

Pensieve header: Testing the FreeLie package.

In[*]:=

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
<< FreeLie.m
$SeriesShowDegree = 4
```

FreeLie` implements / extends

{*, +, **, \$SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords, AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization, Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support, t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ℓ, Δ, σ, ħ, ↦, ↠}.

FreeLie` is in the public domain. Dror Bar-Natan is committed to support it within reason until July 15, 2022. This is version 150814.

Out[*]= 4

{α = RandomLieSeries[{1, 2}], β = RandomLieSeries[{1, 2}]}

$$\left\{ \text{LS} \left[-\overline{1} - 2\overline{2}, -\frac{3\overline{12}}{2}, \frac{1}{3}\overline{112} - \frac{1}{2}\overline{122}, \frac{5}{24}\overline{1112} - \frac{11}{6}\overline{1122} - \frac{7}{6}\overline{1222}, \dots \right], \right. \\ \left. \text{LS} \left[\overline{1} - \overline{2}, \overline{12}, \frac{11}{6}\overline{112} + \frac{1}{6}\overline{122}, -\frac{11}{24}\overline{1112} - \frac{1}{6}\overline{1122} - \frac{35}{24}\overline{1222}, \dots \right] \right\}$$

EulerE[α]

$$\text{LS} \left[-2\overline{2}, -2\overline{12}, \frac{7}{2}\overline{112} + 4\overline{122}, -\frac{13}{2}\overline{1112} - \frac{8}{3}\overline{1122} - \frac{4}{3}\overline{1222}, \dots \right]$$

BCH[LW@x, LW@y]

$$\text{LS} \left[\overline{x} + \overline{y}, \frac{\overline{xy}}{2}, \frac{1}{12}\overline{xxxy} + \frac{1}{12}\overline{xyyy}, \frac{1}{24}\overline{xyxy}, \dots \right]$$

α // CC₁[β]

$$\text{LS} \left[\overline{1} + 2\overline{2}, 3\overline{12}, \frac{11}{6}\overline{112} + \frac{7}{2}\overline{122}, \frac{1}{6}\overline{1112} + \frac{19}{8}\overline{1122} + \frac{31}{24}\overline{1222}, \dots \right]$$

α // RC₁[β]

$$\text{LS} \left[-\overline{1} - 2\overline{2}, -\frac{5\overline{12}}{2}, \frac{11}{6}\overline{112} - \frac{5}{2}\overline{122}, \frac{11}{8}\overline{1112} + \frac{37}{12}\overline{1122} - \frac{31}{12}\overline{1222}, \dots \right]$$

(α // RC₁[β])[3]

$$\frac{7}{6}\text{LW}[1, 1, 2] + \frac{1}{3}\text{LW}[1, 2, 2]$$

In[*]:=

```
Do[Echo@Timing[k → Length[AllLyndonWords[k, {x, y, z}]]], {k, 15}]
```

- » {0., 1 → 3}
- » {0., 2 → 3}
- » {0., 3 → 8}
- » {0., 4 → 18}
- » {0.015625, 5 → 48}
- » {0.046875, 6 → 116}
- » {0.171875, 7 → 312}
- » {0.53125, 8 → 810}
- » {1.8125, 9 → 2184}
- » {4.57813, 10 → 5880}
- » {13.25, 11 → 16104}
- » {44.3125, 12 → 44220}
- » {144.156, 13 → 122640}
- » {472.766, 14 → 341484}
- » {1535.52, 15 → 956576}

In[]:= **Log[3, 956576] // N**

Out[]:= 12.535

Timing[Length[alw = AllLyndonWords[13, {x, y}]]]

{2.496016, 630}

Timing[ℓ /@ alw;]

{10.686068, Null}

ℓ[alw[[1]]]

AW[x, x, x, x, x, x, x, x, x, x, x, x, y] -
 12 AW[x, x, x, x, x, x, x, x, x, x, x, y, x] + 66 AW[x, x, x, x, x, x, x, x, x, x, y, x, x] -
 220 AW[x, x, x, x, x, x, x, x, x, y, x, x, x] + 495 AW[x, x, x, x, x, x, x, x, y, x, x, x, x] -
 792 AW[x, x, x, x, x, x, x, y, x, x, x, x, x] + 924 AW[x, x, x, x, x, x, y, x, x, x, x, x, x] -
 792 AW[x, x, x, x, x, y, x, x, x, x, x, x, x] + 495 AW[x, x, x, x, y, x, x, x, x, x, x, x, x] -
 220 AW[x, x, x, y, x, x, x, x, x, x, x, x, x] + 66 AW[x, x, y, x, x, x, x, x, x, x, x, x, x] -
 12 AW[x, y, x, x, x, x, x, x, x, x, x, x, x] + AW[y, x, x, x, x, x, x, x, x, x, x, x, x]

div[x, alw[[1]]]

-CW[x, x, x, x, x, x, x, x, x, x, x, x, y]

div[y, alw[[1]]]

CW[x, x, x, x, x, x, x, x, x, x, x, x, y]

Timing[div[x, #] & /@ alw;]

{14.586094, Null}

A = LS[{x, y}, As]; dxA = div[x, A]

CWS[As[x] \overline{x} , -As[x, y] \overline{xy} , -As[x, x, y] \overline{xyx} + As[x, y, y] \overline{xyy} ,
 -As[x, x, x, y] \overline{xxxxy} - As[x, x, y, y] \overline{xyxy} + 2 As[x, x, y, y] \overline{xyxy} - As[x, y, y, y] \overline{xyyyy} , ...]

dxA[3]

-As[x, x, y] CW[x, x, y] + As[x, y, y] CW[x, y, y]

Do[Print[Timing[dxA[d]; d]], {d, 13}]

{0., 1}
 {0., 2}
 {0., 3}
 {0., 4}
 {0., 5}
 {0.015600, 6}
 {0.031200, 7}
 {0.078001, 8}
 {0.187201, 9}
 {0.296402, 10}
 {1.107607, 11}
 {2.714417, 12}
 {7.488048, 13}

Do[Print[Timing[A[d]; d]], {d, 13}]

{0., 1}
 {0., 2}
 {0., 3}
 {0., 4}
 {0., 5}
 {0., 6}
 {0., 7}
 {0., 8}
 {0., 9}
 {0., 10}
 {0., 11}
 {0., 12}
 {0., 13}

Do[Print[Timing[div[x, A[d]]; d]], {d, 13}]

```
{0., 1}
{0., 2}
{0., 3}
{0., 4}
{0., 5}
{0., 6}
{0., 7}
{0., 8}
{0., 9}
{0.015600, 10}
{4.695630, 11}
$Aborted
```

```
Do[Print[Timing[div[x, List@@ (A[d])]; d]], {d, 13}]
```

```
{0., 1}
{0., 2}
{0., 3}
{0., 4}
{0.015600, 5}
{0.015600, 6}
{0.062400, 7}
{0.124801, 8}
{0.312002, 9}
{0.702005, 10}
{1.918812, 11}
{4.789231, 12}
{13.416086, 13}
```

```
div[x, List@@ (A[4])]
```

```
{-As[x, x, x, y] CW[x, x, x, y],
 -As[x, x, y, y] CW[x, x, y, y] + 2 As[x, x, y, y] CW[x, y, x, y], -As[x, y, y, y] CW[x, y, y, y]}
```

```
Do[Print[Timing[Total[div[x, List@@ (A[d])]] // Expand; d]], {d, 13}]
```

```
{0., 1}
{0., 2}
{0., 3}
{0., 4}
{0., 5}
{0., 6}
{0., 7}
{0., 8}
{0., 9}
{0.015600, 10}
{0.031200, 11}
{0.078001, 12}
{0.234002, 13}
```

```
Table[(Total[div[x, List@@ (A[d])] // Expand) == div[x, A[d]], {d, 13}] // Timing
{True, As[x, y] - CW[x, y] == -As[x, y] CW[x, y],
 True, True, True, True, True, True, True, True, True, True}
```