Pensieve header: Φ equations in the 2-poles 2-strands universe.

In[•]:=
```
SetDirectory["C:/drorbn/AcademicPensieve/Projects/WKO4"];
<< FreeLie.m;
Φ0s[2, 1] = Φ0s[3, 1] = Φ0s[3, 2] = 0;
Φ0s[3, 1, 2] = 1 / 24; Φ0 = DKS[3, Φ0s];
SeriesSolve[Φ0, Φ0^σ[3,2,1] ≡ -Φ0 && Φ0 ** Φ0^σ[1,23,4] ** Φ0^σ[2,3,4] ≡ Φ0^σ[12,3,4] ** Φ0^σ[1,2,34]];
Φ1s[2, 1] = Φ1s[3, 1] = Φ1s[3, 2] = 0;
Φ1s[3, 1, 2] = 1 / 24;
Φ1s[3, 1, 1, 1, 1, 2] = 1;
Φ1 = DKS[3, Φ1s];
SeriesSolve[Φ1, Φ1^σ[3,2,1] ≡ -Φ1 && Φ1 ** Φ1^σ[1,23,4] ** Φ1^σ[2,3,4] ≡ Φ1^σ[12,3,4] ** Φ1^σ[1,2,34]];
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw,
  CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
  Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
  Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
  t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ħ, ⟞, ⟝}.

FreeLie` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150814.
```

In[•]:=
```
τ = FreeLie`Private`τ
```

Out[•]=
```
FreeLie`Private`τ
```

In[•]:=
```
Φ0@{5}
Φ1@{5}
```

SeriesSolve: In degree 3 arbitrarily setting {Φ0s[3, 1, 1, 2] → 0}.

SeriesSolve: In degree 5 arbitrarily setting {Φ0s[3, 1, 1, 1, 1, 2] → 0}.

Out[•]=

$$\text{DKS}\left[0,\ \frac{1}{24}\,\overline{t_{13}\,t_{23}},\ 0,\ -\frac{7\,\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}}}{5760}+\frac{7\,\overline{t_{13}\,\overline{\overline{t_{13}\,t_{23}}\,t_{23}}}}{5760}-\frac{\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,t_{23}}}}}{1440},\ 0,\ \ldots\right]$$

SeriesSolve: In degree 3 arbitrarily setting {Φ1s[3, 1, 1, 2] → 0}.

Out[•]=

$$\text{DKS}\left[0,\ \frac{1}{24}\,\overline{t_{13}\,t_{23}},\ 0,\ -\frac{7\,\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}}}{5760}+\frac{7\,\overline{t_{13}\,\overline{\overline{t_{13}\,t_{23}}\,t_{23}}}}{5760}-\frac{\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,t_{23}}}}}{1440},\right.$$
$$\left.-\frac{1}{2}\,\overline{t_{13}\,\overline{t_{13}\,t_{23}}\,\overline{t_{13}\,t_{23}}}-\overline{t_{13}\,t_{23}\,\overline{t_{13}\,t_{23}}\,t_{23}}+\overline{t_{13}\,\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}}-\overline{t_{13}\,t_{13}\,\overline{\overline{t_{13}\,t_{23}}\,t_{23}}}+\overline{t_{13}\,\overline{t_{13}\,\overline{\overline{t_{13}\,t_{23}}}}},\ \ldots\right]$$

The same thing, copy-paste ready and machine readable:

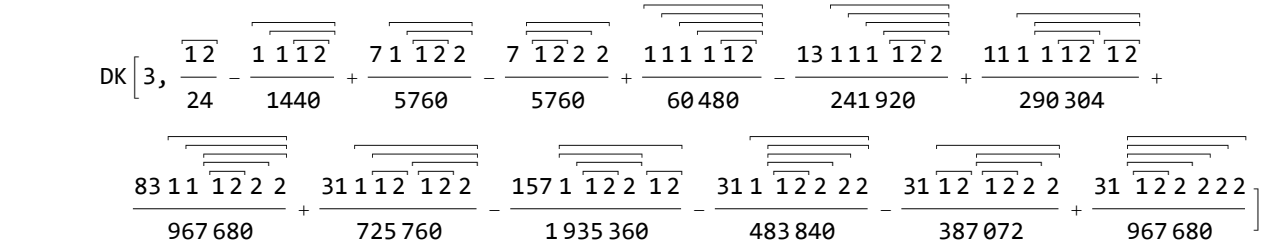
```
In[•]:= Sum[Φ0[k], {k, 6}] // InputForm
```

```
Out[•]//InputForm=
DK[3, LW[1, 2]/24 - LW[1, 1, 1, 2]/1440 + (7*LW[1, 1, 2, 2])/5760 -
  (7*LW[1, 2, 2, 2])/5760 + LW[1, 1, 1, 1, 1, 2]/60480 -
  (13*LW[1, 1, 1, 1, 2, 2])/241920 + (11*LW[1, 1, 1, 2, 1, 2])/290304 +
  (83*LW[1, 1, 1, 2, 2, 2])/967680 + (31*LW[1, 1, 2, 1, 2, 2])/725760 -
  (157*LW[1, 1, 2, 2, 1, 2])/1935360 - (31*LW[1, 1, 2, 2, 2, 2])/483840 -
  (31*LW[1, 2, 1, 2, 2, 2])/387072 + (31*LW[1, 2, 2, 2, 2, 2])/967680]
```

Note that in this context, “LW[1, 1, 2, 2]” (for example) really means "LW[t_{13}, t_{13}, t_{23}, t_{23}]”.

```
In[•]:= Sum[Φ0[k], {k, 6}]
```

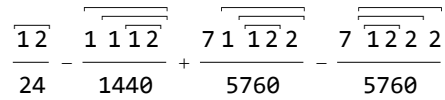
```
Out[•]=
```



```
In[•]:= φ[n_] := Sum[Φ0[k], {k, n}][[2]]
```

```
In[•]:= φ[4]
```

```
Out[•]=
```



```
In[•]:= τ[LW[2], φ[4]]
```

```
Out[•]=
```

$$\frac{AW[1]}{24}-\frac{AW[1,1,1]}{1440}+\frac{7\,AW[1,1,2]}{5760}-\frac{7\,AW[1,2,1]}{2880}-\frac{7\,AW[1,2,2]}{5760}+\frac{7\,AW[2,1,2]}{1920}-\frac{7\,AW[2,2,1]}{1920}$$

In[•]:= $\tau[\mathbf{LW}[2], \varphi[6]]$

Out[•]=

$$\frac{\mathbf{AW}[1]}{24} - \frac{\mathbf{AW}[1,1,1]}{1440} + \frac{7\,\mathbf{AW}[1,1,2]}{5760} - \frac{7\,\mathbf{AW}[1,2,1]}{2880} - \frac{7\,\mathbf{AW}[1,2,2]}{5760} + \frac{7\,\mathbf{AW}[2,1,2]}{1920} - \frac{7\,\mathbf{AW}[2,2,1]}{1920} + \frac{\mathbf{AW}[1,1,1,1,1]}{60\,480} - \frac{13\,\mathbf{AW}[1,1,1,1,2]}{241\,920} + \frac{211\,\mathbf{AW}[1,1,1,2,1]}{1\,451\,520} + \frac{83\,\mathbf{AW}[1,1,1,2,2]}{967\,680} - \frac{11\,\mathbf{AW}[1,1,2,1,1]}{96\,768} - \frac{89\,\mathbf{AW}[1,1,2,1,2]}{414\,720} + \frac{31\,\mathbf{AW}[1,1,2,2,1]}{645\,120} - \frac{31\,\mathbf{AW}[1,1,2,2,2]}{483\,840} + \frac{11\,\mathbf{AW}[1,2,1,1,1]}{145\,152} + \frac{223\,\mathbf{AW}[1,2,1,1,2]}{5\,806\,080} + \frac{31\,\mathbf{AW}[1,2,1,2,1]}{181\,440} + \frac{341\,\mathbf{AW}[1,2,1,2,2]}{1\,935\,360} - \frac{31\,\mathbf{AW}[1,2,2,1,1]}{725\,760} - \frac{31\,\mathbf{AW}[1,2,2,1,2]}{215\,040} + \frac{31\,\mathbf{AW}[1,2,2,2,1]}{322\,560} + \frac{31\,\mathbf{AW}[1,2,2,2,2]}{967\,680} - \frac{157\,\mathbf{AW}[2,1,1,1,2]}{1\,935\,360} + \frac{157\,\mathbf{AW}[2,1,1,2,1]}{967\,680} + \frac{31\,\mathbf{AW}[2,1,1,2,2]}{387\,072} - \frac{157\,\mathbf{AW}[2,1,2,1,1]}{967\,680} - \frac{31\,\mathbf{AW}[2,1,2,1,2]}{129\,024} - \frac{31\,\mathbf{AW}[2,1,2,2,2]}{193\,536} + \frac{157\,\mathbf{AW}[2,2,1,1,1]}{1\,935\,360} + \frac{31\,\mathbf{AW}[2,2,1,2,1]}{129\,024} + \frac{31\,\mathbf{AW}[2,2,1,2,2]}{96\,768} - \frac{31\,\mathbf{AW}[2,2,2,1,1]}{387\,072} - \frac{31\,\mathbf{AW}[2,2,2,1,2]}{96\,768} + \frac{31\,\mathbf{AW}[2,2,2,2,1]}{193\,536}$$

In[•]:= $\tau[\mathbf{LW}[2], \Phi_1[5][[2]]]$

Out[•]=

$$\mathbf{AW}[1,1,1,1] - \mathbf{AW}[1,1,1,2] + \frac{3}{2}\,\mathbf{AW}[1,1,2,1] + \mathbf{AW}[1,1,2,2] + \frac{3}{2}\,\mathbf{AW}[1,2,1,1] - 4\,\mathbf{AW}[1,2,1,2] + 6\,\mathbf{AW}[1,2,2,1] - \mathbf{AW}[2,1,1,1] + \mathbf{AW}[2,1,1,2] - 4\,\mathbf{AW}[2,1,2,1] + \mathbf{AW}[2,2,1,1]$$