big[\frac{\overline{y}}{2}, -\frac{\overline{x\,y}}{24} - \frac{\overline{y\,y}}{48}, 0, \frac{\overline{x\,x\,x\,y}}{1440} - \frac{\overline{x\,x\,y\,y}}{360} + \frac{11\,\overline{x\,y\,x\,y}}{2880} + \frac{\overline{x\,y\,y\,y}}{1440} + \frac{\overline{y\,y\,y\,y}}{5760}, 0,$$

$$-\frac{\overline{x\,x\,x\,x\,y}}{60\,480} + \frac{\overline{x\,x\,x\,y\,y}}{10\,080} - \frac{37\,\overline{x\,x\,x\,y\,x\,y}}{120\,960} - \frac{23\,\overline{x\,x\,x\,y\,y\,y}}{120\,960} + \frac{\overline{x\,x\,y\,x\,y}}{6048} + \frac{\overline{x\,x\,y\,x\,y\,y}}{3024} + \frac{\overline{x\,x\,y\,y\,x\,y}}{3024} +$$

$$\frac{\overline{x\,x\,y\,y\,y\,y}}{10\,080} - \frac{191\,\overline{x\,y\,x\,y\,x\,y}}{362\,880} - \frac{37\,\overline{x\,y\,x\,y\,y\,y}}{120\,960} + \frac{\overline{x\,y\,y\,x\,y\,y}}{6048} - \frac{\overline{x\,y\,y\,y\,y\,y}}{60\,480} - \frac{\overline{y\,y\,y\,y\,y\,y}}{362\,880}, \dots\Big]\Big]\Big]$$

(V₀ ≡ Q ∗∗ ρ₃[V₀]) @{6}

BS[7 True, ...]

(V₀ ≡ ρ₃[V₀ ∗∗ Θs[x, y, -1/4] ∗∗ w1] ∗∗ w1⁻¹ ∗∗ Θs[x, y, 1/4]) @{6}

BS[7 True, ...]

ρρ₃[V_] := ρ₃[V ∗∗ Θs[x, y, -1/4] ∗∗ w1] ∗∗ w1⁻¹ ∗∗ Θs[x, y, 1/4]

(ξₑ ≡ (ξₑ // ρρ₃ // ρρ₃ // ρρ₃)) @{6}

BS[7 True, ...]

$(\rho_3[\xi_e] ** \rho_3[\xi_f]) \equiv \rho_3[\xi_e ** \xi_f]$

$$\text{BS}\left[2 \text{ True}, -6\,\overline{x\,y} == -4\,\overline{x\,y} \,\&\&\, -\frac{5\,\overline{x\,y}}{2} == -\frac{3\,\overline{x\,y}}{2},\right.$$

$$-6\,\overline{x\,y} == -4\,\overline{x\,y} \,\&\&\, -\frac{3}{2}\,\overline{x\,\overline{x\,y}} - \frac{31}{4}\,\overline{\overline{x\,y}\,y} == \frac{3}{2}\,\overline{x\,\overline{x\,y}} - \frac{21}{4}\,\overline{\overline{x\,y}\,y} \,\&\&\,$$

$$-\frac{5\,\overline{x\,y}}{2} == -\frac{3\,\overline{x\,y}}{2} \,\&\&\, -\frac{5}{2}\,\overline{x\,\overline{x\,y}} - \frac{55}{12}\,\overline{\overline{x\,y}\,y} == -2\,\overline{x\,\overline{x\,y}} - \frac{19}{12}\,\overline{\overline{x\,y}\,y},$$

$$-6\,\overline{x\,y} == -4\,\overline{x\,y} \,\&\&\, -\frac{3}{2}\,\overline{x\,\overline{x\,y}} - \frac{31}{4}\,\overline{\overline{x\,y}\,y} == \frac{3}{2}\,\overline{x\,\overline{x\,y}} - \frac{21}{4}\,\overline{\overline{x\,y}\,y} \,\&\&\,$$

$$\frac{47}{12}\,\overline{x\,\overline{x\,\overline{x\,y}}} - \frac{121}{12}\,\overline{x\,\overline{\overline{x\,y}\,y}} - \frac{71}{24}\,\overline{\overline{\overline{x\,y}\,y}\,y} == \frac{1}{4}\,\overline{x\,\overline{x\,\overline{x\,y}}} - \frac{7}{6}\,\overline{x\,\overline{\overline{x\,y}\,y}} - \frac{17}{24}\,\overline{\overline{\overline{x\,y}\,y}\,y} \,\&\&\,$$

$$-\frac{5\,\overline{x\,y}}{2} == -\frac{3\,\overline{x\,y}}{2} \,\&\&\, -\frac{5}{2}\,\overline{x\,\overline{x\,y}} - \frac{55}{12}\,\overline{\overline{x\,y}\,y} == -2\,\overline{x\,\overline{x\,y}} - \frac{19}{12}\,\overline{\overline{x\,y}\,y} \,\&\&\,$$

$$\frac{91}{12}\,\overline{x\,\overline{x\,\overline{x\,y}}} - \frac{275}{24}\,\overline{x\,\overline{\overline{x\,y}\,y}} - \frac{37}{8}\,\overline{\overline{\overline{x\,y}\,y}\,y} == \frac{21}{4}\,\overline{x\,\overline{x\,\overline{x\,y}}} - \frac{155}{24}\,\overline{x\,\overline{\overline{x\,y}\,y}} + \frac{3}{8}\,\overline{\overline{\overline{x\,y}\,y}\,y} \,\&\&\,$$

$$\frac{\overline{xxxx}}{12} + \frac{31\,\overline{xxxy}}{24} + \frac{127\,\overline{xxyy}}{12} - \frac{21\,\overline{xyxy}}{4} + \frac{181\,\overline{xyyy}}{24} + 4\,\overline{yyyy} ==$$

$$\left.\frac{\overline{xxxx}}{12} + \frac{31\,\overline{xxxy}}{24} + \frac{55\,\overline{xxyy}}{12} + \frac{3\,\overline{xyxy}}{4} + \frac{181\,\overline{xyyy}}{24} + 4\,\overline{yyyy}, \ldots\right]$$