# FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS IV: SOME COMPUTATIONS

DROR BAR-NATAN

Abstract. In the previous three papers in this series, [WKO1]–[WKO3], Z. Dancso and I studied a certain theory of "homomorphic expansions" of "w-knotted objects", a certain class of knotted objects in 4-dimensional space. When all layers of interpretation are stripped off, what remains is a study of a certain number of equations written in a family of spaces $\mathcal{A}^w$, closely related to degree-completed free Lie algebras and to degree-completed spaces of cyclic words.

The purpose of this paper is to introduce mathematical and computational tools that enable explicit computations (up to a certain degree) in these $\mathcal{A}^w$ spaces and to use these tools to solve the said equations and verify some properties of their solutions, and as a consequence, to carry out the computation (up to a certain degree) of certain knot-theoretic invariants discussed in [WKO1]–[WKO3] and in my related paper [BN4].

## Contents



*computations below*

*Date*: First edition Nov. 15, 2015, this edition Nov. 22, 2022. Electronic version and related files at [WKO4], `http://drorbn.net/AcademicPensieve/Projects/WKO4`. The arXiv:1511.05624 edition may be older.
2010 *Mathematics Subject Classification.* 57M25.
*Key words and phrases.* w-knots, w-tangles, Kashiwara-Vergne, associators, double tree, Mathematica, free Lie algebras.
This work was partially supported by NSERC grant RGPIN 262178.

# 1. Introduction

Within the previous three papers in this series [WKO1]–[WKO3][1] a number of intricate equations written in various graded spaces related to free Lie algebras and to spaces of cyclic words were examined in detail, for good reasons that were explained there and elsewhere. The purpose of this paper is to introduce mathematical tools (on the upper parts of pages) and computational tools (on the lower parts of pages, below the bold dividing lines[C1]) that allow for the explicit solution of these equations, at least up to a certain degree.

[1]Also within my [BN4], and within papers by Alekseev, Enriquez, and Torossian [AT, AET], and within Kashiwara's and Vergne's [KV], and also within many older papers about Drinfel'd associators (e.g. Drinfel'd's [Dr1, Dr2] and my [BN2].

*computations below*

[C1]If you are not interested in the actual computations, it is safe to ignore the parts of pages below the bold dividing lines and restrict to "strict" mathematics, which is always above these lines. **Alert.** If you are interested in the computations, note that the computational footnotes are sometimes long and crawl across page boundaries. This footnote is the first example.
The programs described in this paper were written in Mathematica [Wo] and are available at [WKO4]. Before starting with any computations, download the packages `FreeLie.m` and `AwCalculus.m` and type within Mathematica: (the interactive Mathematica session demonstrated in this paper is available as [WKO4]/`WKO4Session.nb`)

```
<< FreeLie.m;
<< AwCalculus.m;
$SeriesShowDegree = 4;
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw,
  CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
  Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
  Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
  t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ℏ, ⌐, ⌢}.

FreeLie` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA,
  hm, hS, hΔ, hη, hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed
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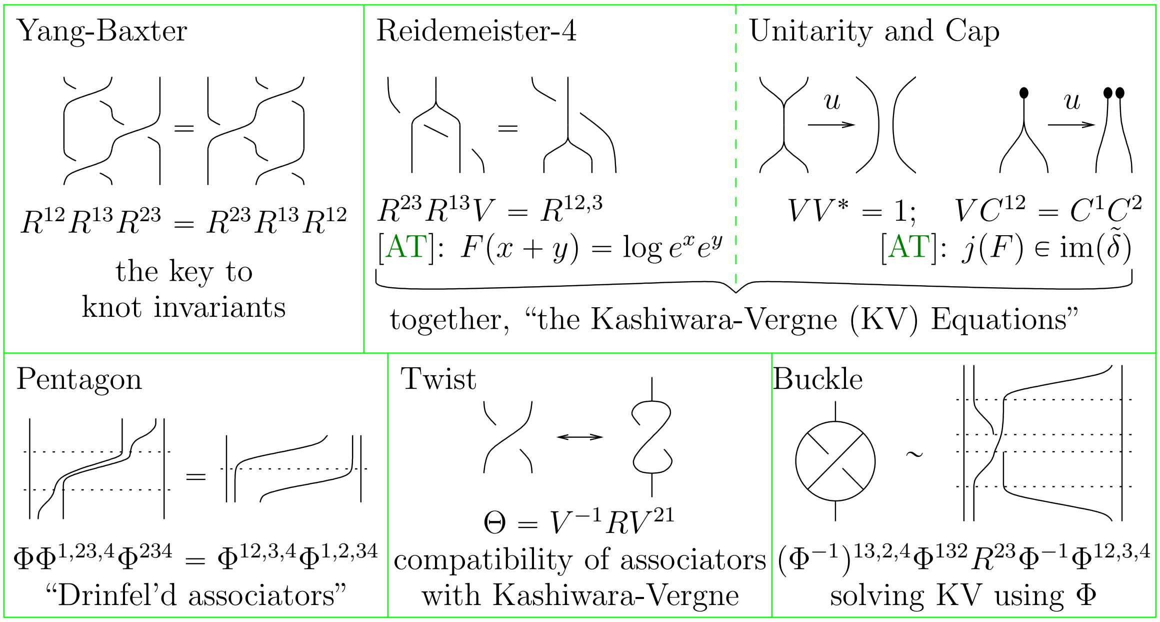



**Figure 1.1.** The most important equations.

Why bother? What do limited explicit computations add, given that these intricate equations are known to be soluble, and given that the conceptual framework within which these equations make sense is reasonably well understood [WKO1]–[WKO3]? My answers are three:

(1) Personally, my belief in what I can't compute decays quite rapidly as a function of the complexity involved. Even if the overall picture is clear, the details will surely go wrong, and sooner or later, something bigger than a detail will go wrong. Even a limited computation may serve as a wonderful sanity check. In situations such as ours, where many signs and conventions need to be decided and may well go wrong, even a low-degree computation increases my personal confidence level by a great degree. Given computations that work to degree 6 (say), it is hard to imagine that a detail was missed or that conventions were established in an inconsistent manner. In fact, if the computer programs are clear enough and are shown to work, these programs become the authoritative declarations of the details and conventions.
(2) The computational tools introduced here may well be useful in other contexts where free Lie algebras and/or cyclic words arise.
(3) The papers [WKO1, WKO2] (and likewise [BN4]) are about equations, but even more so, about the construction of certain knot and tangle invariants. With the tools presented here, the invariants of arbitrary knotted objects of the types studied in [WKO1, WKO2, BN4] may be computed.

The equations of [WKO1]–[WKO3] always involve group-like, or "exponential" elements, and are written in some spaces of "arrow diagrams" that go under the umbrella name $\mathcal{A}^w$. Hence a crucial first step is to find convenient presentations for the group-like elements $\mathcal{A}^w_{\exp}$ in $\mathcal{A}^w$-spaces. It turns out that there are (at least) two such presentations, each with its own advantages and disadvantages. Hence in Section 2 we recall $\mathcal{A}^w$ briefly (2.1), then discuss

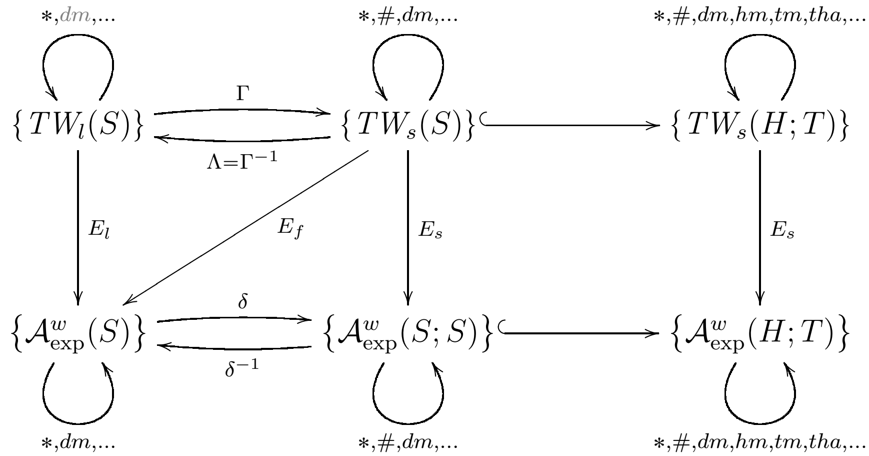



**Figure 1.2.** The main spaces and maps appearing in this paper.

some free-Lie-algebra preliminaries (2.2), then describe the Alekseev-Torossian-[AT]-inspired "lower-interlaced" presentation $E_l$ of $\mathcal{A}^w_{\exp}$ (2.3), then describe the [BN4]-inspired "factored" presentation $E_f$ of $\mathcal{A}^w_{\exp}$ and its stronger precursor "split" presentation $E_s$ (2.4), and then describe how to convert between the two primary presentations (2.5).

We then present our computations in Section 3: Some knot and tangle invariants are computed in Section 3.1 and solutions of the Kashiwara-Vergne (KV) equations in Section 3.2. In Section 3.3 we discuss the "Twist Equation" and compute dimensions of spaces of solutions of the linearized KV equations, with and without the Twist Equation. In Section 3.4 we compute a Drinfel'd associator, in Section 3.5 we compute associators in $\mathcal{A}^w$ starting from a solution of the KV equations, and in Section 3.6 we show how to compute a solution of KV from a Drinfel'd associator. The last computational result is in Section 3.7, where we give computational support to the existence of an action of the symmetric group $S_4$ on the set of solutions of the Kashiwara-Vergne Equations.

We conclude this introduction with a description of the commutative diagram in Figure 1.2 which displays the main spaces and maps appearing in this paper, as described in detail in Section 2. The bottom row of this diagram consists of spaces of "group-like" elements inside spaces $\mathcal{A}^w$ of "arrow diagrams"; these are the spaces that have direct knot-theoretic significance. The top row are spaces of "trees and wheels", or more precisely, various elements of free Lie algebras and various cyclic words. They are the spaces of "primitives" corresponding to the group-like elements at the bottom, via various "exponentiation" maps $E_l$, $E_f$, and $E_s$. In this paper we study[C2] the spaces on the bottom row by means of their presentations by elements in the top row.

The collection $\{\mathcal{A}^w_{\exp}(S)\}$ of spaces we primarily wish to study (and in which most of the equations of Figure 1.1 are written) appears on the bottom left. There are many binary and unary operations acting on the spaces within $\{\mathcal{A}^w_{\exp}(S)\}$ as indicated by the circular

*computations below*

The last input ("human") line above declares that by default we wish the computer to print series within graded spaces (such as free Lie algebras) to degree 4. Note that we highlight in pink input lines that affect later computations.

[C2]Or "implement", in computer-speak.

self-arrow appearing there, which is labelled with the most important of these operations, the binary $*$ and the unary $dm$. On the top left of the diagram are the spaces $\{TW_l(S)\}$ of trees and wheels which represent $\{\mathcal{A}^w_{\exp}(S)\}$ via the $E_l$ presentation. The same collection of operations acts here too, though notice that the operation $dm$ is grayed-out, because we have no direct implementation for it in $TW_l$ language.

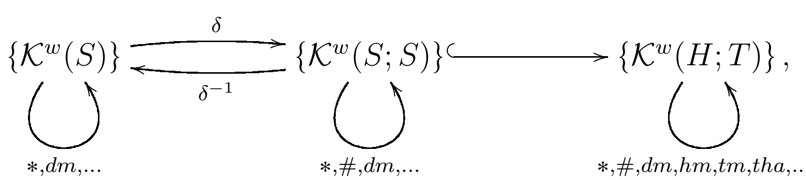
On the bottom right is a bigger collection of spaces, $\{\mathcal{A}^w_{\exp}(H;T)\}$, which contains as a subset the collection $\{\mathcal{A}^w_{\exp}(S;S)\}$ (bottom middle), which is isomorphic in a non-trivial manner (via $\delta$ and $\delta^{-1}$) to $\{\mathcal{A}^w_{\exp}(S)\}$. A richer collection of operations act on $\{\mathcal{A}^w_{\exp}(H;T)\}$, and the most important of those are $*$, $\#$, $dm$, $hm$, $tm$, and $tha$.

On the top right is the collection $\{TW_s(H;T)\}$ of spaces of trees and wheels which represent $\{\mathcal{A}^w_{\exp}(H;T)\}$ via the $E_s$ presentation. When restricted to $H=T=S$, this is the collection $\{TW_s(S)\}$ representing $\{\mathcal{A}^w_{\exp}(S;S)\}$, and representing our primary interest $\{\mathcal{A}^w_{\exp}(S)\}$ via $E_f$, the composition of $E_s$ with $\delta^{-1}$.

Note that $TW_l$ and $TW_s$ are set-theoretically the same spaces of trees and wheels. Yet the operations $*$, $dm$, etc. act on them in a different manner, and hence they deserve to have different names[2]. Note also that $TW_l$ and $TW_s$ are in fact isomorphic via structure-preserving isomorphisms (denoted $\Gamma$ and $\Lambda=\Gamma^{-1}$). These isomorphisms are compositions of the relatively simple-minded $\delta$ and $\delta^{-1}$ with the more complex "exponentiations" $E_l$ and $E_s$ and their inverses. Thus the isomorphisms $\Gamma$ and $\Lambda$ are non-linear and quite complicated.

[AT] We will occasionally comment on the relationship between the constructs appearing in this papers and three related topics: "topology", or more precisely certain aspects of the theory of 2-knots, "Lie theory", or more precisely certain classes of formulas that make sense in arbitrary finite-dimensional Lie algebras, and "Alekseev-Torossian", or more precisely, issues related to the paper [AT]. These comments will in general be incomplete and should be regarded as "hints for the already initiated" — people familiar with the papers [WKO1, WKO2, WKO3, BN4, AT] will hopefully find that these comments help to put the current paper in context. These comments will always be labelled by one (or more) of the three logos at the head of this paragraph, which correspond, in order, to "topology", "Lie theory", and "Alekseev-Torossian".

Within the study of simply-knotted (ribbon) 2-knots, or more precisely w-knotted-objects as they appear in [WKO1, WKO2, BN4], the rows of Figure 1.2 correspond to the extra row




via the "associated graded" procedure described in [WKO2]. Here $\mathcal{K}^w(S)$ is the set of $S$-labelled w-tangles [WKO2], $\mathcal{K}^w(H;T)$ is the set of w-knotted $H$-labelled hoops and $T$-labelled balloons [BN4], $\mathcal{K}^w(S;S)$ is the same but with $H=T=S$, and $\delta$ is the same as in [BN4]. This correspondence is further recalled throughout the rest of this paper.

[2]Much as in group theory, a direct product $N\times H$ is set-theoretically the same as a semi-direct product $N\rtimes H$, yet it is wrong to refer to them by the same name.

The corresponding Lie-theoretic spaces (compare [WKO1, Section 3.5]) are

$$\left\{\mathcal{U}(I\mathfrak{g})^{\otimes S}\right\} \underset{\delta^{-1}}{\overset{\delta}{\rightleftarrows}} \left\{\mathcal{U}(\mathfrak{g})^{\otimes S}\otimes\mathcal{S}(\mathfrak{g}^*)^{\otimes S}\right\} \hookrightarrow \left\{\mathcal{U}(\mathfrak{g})^{\otimes H}\otimes\mathcal{S}(\mathfrak{g}^*)^{\otimes T}\right\}.$$

$$\circlearrowright_{*,dm,\dots} \qquad \circlearrowright_{*,\#,dm,\dots} \qquad \circlearrowright_{*,\#,dm,hm,tm,tha,\dots}$$

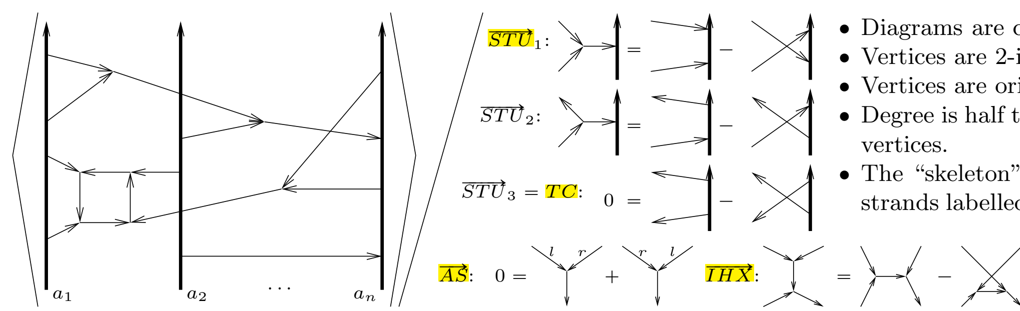
This correspondence is further recalled throughout the rest of this paper.

[AT] In [AT] there is no good counterparts for last two columns of our diagram. The counterpart of the first (and primary) column is a mixture $\hat{\mathcal{U}}((\mathfrak{a}_n\oplus\mathfrak{tder}_n)\ltimes\mathfrak{tr}_n)$ containing the most important spaces occurring in [AT]. More in the next section.

1.1. **Acknowledgement.** This paper was written almost entirely with Z. Dancso in the room (physically or virtually via Skype), working on various parts of our joint series [WKO1]–[WKO3]. Hence her indirect contribution to it, in a huge number of routine consultations, should be acknowledged in capitals: THANKS, ZSUZSI. I would like to further thank A. Alekseev and S. Morgan for their comments and suggestions.

## 2. Group-like elements in $\mathcal{A}^w$

2.1. **A brief review of $\mathcal{A}^w$.** Let $S=\{a_1,a_2,\dots\}$[3] be a finite set of "strand labels". The space $\mathcal{A}^w(S)$ is the completed graded vector space[4] of diagrams made of (vertical) "strands" labelled by the elements of $S$, and "arrows" as summarized by the following picture:




- Diagrams are connected.
- Vertices are 2-in 1-out.
- Vertices are oriented.
- Degree is half the number of trivalent vertices.
- The "skeleton" is a union of vertical strands labelled by the elements of $S$.

When $S=\{1,2,\dots,n\}$ we abbreviate $\mathcal{A}^w(\uparrow_n):=\mathcal{A}^w(S)$.

[AT] In topology, elements of $\mathcal{A}^w(S)$ are closely related to (finite type invariants of) simply knotted 2-dimensional tubes in $\mathbb{R}^4$ ([WKO1]–[WKO3], [BN4]). In Lie theory, they represent "universal" $\mathfrak{g}$-invariant tensors in $\mathcal{U}(I\mathfrak{g})^{\otimes S}$, where $I\mathfrak{g}:=\mathfrak{g}\ltimes\mathfrak{g}^*$[5] and $\mathfrak{g}$ is some finite dimensional Lie algebra ([WKO1]–[WKO3]). Readers of Alekseev and Torossian [AT] may care about $\mathcal{A}^w$ because using notation from [AT], $\mathcal{A}^w(\uparrow_n)$ is the completed universal enveloping algebra of $(\mathfrak{a}_n\oplus\mathfrak{tder}_n)\ltimes\mathfrak{tr}_n$ (see [WKO2]), and hence much of the [AT] story can be told within $\mathcal{A}^w$. Several significant Lie theoretic problems (e.g., the Kashiwara-Vergne problem, [KV, AT, WKO2]) can be interpreted as problems about $\mathcal{A}^w(\uparrow_n)$.

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[3] Yellow highlighting corresponds to the glossary, Section 4.

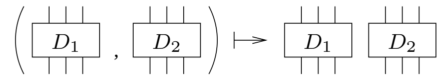
[4] For simplicity we always work over $\mathbb{Q}$.

[5] In earlier papers we have used the order $I\mathfrak{g}=\mathfrak{g}^*\rtimes\mathfrak{g}$.

*Comment* 2.1. Using the $\overrightarrow{STU}_2$ relation one may sort the skeleton vertices in every $D \in \mathcal{A}^w(S)$ so that along every skeleton component all arrow heads appear ahead of all arrow tails, and by a diagrammatic analogue of the PBW theorem (compare [BN1, Theorem 8]), this sorted form is unique modulo $\overrightarrow{STU}_1$, $TC$, $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations.

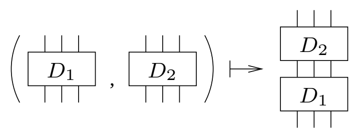
**Definition 2.2.** A number of operations are defined on elements of the $\mathcal{A}^w(S)$ spaces:

(1) If $S_1$ and $S_2$ are disjoint, then given $D_1 \in \mathcal{A}^w(S_1)$ and $D_2 \in \mathcal{A}^w(S_2)$, their union $D_1 D_2 = D_1 \sqcup D_2 \in \mathcal{A}^w(S)$, where $S = S_1 \sqcup S_2$, is obtained by placing them side by side as illustrated on the right.




In topology, $\sqcup$ corresponds to the disjoint union of 2-tangles[6]. In Lie theory, it corresponds to the map $\mathcal{U}(I\mathfrak{g})^{\otimes S_1} \otimes \mathcal{U}(I\mathfrak{g})^{\otimes S_2} \to \mathcal{U}(I\mathfrak{g})^{\otimes (S_1 \sqcup S_2)}$.

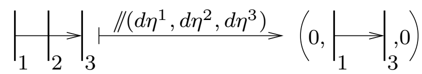
(2) Given $D_1 \in \mathcal{A}^w(S)$ and $D_2 \in \mathcal{A}^w(S)$, their product $D_1 * D_2 \in \mathcal{A}^w(S)$ is obtained by "stacking $D_2$ on top of $D_1$":

$(D_1, D_2) =$  $= D_1 * D_2. \qquad (1)$


[AT] In topology, the stacking product corresponds to the concatenation operation on knotted tubes, akin to the standard stacking product of tangles. In Lie theory, it comes from the algebra structure of $\mathcal{U}(I\mathfrak{g})^{\otimes S}$. In [AT], it is the product of the completed universal enveloping algebra $\hat{\mathcal{U}}((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n)$.

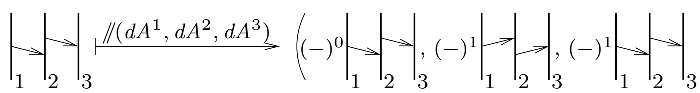
Note that below and throughout this paper we use $/\!/$ for postfix operator application and for "composition done right". Meaning that $x/\!/f$ is equivalent to $f(x)$ and $f/\!/g$ is $g \circ f$ is "do $f$ then do $g$".

(3) Given $D \in \mathcal{A}^w(S)$ and $a \in S$, $D/\!/d\eta^a$ is the result of deleting strand $a$ from $D$ and mapping it to 0 if any arrow connects to $a$, as illustrated on the right.




In topology, $d\eta^a$ is the removal of one component from a 2-tangle. In Lie theory it corresponds to the co-unit $\eta\colon \mathcal{U}(I\mathfrak{g}) \to \mathbb{Q}$.

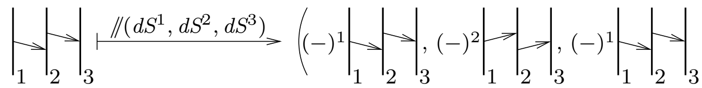
(4) Given $D \in \mathcal{A}^w(S)$ and $a \in S$, $D/\!/dA^a$ is the result of "flipping over stand $a$ and multiplying by a $(-)$ sign for each arrow whose head connects to $a$", as illustrated above. We denote by $dA$ the operation of likewise flipping (with signs) *all* strands: $dA = dA^S := \prod_{a \in S} dA^a$.




In topology, $dA^a$ is the reversal of the 1D orientation of a knotted tube [WKO2]. In Lie theory, it is the antipode of $\mathcal{U}(I\mathfrak{g})$ combined with the sign reversal $\varphi \to -\varphi$ acting on the $\mathfrak{g}^*$ factor of $I\mathfrak{g}$. When elements of $\mathcal{U}(I\mathfrak{g})^{\otimes S}$ are interpreted as differential operators acting on functions on $\mathfrak{g}^S$, $dA$ corresponds to the $L^2$ adjoint.

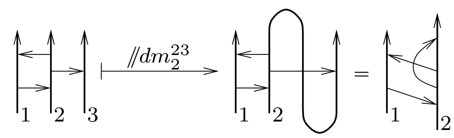
[6]To be clear, the "2" in "2-tangles" refers to the dimension of the things being knotted, and not to the number of components.

(5) Similarly, $D /\!\!/ dS^a$ is the result of "flipping over stand $a$ and multiplying by a $(-)$ sign for each arrow head or tail that connects to $a$", as illustrated above[7].



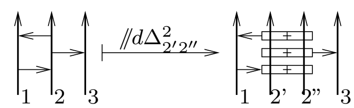

In topology, $dS^a$ is the reversal of both the 1D and the 2D orientation of a knotted tube [WKO2]. In Lie theory, it is the antipode of $\mathcal{U}(I\mathfrak{g})$.

(6) Given $D \in \mathcal{A}^w(S)$, given $a, b \in S$, and given $c \notin S\backslash\{a,b\}$, $D /\!\!/ dm^{ab}_c$ is the result of "stitching strands $a$ and $b$ and calling the resulting strand $c$", as illustrated on the right.




In topology, $dm^{ab}_c$ is the "internal stitching" of two tubes within a single 2-link, akin to the "stitching" operation that combines two strands of an ordinary tangle into a single "longer" one. In Lie theory, it is an "internal product" $\mathcal{U}(I\mathfrak{g})^{\otimes n} \to \mathcal{U}(I\mathfrak{g})^{\otimes(n-1)}$ which "merges" two factors within $\mathcal{U}(I\mathfrak{g})^{\otimes n}$.

(7) Given $D \in \mathcal{A}^w(S)$, given $a \in S$, and given $b, c \notin S\backslash a$, $D /\!\!/ d\Delta^a_{bc}$ is the result of "doubling" strand $a$, calling the resulting "daughter strands" $b$ and $c$, and summing over all ways of lifting the arrows that were connected to $a$ to either $b$ or $c$ (so if there are $k$ arrows connected to $a$, $D /\!\!/ d\Delta^a_{bc}$ is a sum of $2^k$ diagrams).




[AT] In topology, $d\Delta$ is the operation of "doubling" one component in a 2-link. In Lie theory, it is the co-product $\Delta\colon \mathcal{U}(I\mathfrak{g}) \to \mathcal{U}(I\mathfrak{g})^{\otimes 2}$ acting on the $a$ factor in $\mathcal{U}(I\mathfrak{g})^{\otimes S}$, extended by the identity acting on all other factors. In [AT], it is the coface maps of [AT, Example 3.14].

(8) Finally, the operation $d\sigma^a_b\colon \mathcal{A}(S) \to \mathcal{A}(S\backslash\{a\} \sqcup \{b\})$ does nothing but renaming the strand $a$ to $b$ (assuming $a \in S$ and $b \notin S\backslash\{a\}$). 2.2

We note that the product operation $(D_1, D_2) \mapsto D_1 * D_2$ can be implemented using the union operation $\sqcup$, the stitching operation $dm$, and some renaming — namely, if $\bar{S} = \{\bar{a}\colon a \in S\}$ is some set of "temporary" labels disjoint from $S$ but in a bijection with $S$, then

$$D_1 * D_2 = \left( D_1 \sqcup \left( D_2 /\!\!/ \prod_a d\sigma^a_{\bar{a}} \right) \right) /\!\!/ \prod_a dm^{a\bar{a}}_a. \tag{2}$$

Therefore below we will sometimes omit the implementation of $(D_1, D_2) \mapsto D_1 D_2$ provided all other operations are implemented.

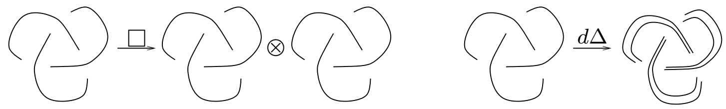
We note that $\mathcal{A}^w(S)$ is a co-algebra, with the co-product $\square(D)$, for a diagram $D$ representing an element of $\mathcal{A}^w(S)$, being the sum of all ways of dividing $D$ between a "left co-factor" and a "right co-factor" so that connected components of $D\backslash(\uparrow \times S)$ ($D$ with its skeleton removed) are kept intact (compare with [BN1, Definition 3.7]).

**Definition 2.3.** An element $Z$ of $\mathcal{A}^w(S)$ is "group-like" if $\square(Z) = Z \otimes Z$. We denote the set of group-like elements in $\mathcal{A}^w(S)$ by $\mathcal{A}^w_{\exp}(S)$.

[7] The letter $S$ is used here for both "a set of strands" and "an operation similar to an antipode". Hopefully no confusion will arise.

We leave it for the reader to verify that all the operations defined above restrict to operations $\mathcal{A}^w_{\exp} \to \mathcal{A}^w_{\exp}$.

In topology, $\square$ is the operation of "cloning" an entire 2-link. It is not to be confused with $d\Delta$; one dimension down and with just one component, the pictures are:




[AT] In [AT], $\square$ is the co-product of $\hat{\mathcal{U}}((\mathfrak{a} \oplus \mathfrak{tder}) \ltimes \mathfrak{tr})$ and moding out by wheels, $\mathcal{A}^w_{\exp}$ is TAut.

In Lie theory, $\square$ is *not* the co-product $\Delta\colon \mathcal{U}(I\mathfrak{g}) \to \mathcal{U}(I\mathfrak{g})^{\otimes 2}$. Rather, given two finite dimensional Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_1$, $\square$ corresponds to the map

$$\square\colon \mathcal{U}(I(\mathfrak{g}_1 \oplus \mathfrak{g}_2))^{\otimes S} \to \mathcal{U}(I\mathfrak{g}_1)^{\otimes S} \otimes \mathcal{U}(I\mathfrak{g}_2)^{\otimes S}.$$

*Discussion* 2.4. We seek to have efficient descriptions of the elements of $\mathcal{A}^w_{\exp}(S)$ and efficient means of computing the above operations on such elements.

Let $\mathcal{A}^w_{\mathrm{prim}}(S)$[8] denote the set of primitives of $\mathcal{A}^w(S)$: these are the elements $\zeta \in \mathcal{A}^w(S)$ satisfying $\square(\zeta) = \zeta \otimes 1 + 1 \otimes \zeta$. Let $FL(S)$ denote the degree-completed free Lie algebra with generators $S$, and let $CW(S)$ denote the degree-completed vector space spanned by non-empty cyclic words on the alphabet $S$. In [WKO2, Proposition 3.19] we have shown that there is a short exact sequence of vector spaces

$$0 \to CW(S) \to \mathcal{A}^w_{\mathrm{prim}}(S) \to FL(S)^S \to 0, \tag{3}$$

where $FL(S)^S$ denotes the set of all functions $S \to FL(S)$. Hence $\mathcal{A}^w_{\mathrm{prim}}(S) \simeq FL(S)^S \oplus CW(S)$ (not canonically!). Often in bi-algebras there is a bijection given by $\zeta \mapsto e^\zeta$ between primitive elements $\zeta$ and group-like elements $e^\zeta$. Hence we may expect to be able to present elements of $\mathcal{A}^w_{\exp}(S)$ as formal exponentials of combinations of "trees" (elements of $FL(S)^S$) and "wheels" (elements of $CW(S)$)[9]:

$$\mathcal{A}^w_{\exp}(S) \sim TW(S) := FL(S)^S \times CW(S) = \left\{ (\lambda; \omega) \colon \begin{array}{c} \lambda = \{a \to \lambda_a\}_{a \in S},\ \lambda_a \in FL(S) \\ \omega \in CW(S) \end{array} \right\}. \tag{4}$$

We implement Equation (4) in a more-or-less straightforward way in Section 2.3 and in a less straightforward but somewhat stronger way in Section 2.4. [2.4]

*Discussion* 2.5. Why are there two presentations for elements of $\mathcal{A}^w_{\exp}$?

Because as we shall see, $\mathcal{A}^w$ is a bi-algebra in two ways, using two different products, yet with the same co-product $\square$. In $\mathcal{A}^w$, the notions "primitive" and "group-like", whose definition involves only $\square$, are canonical. Yet the bijection between primitive and group-like elements, $\zeta \leftrightarrow e^\zeta$, depends also on the product used within the power-series interpretation of $e^\zeta$. Thus there are two different ways to describe the group-like elements $\mathcal{A}^w_{\exp}$ of $\mathcal{A}^w$ in terms of its primitives $TW$.

---

[8] $\mathcal{A}^w_{\mathrm{prim}}$ is elsewhere denoted $\mathcal{P}^w$.

[9] We use the set-theoretic notation "$\times$" rather than the linear-algebraic "$\oplus$" in Equation (4) to emphasize that the two sides of that equation are only expected to be set-theoretically isomorphic. The left-hand-side, in fact, is not even a linear space in a natural way.

The first product on $\mathcal{A}^w$ is the stacking product of Equation (1). The second will be introduced later, in Equations (18) and (42).

Very roughly speaking, $\mathcal{A}^w$ is a combinatorial model of "$\pi_1 \ltimes \pi_2$" (with homotopies replaced by isotopies; see [BN4]). The other product on $\mathcal{A}^w$ is the one coming from the direct product "$\pi_1 \times \pi_2$".

Very roughly speaking, $\mathcal{A}^w$ is a combinatorial model of (tensor powers of a completion of) $\mathcal{U}(I\mathfrak{g})$. By PBW, $\mathcal{U}(I\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g}^*)$ as co-algebras but not as algebras. The other product on $\mathcal{A}^w$ is the one corresponding to the natural product on $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g}^*)$. The reality is a bit more delicate, though. $\mathcal{A}^w$ is only a model of (a small part of) the $\mathfrak{g}$-invariant part of $\mathcal{U}(I\mathfrak{g})$, and the co-product $\square$ of $\mathcal{A}^w$ does not correspond to the co-product $\Delta$ of $\mathcal{U}(I\mathfrak{g})$. 2.5

2.2. **Some preliminaries about free Lie algebras and cyclic words.** It should be clear from Discussion 2.4 that free Lie algebras and cyclic words play a prominent role in this paper. For the convenience of our readers we collect in this section some preliminaries about about these topics. Almost everything in this section comes either from Alekseev-Torossian's [AT], or from [WKO2, BN4], and the detailed proofs of the assertions made here can be found in these papers.

Note that Lie algebras appear in two distinct roles in this paper. *Free* Lie algebras $FL$ appear along with cyclic words $CW$ as the primitives of $\mathcal{A}^w$ (Equation (3)). *Finite dimensional* Lie algebras $\mathfrak{g}$ appear only as motivational comments, always marked with a symbol. As already indicated, elements in $\mathcal{A}^w$, and hence elements of $FL$ and of $CW$ can represent "universal" formulas that make sense in any finite dimensional Lie algebra $\mathfrak{g}$. Hence part of our discussion of $FL$ and $CW$ is a discussion of things that make sense universally for all finite dimensional Lie algebras.

Recall that $FL(S)$ denotes the graded completion of the free Lie algebra over a set of generators $S$, all considered to have degree 1. In the case when $S = \{x_1, \ldots, x_n\}$, Alekseev and Torossian [AT] denote this space $\mathfrak{lie}_n$.[C3]

*computations below*

[C3] In computer talk, generators of $FL(S)$ are always single-character "Lyndon words" (e.g. [Re]); in our case we set $x$ and $y$ to be the single-character words "$x$" and "$y$", and then $\alpha$, $\beta$, and $\gamma$ to be the Lie series $x+[x,y]$, $y-[x,[x,y]]$, and $x+y-2[x,y]$ (elements of $FL$ are infinite series, in general, but these examples are finite):

```
x = LW@"x"; y = LW@"y";
{α, β, γ} = LS /@ {x + b[x, y], y - b[x, b[x, y]], x + y - 2 b[x, y]}
```

{LS[$\overline{x}$, $\overline{xy}$, 0, 0, ...], LS[$\overline{y}$, 0, $-\overline{x\overline{xy}}$, 0, ...], LS[$\overline{x}+\overline{y}$, $-2\overline{xy}$, 0, 0, ...]}

Note that as we requested earlier, our example series are printed to degree 4. Note also that they are printed using "top bracket" $\overline{xy} := [x,y]$ notation, which is easier to read when many brackets are nested.
We then compute $[\alpha, \beta]$ and verify the Jacobi identity for $\alpha$, $\beta$, and $\gamma$:

```
{b[α, β], b[α, b[β, γ]] + b[β, b[γ, α]] + b[γ, b[α, β]]}
```

{LS[0, $\overline{xy}$, $\overline{\overline{xy}y}$, $-\overline{x\overline{x\overline{xy}}}$, ...], LS[0, 0, 0, 0, ...]}

A noteworthy element of $FL(x,y)$ is the Baker-Campbell-Hausdorff series,[C4]

$$\mathrm{BCH}(x,y) := \log(e^x e^y) = x + y + \frac{[x,y]}{2} + \frac{[x,[x,y]] + [[x,y],y]}{12} + \dots.$$

Recall also that $CW(S)$ ($\mathfrak{tr}_n$, in [AT]) denotes the graded completion of the vector space spanned by non-empty cyclic words in the alphabet $S$. Our convention is to crown cyclic words with an "arch"; thus $\widehat{uvw} = \widehat{vwu}$[C5]. Note that there is a map $CW(FL(S)) \to CW(S)$ by interpreting brackets within elements of $FL(S)$ as commutators and then mapping "long" words to cyclic words. E.g., $\widehat{u[v,w]} = \widehat{uvw} - \widehat{uwv}$.

We denote by $h^{\deg}$ the operations $FL \to FL$ and $CW \to CW$ which multiply any degree $k$ element by $h^k$. In particular, $(-1)^{\deg}$ acts on $FL/CW$ as the identity in even degrees and as minus the identity in odd degrees.[C6]

Let $\mathrm{der}_S$ denote the Lie algebra of all derivations of $FL(S)$ ($\mathfrak{der}_n$ in [AT]). There is a linear map $\partial\colon FL(S)^S \to \mathrm{der}_S$ which assigns to every $\lambda = (\lambda_a)_{a\in S} \in FL(S)^S$ the unique derivation $\partial_\lambda$ for which $\partial_\lambda(a) = [a, \lambda_a]$ for every $a \in S$.[10] [C7] The image of $\partial$ is a subalgebra of $\mathrm{der}_S$ denoted $\mathrm{tder}_S$ ($\mathfrak{tder}_n$ in [AT]); the elements of $\mathrm{tder}_S$ are called "tangential derivations". The kernel of $\partial$ can be identified as the Abelian Lie algebra $A_S$ generated by $S$ ($\mathfrak{a}_n$ in [AT]),

*computations below*

[C4]In computer talk:

```
bch = BCH[x, y]
```

LS[ $\overline{x} + \overline{y}$, $\frac{\overline{xy}}{2}$, $\frac{1}{12}\overline{x\overline{xy}} + \frac{1}{12}\overline{\overline{xy}y}$, $\frac{1}{24}\overline{x\overline{\overline{xy}y}}$, ... ]

Fuller output: [WKO4]/bch.nb

Just to show that we can, here are the lexicographically middle three of the 2,181 terms of the BCH series in degree 16, along with the time in seconds it took my humble laptop to compute it:

```
Timing@{Length@(bch@16), (bch@16)[[1090 ;; 1092]]}
```

{39.5313, {2181, $-\frac{17\, x\, x\, x\overline{xy}\,\overline{xy}\,x\overline{xy}\,x\,x\,\overline{xy}\,y\,y}{179\,625\,600}$ + $\frac{389\, x\, x\, x\overline{xy}\,\overline{xy}\,x\overline{xy}\,x\overline{xy}\,x\overline{xy}}{1\,320\,883\,200}$ + $\frac{53\, x\, x\, x\overline{xy}\,\overline{xy}\,x\overline{xy}\,x\overline{xy}\,\overline{xy}\,y}{1\,089\,728\,640}$}}

(In a few hours my laptop computed the BCH series to degree 23; in as much as I know, the farthest it was ever computed. See [BN4, CM].)

[C5]Cyclic words in computer talk:

```
{ω1, ω2} = CWS /@ {cw[x] - 3 cw[y, x, x],  cw[y] + cw[y, y]}
```

{CWS[ $\widehat{x}$, 0, -3 $\widehat{xxy}$, 0, ...], CWS[ $\widehat{y}$, $\widehat{yy}$, 0, 0, ...]}

[C6]In computer talk:

```
DegreeScale[h] /@ {ω1, ω2}
```

{CWS[ h $\widehat{x}$, 0, -3 $h^3$ $\widehat{xxy}$, 0, ...], CWS[ h $\widehat{y}$, $h^2$ $\widehat{yy}$, 0, 0, ...]}

which is linearly embedded in $FL(S)^S$ as the set of all sequences $\lambda\colon S \to FL(S)$ for which $\lambda_a$ is a scalar multiple of $a$ for every $a \in S$. Thus we have a short exact sequence of vector spaces

$$0 \to A_S \to FL(S)^S \xrightarrow{\partial} \mathrm{tder}_S \to 0. \tag{5}$$

The map $FL(S)^S \ni \lambda = (\lambda_a) \mapsto \sum_a \langle \lambda_a, a\rangle a \in A_S$, where $\langle \lambda_a, a\rangle$ is the coefficient of $a$ in $\lambda_a$ is a splitting of the above sequence, and hence $FL(S)^S \simeq A_S \oplus \mathrm{tder}_S$ in a canonical manner.

There is a unique Lie bracket $[\cdot,\cdot]_{tb}$ (the "tangential bracket") on $FL(S)^S$ which makes (5) a split exact sequence of Lie algebras, and hence $(FL(S)^S, [,]_{tb}) \simeq A_S \oplus \mathrm{tder}_S$ as Lie algebras. With $[\cdot,\cdot]$ denoting the ordinary direct-sum bracket on $FL(S)^S$ and with the action of $\partial_\lambda$ extended to $\partial_\lambda\colon FL(S)^S \to FL(S)^S$ in the obvious manner, we have[C8]

$$[\lambda_1, \lambda_2]_{tb} = [\lambda_1, \lambda_2] + \partial_{\lambda_1}\lambda_2 - \partial_{\lambda_2}\lambda_1.$$

The $\lambda \mapsto \partial_\lambda$ action of $(FL(S)^S, [,]_{tb})$ on $FL(S)$ extends to an action on the universal enveloping algebra of $FL(S)$, the free associative algebra $FA(S)$ on $S$ generators, and then descends to the vector-space quotient of $FA(S)$ by commutators, namely to cyclic words. Leaving aside the empty word, we find that $(FL(S)^S, [,]_{tb})$ acts on $CW(S)$, and hence also on $TW(S)$.[C9]

There are two ways to assign an automorphism of the free Lie algebra $FL(S)$ to an element $\lambda \in FL(S)^S$:

---

[10]Using the notation of [BN4], $\partial_\lambda = -\sum_{a\in S} \mathrm{ad}_a^{\lambda_a} = -\sum_{a\in S} \mathrm{ad}_a\{\lambda_a\}$. I apologize for the minus sign which stems from a bad choice made in [BN4].

*computations below*

[C7]An example:

```
{λ = ⟨x → α, y → β⟩, γ // Dλ}
```

$\{\langle \overline{x} \to \mathrm{LS}[\overline{x}, \overline{xy}, 0, 0, \ldots], \overline{y} \to \mathrm{LS}[\overline{y}, 0, -\overline{x\overline{xy}}, 0, \ldots]\rangle, \mathrm{LS}[0, 0, \overline{x\overline{xy}}, -\overline{x\overline{\overline{xy}y}}, \ldots]\}$

[C8]For example:

```
λ1 = λ; λ2 = ⟨x → β, y → γ⟩; tb[λ1, λ2]
```

$\langle \overline{x} \to \mathrm{LS}[0, 0, \overline{x\overline{xy}}, -\overline{x\overline{\overline{xy}y}}, \ldots], \overline{y} \to \mathrm{LS}[0, 0, \overline{x\overline{xy}}, -\overline{x\overline{\overline{xy}y}}, \ldots]\rangle$

[C9]We check that up to degree 8, $\partial_{[\lambda_1,\lambda_2]_{tb}}(\omega_1) = [\partial_{\lambda_1}, \partial_{\lambda_2}](\omega_1)$ (for our choice of $\lambda_1$, $\lambda_2$, and $\omega_1$, both sides vanish below degree 8):

```
lhs = D_tb[λ1,λ2][ω1]; rhs = b[Dλ1, Dλ2][ω1];
{lhs@{8}, (lhs ≡ rhs)@{8}}
```

$\{\mathrm{CWS}[0, 0, 0, 0, 0, 0, 0, 18\,\overline{xxxxyxyy} - 18\,\overline{xxxxyyxy} - 36\,\overline{xxxyxxyy} + 36\,\overline{xxxyyxxy}, \ldots],$
$\mathrm{BS}[9\ \mathrm{True}, \ldots]\}$

Note that the comparison operator ≡ returns a "Boolean Sequence" (`BS`) rather than a single `True`/`False` value, as the computer has no way of knowing whether two series are equal without computing them up to a given degree. In our case, we've asked for the comparison of `lhs` with `rhs` up to degree 8, and the output, including degree 0, is a sequence of 9 affirmations, summarized as "`9 True`".

(1) One may exponentiate the derivation $\partial_\lambda$ to get $e^{\partial_\lambda}\colon FL(S) \to FL(S)$.
(2) One may define an automorphism $C^\lambda\colon FL(S) \to FL(S)$ by setting its values on the generators by $C^\lambda(a) := e^{\lambda_a} a e^{-\lambda_a} = e^{\operatorname{ad}\lambda_a} a$. We denote the inverse of $C^\lambda$ by $RC^{-\lambda}$ and note that it is *not* $C^{-\lambda}$.

[AT] In [AT], (1) corresponds to the presentation of elements of the automorphism group $\mathrm{TAut}_n$ as exponentials of elements of its Lie algebra $\mathrm{tder}_n$, while (2) corresponds to its presentation in terms of "basis conjugating automorphisms" $x_i \mapsto g_i^{-1} x_i g_i$ where $g_i = e^{-\lambda_i}$. Compare with [AT, Section 5.1].

The following pair of propositions, which we could not find elsewhere, relates these two automorphisms:

**Proposition 2.6.** *Given $\lambda \in FL(S)^S$, let $t$ be a scalar-valued formal variable and let $\Gamma_t(\lambda) \in FL(S)^S$ be the (unique) solution of the ordinary differential equation*

$$\Gamma_0(\lambda) = 0 \qquad \text{and} \qquad \frac{d\Gamma_t(\lambda)}{dt} = \lambda /\!\!/ e^{-t\partial_\lambda} /\!\!/ \frac{\operatorname{ad}\Gamma_t(\lambda)}{e^{\operatorname{ad}\Gamma_t(\lambda)} - 1}. \tag{6}$$

*Then*

$$e^{-t\partial_\lambda} = C^{\Gamma_t(\lambda)}.\text{[C10]} \tag{7}$$

*Proof.* The two sides $L_t$ and $R_t$ of Equation (7) are power-series perturbations of the identity automorphism of $FL(S)$. More fully, $L_t$ can be written $L_t = \sum_{d\geqslant 0} t^d L(d)$ where $L(d)\colon FL(S) \to FL(S)$ raises degrees by at least $d$ (and so the sum converges), and where $L(0)$ is the identity. $R_t$ can be written in a similar way. We claim that it is enough to prove that

$$A_t := \left(\frac{dL_t}{dt}\right) /\!\!/ L_t^{-1} = \left(\frac{dR_t}{dt}\right) /\!\!/ R_t^{-1} =: B_t. \tag{8}$$

Indeed, if otherwise $L_t \neq R_t$, consider the minimal $d$ for which $L(d) \neq R(d)$. Then $d > 0$ and the least-degree term in $A_t - B_t$ is the degree $d-1$ term, which equals $dt^{d-1}L(d) /\!\!/ L_t^{-1} - dt^{d-1}R(d) /\!\!/ R_t^{-1} = dt^{d-1}(L(d) - R(d)) /\!\!/ L_t^{-1} \neq 0$ (the last equality is because $L_t^{-1} = R_t^{-1}$ to degree $d$), contradicting Equation (8). Note that in fact we have shown that if $A_t = B_t$ to degree $d$ in $t$, then Equation (7) holds to degree $d+1$.

*computations below*

[C10] We verify that the computer-calculated $\Gamma_t(\lambda)$ satisfies the ODE in (6) and then that the operator equality (7) holds, at least when evaluated on "our" $\gamma$:

```
lhs = ∂t Γt[λ]; rhs = λ // e^(-t Dλ) // adSeries[ad/(e^ad - 1), Γt[λ]];
{Γ0[λ], lhs, (lhs ≡ rhs)@{6}}
```

```
{⟨x → LS[0, 0, 0, 0, ...], y → LS[0, 0, 0, 0, ...]⟩,
 ⟨x → LS[x, xy, -t xxy, 1/4 t² x xxy - t x xyy, ...], y → LS[y, 0, -xxy, -t x xyy, ...]⟩,
 BS[7 True, ...]}
```

```
{γ // e^(-t Dλ), γ // CC[Γt[λ]]}
```

```
{LS[x + y, -2 xy, -t xxy, t x xyy, ...], LS[x + y, -2 xy, -t xxy, t x xyy, ...]}
```

To compute $B_t$ we need the differential of $C^\mu$ (at $\mu = \Gamma_t(\lambda)$) and the chain rule. The differential of $C^\mu$ is quite difficult; fortunately, we have computed it in the case where $\mu = (u \to \gamma)$ is supported on just one $u \in S$, in [BN4, Lemma 10.7]. Both the result and its proof generalize simply, and so we have

$$\delta C^\mu = -\partial\left\{\delta\mu /\!\!/ \frac{e^{\operatorname{ad}\mu}-1}{\operatorname{ad}\mu} /\!\!/ RC^{-\mu}\right\} /\!\!/ C^\mu,$$

where we have written $\partial\{\text{mess}\}$ instead of $\partial_{\text{mess}}$ because mess is too big to fit as a subscript. Hence by the chain rule and then by Equation (6),

$$B_t = -\partial\left\{\frac{d\Gamma_t(\lambda)}{dt} /\!\!/ \frac{e^{\operatorname{ad}\mu}-1}{\operatorname{ad}\mu} /\!\!/ RC^{-\mu}\right\}\Bigg|_{\mu=\Gamma_t(\lambda)} = -\partial\left\{\lambda /\!\!/ e^{-t\partial_\lambda} /\!\!/ RC^{-\Gamma_t(\lambda)}\right\} = -\partial_{\lambda /\!\!/ e^{-t\partial_\lambda} /\!\!/ RC^{-\Gamma_t(\lambda)}}.$$

On the other hand, computing $A_t$ is a simple differentiation, and we get that $A_t = -\partial_\lambda$. Comparing with the line above, we find that if Equation (7) holds to degree $d$, then Equation (8) also holds to degree $d$. But then as we noted, (7) holds to degree $d+1$. As Equation (7) clearly holds at $t = 0$, we find that it holds to all orders. □

*Comment* 2.7. It is easier (though insufficient) to assume that there is a solution $\Gamma_t(\lambda)$ to Equation (7) and deduce that it must satisfy the differential equation (6): simply differentiate (7) with respect to $t$ and simplify as much as you can allowing yourself to use (7) as needed within the simplification process. The result is (6), and the steps follow the computational steps of the above proof rather closely. The actual proof is a bit harder because if we cannot assume (7) while deriving it, so we have to resort to an inductive process.

**Proposition 2.8.** *As in the previous proposition, let* $\Lambda_t(\lambda)$ *be the (unique) solution of*

$$\Lambda_0(\lambda) = 0 \qquad \text{and} \qquad \frac{d\Lambda_t(\lambda)}{dt} = \lambda /\!\!/ e^{\partial_{\Lambda_t(\lambda)}} /\!\!/ \frac{\operatorname{ad}_{tb}\Lambda_t(\lambda)}{e^{\operatorname{ad}_{tb}\Lambda_t(\lambda)}-1}. \tag{9}$$

*Then*

$$C^{t\lambda} = e^{-\partial_{\Lambda_t(\lambda)}}. \tag{10}$$

The proof of this proposition is very similar and not even a tiny bit nicer than the proof of the previous one. So we skip it and instead include a computer verification.[C11]

As special cases, we denote $\Gamma_1(\lambda)$ by $\Gamma(\lambda)$ and $\Lambda_1(\lambda)$ by $\Lambda(\lambda)$.

One special case of $C^\lambda$ deserves to be named:

*computations below*

[C11] We verify that the computer-calculated $\Lambda_t(\lambda)$ satisfies the ODE in (9) and then that the operator equality (10) holds, at least when evaluated on "our" $\gamma$:

```
lhs = ∂t Λt[λ]; rhs = λ // e^(DΛt[λ]) // adSeries[ad/(e^ad - 1), Λt[λ], tb];
{Λ0[λ], lhs, (lhs ≡ rhs) @ {6}}
```

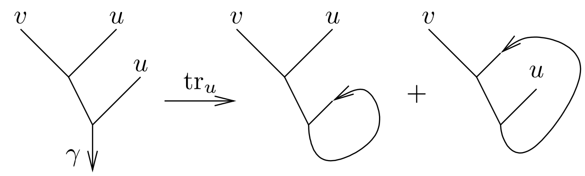
$$\left\{\langle \overline{x} \to \mathrm{LS}[0, 0, 0, 0, \ldots], \overline{y} \to \mathrm{LS}[0, 0, 0, 0, \ldots]\rangle,\right.$$
$$\left\langle \overline{x} \to \mathrm{LS}\left[\overline{x}, \overline{xy}, t\,\overline{x\overline{xy}}, \tfrac{1}{2}t^2\,\overline{x\overline{x\overline{xy}}} + t\,\overline{x\overline{\overline{xy}y}}, \ldots\right], \overline{y} \to \mathrm{LS}\left[\overline{y}, 0, -\overline{x\overline{xy}}, t\,\overline{x\overline{\overline{xy}y}}, \ldots\right]\right\rangle,$$
$$\left.\mathrm{BS}[7\ \mathrm{True}, \ldots]\right\}$$

```
{γ // CC[t λ], γ // e^(-DΛt[λ])}
```

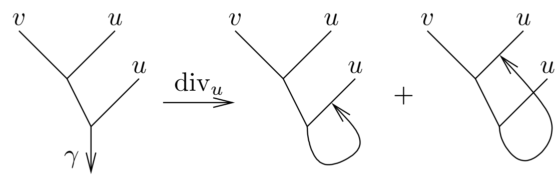
**Definition 2.9.** (Compare [BN4, Section 4.2]) Given $u \in S$ and $\gamma \in FL(S)$ let $C_u^\gamma$ denote the automorphism of $FL(S)$ defined by mapping the generator $u$ to its "conjugate" $e^\gamma u e^{-\gamma} = e^{-\operatorname{ad}\gamma}(u)$ (this is simply $C^\lambda$, where $\lambda$ is the length 1 sequence $(u \to \gamma)$). Let $RC_u^{-\gamma}$ be the inverse of $C_u^\gamma$ (which is *not* $C_u^{-\gamma}$).[C12]

Last we define/recall a number of functionals $FL(S) \to CW(S)$:

**Definition 2.10.** For $u \in S$ we let $\operatorname{tr}_u\colon FL(S) \to CW(S)$ be the sum of all ways of connecting the head of $\gamma$ to any of its $u$-labelled tails and regarding the result as an element of $CW(FL(S)) \to CW(S)$. The example on the right corresponds to the specific computation $\operatorname{tr}_u[[v,u],u] = \widehat{[v,u]} + \widehat{v(-u)} = -\widehat{uv}$[C13]




**Definition 2.11.** (Compare [BN4, Section 5.1]) For $u \in S$ we let $\operatorname{div}_u\colon FL(S) \to CW(S)$ be the functional defined schematically by the picture on the right, which corresponds to the specific computation $\operatorname{div}_u[[v,u],u] = \widehat{u[v,u]} + \widehat{uv(-u)} = -\widehat{uuv}$[C14] (more details in [BN4]). Given also $\gamma \in FL(S)$, set




$$J_u(\gamma) := \int_0^1 ds\ \operatorname{div}_u(\gamma /\!\!/ RC_u^{s\gamma}) /\!\!/ C_u^{-s\gamma}.$$ [C15]

computations below

$\{\mathrm{LS}[\overline{x}+\overline{y},\ -2\,\overline{xy},\ -t\,\overline{x\overline{xy}},\ -\frac{1}{2}t^2\,\overline{x\overline{x\overline{xy}}} + t\,\overline{x\overline{\overline{xy}y}},\ \ldots],$
$\mathrm{LS}[\overline{x}+\overline{y},\ -2\,\overline{xy},\ -t\,\overline{x\overline{xy}},\ -\frac{1}{2}t^2\,\overline{x\overline{x\overline{xy}}} + t\,\overline{x\overline{\overline{xy}y}},\ \ldots]\}$

C12 Just testing:

```
{α // CCx[-γ], α // CCx[-γ] // RCx[γ], α // CCx[-γ] // CCx[γ]}
```

$\{\mathrm{LS}[\overline{x},\ 2\,\overline{xy},\ -\frac{5}{2}\,\overline{x\overline{xy}} + \frac{3}{2}\,\overline{\overline{xy}y},\ \frac{7}{6}\,\overline{x\overline{x\overline{xy}}} - \frac{23}{6}\,\overline{x\overline{\overline{xy}y}} + \frac{2}{3}\,\overline{\overline{\overline{xy}y}y},\ \ldots],$
$\mathrm{LS}[\overline{x},\ \overline{xy},\ 0,\ 0,\ \ldots],\ \mathrm{LS}[\overline{x},\ \overline{xy},\ -\overline{x\overline{xy}},\ 2\,\overline{x\overline{x\overline{xy}}} + \overline{x\overline{\overline{xy}y}},\ \ldots]\}$

C13 In computer talk, and using a temporary value for $\gamma$, so as not to interfere with its existing value:

```
u = LW@"u"; v = LW@"v";
With[{γ = b[b[v, u], u]}, tru[γ]]
```

$-\widehat{uv}$

C14 In computer talk:

```
With[{γ = u + b[b[v, u], u]}, divu[γ]]
```

$\widehat{u} - \widehat{uuv}$

**Definition 2.12.** Let div: $FL(S) \to CW(S)$ be the Alekseev-Torossian "divergence" functional, as in [AT, Section 5.1], but extended by 0 on $A_S$. In our language, $\operatorname{div}\lambda = \sum_{u\in S}\operatorname{div}_u\lambda$. Let $j\colon FL(S) \to CW(S)$ is the Alekseev-Torossian "logarithm of the Jacobian": $j(\lambda) = \frac{e^{\partial_\lambda}-1}{\partial_\lambda}(\operatorname{div}\lambda)$.[C16]

Alekseev and Torossian prove in [AT] that $j$ is the unique functional $j\colon FL(S) \to CW(S)$ satisfying the "cocycle condition" $j\left(\mathrm{BCH}_{tb}(\lambda_1,\lambda_2)\right) = j(\lambda_1) + e^{\partial_{\lambda_1}} j(\lambda_2)$, where $\mathrm{BCH}_{tb}$ stands for the BCH formula using the tangential bracket $[\cdot,\cdot]_{tb}$ on $FL(S)^S$:

$$\mathrm{BCH}_{tb}(\lambda_1,\lambda_2) = \lambda_1 + \lambda_1 + \frac{1}{2}[\lambda_1,\lambda_2]_{tb} + \ldots,$$

and the "initial condition" $\frac{\partial}{\partial\epsilon} j(\epsilon\lambda) = \operatorname{div}\lambda$.[C17]

2.3. **The lower-interlaced presentation $E_l$ of $\mathcal{A}^w_{\exp}$.** For a finite set $S$ let $TW_l(S)$ be set-theoretically the same as $TW(S) = FL(S)^S \times CW(S)$ — we only add the "$l$" subscript to emphasize that $TW_l$ carries an algebraic structure, and that it is different from the algebraic

*computations below*

[C15]We quote the implementation of $J$ in FreeLie.m (FL) and, reverting to the "old" $\gamma$, compute $J_1(\gamma)$:

FL
```
Ju_[γ_] := Ju[γ] = Module[{s}, ∫_0^1 (γ // RCu[s γ] // divu // CCu[-s γ]) ⅆs];
```

```
Jx[γ]
```

```
CWS[x, 5 xy/2, -7 xxy/6 + 7 xyy/6, 3 xxxy/8 - 11 xxyy/4 - 3 xyxy/4 + 3 xyyy/8, ...]
```

[C16]A quote of the computer-definition, and then $\operatorname{div}\lambda$ and $j(\lambda)$, computed to degree 5:

FL
```
div[λ_AngleBracket] := Sum[diva[λa], {a, Support[λ]}];
j[λ_AngleBracket] := div[λ] // DerivationSeries[(e^der - 1)/der, Dλ];
```

```
{div[λ]@{5}, j[λ]@{5}}
```

```
{CWS[x + y, -xy, -xxy, 0, 0, ...], CWS[x + y, -xy, -xxy, -xxyy + xyxy, -xxxyy + xxyxy, ...]}
```

[C17]We verify the cocycle condition and the initial condition. For the latter, we first declare $\epsilon$ to be "an infinitesimal" by declaring that $\epsilon^2 = 0$, and then we verify that $j(\epsilon\lambda) = \epsilon \operatorname{div}\lambda$:

```
lhs = j[BCHtb[λ1, λ2]]; rhs = j[λ1] + e^Dλ1[j[λ2]];
{lhs, (lhs ≡ rhs)@{8}}
```

```
{CWS[x + 2 y, -3 xy, 0, -9 xxyy + 9 xyxy, ...], BS[9 True, ...]}
```

```
ϵ /: ϵ^2 = 0;
{j[ϵ λ], j[ϵ λ] ≡ ϵ div[λ]}
```

```
{CWS[ϵ x + ϵ y, -ϵ xy, -ϵ xxy, 0, ...], BS[5 True, ...]}
```

structure on $TW_s$, which we will study later. Elements of $TW_l(S)$ are ordered pairs $(\lambda;\,\omega)_l$, where $\lambda \in FL(S)^S$, $\omega \in CW(S)$, and the subscript $l$ is there only to remind us of the context.

Set

$$E_l(\lambda;\,\omega)_l := \exp(l\lambda) * \exp(\iota\omega) \in \mathcal{A}^w_{\exp}(S), \qquad \left(\begin{matrix}\text{“}E_l\text{” for “\underline{E}xponentiation}\\ \text{after using }\underline{l}\text{”}\end{matrix}\right)$$

where $l\colon FL(S)^S = A_S \oplus \mathrm{tder}_S \to \mathcal{A}^w(S)$ is the “lower” Lie embedding[11] of trees into $\mathcal{A}^w(S)$ (see [WKO2, Section 3.2]), where $\iota$ is the obvious inclusion of wheels ($= CW(S) = \mathfrak{tr}_S$) into $\mathcal{A}^w(S)$, and where exponentiation is taken using the stacking product (1) of $\mathcal{A}^w(S)$. A pictorial representation of $E_l(\lambda;\,\omega)_l$ appears on the right: Reading from the bottom up, we see “exponentially many” copies of $\lambda$ (meaning, a sum over $n$ of $n$ copies with coefficient $1/n!$). Each $\lambda$ is a linear combination of trees with one head and many tails, which are attached to the strands in $T$ with the head below the tails. Each copy of $\lambda$ appears on the right as a gray “wizard's cap” whose tip corresponds to the head of $\lambda$, and is therefore tipped downward. Above $\exp(l\lambda)$ is our symbolic representation of $\exp(\iota\omega)$.

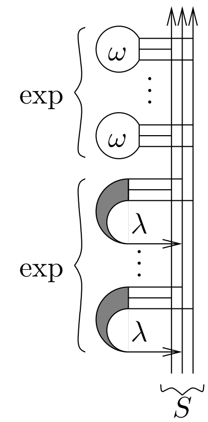



**Figure 2.13.** $E_l(\lambda;\,\omega)_l$.

Figure 2.13 also explains the name “interlaced” for this presentation, for in it heads and tails are interlaced along the strands of $S$ (contrast with $E_s$ in Figure 2.19 and with $E_f$ in Figure 2.28).

It follows from the results of [WKO2, Section 3.2] that the map $E_l\colon TW_l(S) \to \mathcal{A}^w_{\exp}(S)$ is a set-theoretic bijection. Hence the operations of Definition 2.2 induce corresponding operations on $TW_l(S)$. We list these within the (long!) definition-proposition below.

**Definition-Proposition 2.14.** *The bijection $E_l$ intertwines the operations defined below with the operations in Definition 2.2:*[C18]

*(1) If $S_1 \cap S_2 = \varnothing$ and $(\lambda_i;\,\omega_i)_l \in TW_l(S_i)$,*

$$(\lambda_1;\,\omega_1)_l(\lambda_2;\,\omega_2)_l = (\lambda_1;\,\omega_1)_l \sqcup (\lambda_2;\,\omega_2)_l := (\lambda_1 \sqcup \lambda_2;\,\omega_1 + \omega_2)_l, \tag{11}$$

[11]We could equally well used the “upper” Lie embedding $u$, setting $E_u(\lambda;\,\omega)_u := \exp(\iota\omega)\exp(u\lambda)$, with only minor modifications to the formulas that follow.

*computations below*

[C18]We cannot verify Definition-Proposition 2.14 *per se* on the computer, as we have no direct computer implementation of $\mathcal{A}^w$. Indeed, the whole point of this paper is to provide an implementation of $\mathcal{A}^w$ by means of $E_l$ (and later, $E_s$ and $E_f$). Instead, we verify below that many properties of operations on $\mathcal{A}^w$ (the associativity of the stacking product, etc.) indeed hold for their $E_l$ implementations. We start by setting the values of some “sample” elements on which we will run our tests (note that on the computer we represent $(\lambda;\,\omega)_l$ as `El[λ,ω]`):

```
x = LW@"x"; y = LW@"y";
{ζa = El[⟨x → LS[x + b[x, y]], y → LS[y - b[x, b[x, y]]]⟩, CWS[cw[x] - 3 cw[x, y, x]]],
 ζb = El[⟨x → LS[y - b[x, y]], y → LS[x + y + b[y, b[x, y]]]⟩, CWS[cw[y] - 2 cw[x, y]]],
 ζc = El[⟨x → LS[x - b[b[x, y], b[x, y]]], y → LS[y + 3 b[x, b[x, y]]]⟩,
   CWS[cw[x] - 2 cw[x, y] + cw[x, y, x]]]}
```

*where* $\sqcup\colon FL(S_1)^{S_1} \times FL(S_2)^{S_2} \to FL(S_1 \sqcup S_2)^{S_1 \sqcup S_2}$ *is the union operation of functions (or, in computer speak, the concatenation of associative arrays) followed by the inclusions* $FL(S_i) \to FL(S_1 \sqcup S_2)$*, and* $\omega_1 + \omega_2$ *is defined using the inclusions* $CW(S_i) \to CW(S_1 \sqcup S_2)$*.*

*(2) If* $(\lambda_i; \omega_i)_l \in TW_l(S)$*,*

$$(\lambda_1; \omega_1)_l * (\lambda_2; \omega_2)_l := (\mathrm{BCH}_{tb}(\lambda_1, \lambda_2); e^{-\partial_{\lambda_2}}(\omega_1) + \omega_2)_l.^{C19} \tag{12}$$

*(3) If* $(\lambda; \omega)_l \in TW_l(S)$ *and* $a \in S$*,*

$$(\lambda; \omega)_l /\!\!/ d\eta^a := ((\lambda \backslash a) /\!\!/ (a \to 0); \omega /\!\!/ (a \to 0))_l, \tag{13}$$

*where* $\lambda \backslash a$ *denotes the function* $\lambda$ *with the element* $a$ *removed from its domain (in computer talk, "remove the key* $a$*"), and* $(a \to 0)$ *denotes the substitution* $a = 0$*, which is defined on both* $FL$ *and* $CW$ *and maps* $FL(S) \to FL(S \backslash a)$ *and* $CW(S) \to CW(S \backslash a)$*.*[C20]

*(4) For a single* $a \in S$*, I don't know a simple description of the operation* $dA^a$ *in* $E_l$ *language*[12]*. Yet the composition* $dA := dA^S := \prod_{a \in S} dA^a$ *is manageable: (*$j$ *is defined in Definition 2.12)*

$$(\lambda; \omega)_l /\!\!/ dA^S := (-\lambda; e^{\partial_\lambda}(\omega) - j(\lambda))_l.^{C21} \tag{14}$$

[12] A not-so-simple description would be to use the language of the factored presentation of Section 2.4, converting back and forth using the results of Section 2.5.

*computations below*

```
{El[⟨x̄ → LS[x̄, x̄y, 0, 0, ...], ȳ → LS[ȳ, 0, -x x̄y, 0, ...]⟩, CWS[x̂, 0, -3 x̂xy, 0, ...]],
 El[⟨x̄ → LS[ȳ, -x̄y, 0, 0, ...], ȳ → LS[x̄ + ȳ, 0, -x̄yy, 0, ...]⟩, CWS[ŷ, -2 x̂y, 0, 0, ...]],
 El[⟨x̄ → LS[x̄, 0, 0, 0, ...], ȳ → LS[ȳ, 0, 3 x x̄y, 0, ...]⟩, CWS[x̂, -2 x̂y, x̂xy, 0, ...]]}
```

[C19] We quote the $E_l$ implementation of the stacking product from AwCalculus.m (AC) and verify that it is associative, at least to degree 8:

```
AC El /: El[λ1_, ω1_] ** El[λ2_, ω2_] /; Support[λ1] == Support[λ2] :=
    El[BCH_tb[λ1, λ2], e^(-D_λ2)[ω1] + ω2];
```

```
lhs = ζa ** (ζb ** ζc); rhs = (ζa ** ζb) ** ζc;
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

```
{El[⟨x̄ → LS[2 x̄ + ȳ, 0, 1/2 x x̄y, ...], ȳ → LS[x̄ + 3 ȳ, 0, 5/2 x x̄y - x̄yy, ...]⟩,
  CWS[2 x̂ + ŷ, -4 x̂y, -2 x̂xy, ...]], BS[9 True, ...]}
```

[C20] Example:

```
{ζa // dη^x, ζa // dη^y}
```

```
{El[⟨ȳ → LS[ȳ, 0, 0, 0, ...]⟩, CWS[0, 0, 0, 0, ...]],
 El[⟨x̄ → LS[x̄, 0, 0, 0, ...]⟩, CWS[x̂, 0, 0, 0, ...]]}
```

(5) *For a single $a \in S$, I don't know a simple description of the operation $dS^a$ in $E_l$ language*[12]*. Yet the composition $dS := dS^S := \prod_{a \in S} dS^a$ is manageable:*

$$(\lambda;\, \omega)_l /\!\!/ dS^S := (-\lambda /\!\!/ (-1)^{\deg};\ (e^{\partial_\lambda}(\omega) - j(\lambda)) /\!\!/ (-1)^{\deg})_l.^{C22} \qquad (15)$$

(6) *I don't know a simple description of the operation $dm^{ab}_c$ in $E_l$ language*[12]*. Yet note that Equation (2) implies that "applying $dm$ to all strands" is manageable, being the stacking product described in (12).*

(7) *We have*

$$(\lambda;\, \omega)_l /\!\!/ d\Delta^a_{bc} := ((\lambda \backslash a) \sqcup (b \to \lambda_a,\, c \to \lambda_a) /\!\!/ (a \to b + c);\ \omega /\!\!/ (a \to b + c))_l, \qquad (16)$$

*where $(a \to b + c)$ denotes the obvious replacement of the generator $a$ with the sum $b + c$. It represents morphisms $FL(S) \to FL((S \backslash a) \sqcup \{b, c\})$, $FL(S)^H \to FL((S \backslash a) \sqcup \{b, c\})^H$ (for any set $H$), and $CW(S) \to CW((S \backslash a) \sqcup \{b, c\})$.*[C23]

computations below

[C21] We quote the computer-definition of $dA$, compute an example, verify that $dA$ is an involution, and then that it is an anti-homomorphism relative to the stacking product:

```
AC El[λ_, ω_] // dA := El[-λ, e^(Dλ)[ω] - j[λ]];
```

```
{ξd = El[λ, CWS[0]], ξd // dA}
```

$\{El[\langle \overline{x} \to LS[\overline{x}, \overline{xy}, 0, 0, \ldots], \overline{y} \to LS[\overline{y}, 0, -\overline{x\overline{xy}}, 0, \ldots]\rangle, CWS[0, 0, 0, 0, \ldots]],$
$El[\langle \overline{x} \to LS[-\overline{x}, -\overline{xy}, 0, 0, \ldots], \overline{y} \to LS[-\overline{y}, 0, \overline{x\overline{xy}}, 0, \ldots]\rangle,$
$CWS[-\widehat{x} - \widehat{y}, \widehat{xy}, \widehat{xxy}, \widehat{xxyy} - \widehat{xyxy}, \ldots]]\}$

```
(ξd ≡ (ξd // dA // dA))@{8}
```

BS[9 True, ...]

```
lhs = (ξa ** ξb) // dA; rhs = (ξb // dA) ** (ξa // dA);
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

$\{El[\langle \overline{x} \to LS[-\overline{x} - \overline{y}, 0, -\frac{1}{2}\overline{x\overline{xy}}, \ldots], \overline{y} \to LS[-\overline{x} - 2\overline{y}, 0, \frac{1}{2}\overline{x\overline{xy}} + \overline{\overline{xy}y}, \ldots]\rangle,$
$CWS[-\widehat{y}, -2\widehat{xy}, -2\widehat{xxy} - \widehat{xyy}, \ldots]], BS[9 True, \ldots]\}$

[C22] An example:

```
ξd // dS
```

$El[\langle \overline{x} \to LS[\overline{x}, -\overline{xy}, 0, 0, \ldots], \overline{y} \to LS[\overline{y}, 0, -\overline{x\overline{xy}}, 0, \ldots]\rangle,$
$CWS[\widehat{x} + \widehat{y}, \widehat{xy}, -\widehat{xxy}, \widehat{xxyy} - \widehat{xyxy}, \ldots]]$

[C23] The computer-definition, an example, and then a verification that $d\Delta$ is homomorphism relative to the stacking product:

(8) *We have*

$$(\lambda;\omega)_l /\!\!/ d\sigma^a_b := (((\lambda \backslash a) \sqcup (b \to \lambda_a)) /\!\!/ (a \to b); \omega /\!\!/ (a \to b))_l, \tag{17}$$

*where* $(a \to b)$ *denotes the obvious "generator renaming" morphisms* $FL(S) \to FL((S\backslash a) \sqcup b)$, $FL(S)^H \to FL((S\backslash a) \sqcup b)^H$ *(for any set* $H$*), and* $CW(S) \to CW((S\backslash a) \sqcup b)$.

*Proof.* Equations (11), (13), (16), and (17) are trivial and were stated only to introduce notation. The tree-level part of Equation (12) follows from the fact that $l$ is a morphism of Lie algebras (see within the proof of [WKO2, Proposition 3.19]). The wheels part of Equation (12) follows from [WKO2, Remark 3.24]. Equation (14) follows from the observation that $dA^S$ is the adjoint map $*$ of [WKO2, Definition 3.26] and then from [WKO2, Proposition 3.27]. Equation (15) is the easily-established fact that on $\mathcal{A}^w$, $dS^S = (-1)^{\deg} dA^S$. □

Note that the absence of simple descriptions of $dA^a$, $dS^a$, and $dm^{ab}_c$ in the $E_l$ language is fatal for its applicability to knot theory, as these operations are needed within the computation of knot and tangle invariants. See Section 3.1.

[AT] *Comment* 2.15. Let $\pi_T \colon TW(S) \to FL(S)^S$ denote the projection onto the first factor ("trees") of $TW(S) = FL(S)^S \times CW(S)$, and recall that up to a minor central factor, $(FL(S)^S, tb)$ is $\text{tder}_S$. Recall also that $\text{tder}_S$ is the Lie algebra of $\text{TAut}_S$, and that elements of $\text{tder}_S$ represent elements of $\text{TAut}_S$ by exponentiation. With this in mind, the tree part of Equation (12) becomes the product of $\text{TAut}_S$. In other words, the diagram

$$\begin{array}{ccc} TW_l(S) \times TW_l(S) & \xrightarrow{\ *\ } & TW_l(S) \\ \big\downarrow {\scriptstyle (\pi_T /\!\!/ \exp)\times(\pi_T /\!\!/ \exp)} & & \big\downarrow {\scriptstyle \pi_T /\!\!/ \exp} \\ \text{TAut}_S \times \text{TAut}_S & \xrightarrow{\text{mult.}} & \text{TAut}_S \end{array}$$

computations below

```
AC El[λ_, ω_] // dΔ[a_, b_, c_] := El[
     (λ \ a) ⋃ ⟨b → λa, c → λa⟩ // LieMorphism[LW@a → LW@b + LW@c],
     ω // LieMorphism[LW@a → LW@b + LW@c]]
```

```
{ζa, ζa // dΔ[y, y, z]}
```

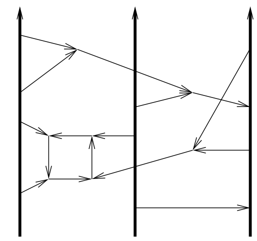
```
{El[⟨x̄ → LS[x̄, x̄ȳ, 0, 0, ...], ȳ → LS[ȳ, 0, -x̄x̄ȳ, 0, ...]⟩, CWS[x̂, 0, -3 x̂x̂ŷ, 0, ...]],
 El[⟨z → LS[ȳ + z̄, 0, -x̄x̄ȳ - x̄x̄z̄, 0, ...], x̄ → LS[x̄, x̄ȳ + x̄z̄, 0, 0, ...],
   ȳ → LS[ȳ + z̄, 0, -x̄x̄ȳ - x̄x̄z̄, 0, ...]⟩, CWS[x̂, 0, -3 x̂x̂ŷ - 3 x̂x̂ẑ, 0, ...]]}
```

```
lhs = (ζa ** ζb) // dΔ[y, y, z]; rhs = (ζa // dΔ[y, y, z]) ** (ζb // dΔ[y, y, z]);
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

```
{El[⟨z → LS[x̄ + 2 ȳ + 2 z̄, 0, -1/2 x̄x̄ȳ - 1/2 x̄x̄z̄ - x̄ȳz̄ - x̄ȳȳ - 2 x̄z̄ȳ - x̄z̄z̄, ...],
   x̄ → LS[x̄ + ȳ + z̄, 0, 1/2 x̄x̄ȳ + 1/2 x̄x̄z̄, ...],
   ȳ → LS[x̄ + 2 ȳ + 2 z̄, 0, -1/2 x̄x̄ȳ - 1/2 x̄x̄z̄ - x̄ȳz̄ - x̄ȳȳ - 2 x̄z̄ȳ - x̄z̄z̄, ...]⟩,
  CWS[x̂ + ŷ + ẑ, -2 x̂ŷ - 2 x̂ẑ, -3 x̂x̂ŷ - 3 x̂x̂ẑ, ...]], BS[9 True, ...]}
```

is commutative. Hence the $E_l$ presentation is valuable for [AT] as many of the [AT] equations involve the group structure of $\mathrm{TAut}_S$.

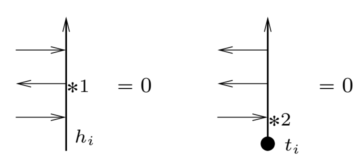
2.4. **The factored presentation $E_f$ of $\mathcal{A}^w_{\exp}$ and its stronger precursor $E_s$.** Following [BN4], in the "factored" presentation $E_f$ of $\mathcal{A}^w_{\exp}$ arrow heads are treated separately from arrow tails in diagrams such as the one on the right. This presentation of $\mathcal{A}^w_{\exp}$ is more complicated than the previous one, yet it is also more powerful, and in some sense, it is made of simpler ingredients. We first enlarge the collection of spaces $\{\mathcal{A}^w(S)\}$ to a somewhat bigger collection $\{\mathcal{A}^w(H;T)\}$ on which a larger class of operations act. The new operations are more "atomic" than the old ones, in the sense that each of the operations of Definition 2.2 is a composition of 2-3 of the new operations. The advantage is that the new operations all have reasonably simple descriptions as operations on the group-like subsets $\{\mathcal{A}^w_{\exp}(H;T)\}$ (the "split" presentation $E_s$ below), and hence even the few operations whose description in the $E_l$ presentation was omitted in Definition-Proposition 2.14 can be fully described and computed in the $E_f$ presentation.



A sketch of our route is as follows: In Section 2.4.1, right below, we describe the spaces $\{\mathcal{A}^w(H;T)\}$. In Section 2.4.2 we describe the zoo of operations acting on $\{\mathcal{A}^w(H;T)\}$. Section 2.4.3 is the tofu of the matter — we describe the operations of the previous section in terms of spaces $\{TW_s(H;T)\}$ of trees and wheels, whose elements are in a bijection $E_s$ with the group like elements of $\{\mathcal{A}^w(H;T)\}$. Finally in Section 2.4.4 we explain how the system of spaces $\{\mathcal{A}^w(S)\}$ includes into the system $\{\mathcal{A}^w(H;T)\}$ and how the operations of the former are expressed in terms of the latter, concluding the description of $E_f$.

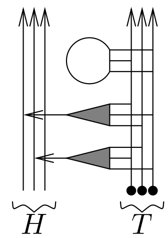
2.4.1. *The family* $\{\mathcal{A}^w(H;T)\}$. Let $H = \{h_1, h_2, \ldots\}$ be some finite set of "head labels" and let $T = \{t_1, t_2, \ldots\}$ be some finite set of "tail labels" (these sets need not be of the same cardinality). Let $\mathcal{A}^w(H;T)$ be $\mathcal{A}^w(H \sqcup T)$[13] moded out by the following further relations:

- If an arrow tail lands anywhere on a head strand ($*1$ on the right), the whole diagram is zero.
- The $CP$ relation: If an arrow head is the lowest vertex on a tail strand ($*2$ on the right), the whole diagram is zero. (As on the right, we indicate the bottom ends of tail strands with bullets "•").




[13] We will often use sets of labels $H$ and $T$ that are *not* disjoint. The notation "$H \sqcup T$" stands for the union of $H$ and $T$, made disjoint by brute force; for example, by setting $H \sqcup T := (\{h\} \times H) \cup (\{t\} \times T)$, where $h$ and $t$ are two distinct labels chosen in advance to indicate "heads" and "tails". In practise we will keep referring to the images of the elements of $H$ within $H \sqcup T$ as $h_i$ rather than $(h, h_i)$, and likewise for the $t_i$'s. We will mostly avoid the confusion that may arise when $H \cap T \neq \varnothing$ by labelling operations as "head operations" which will always refer to labels in $H \hookrightarrow H \sqcup T$ or as "tail operations", when referring to labels in $T \hookrightarrow H \sqcup T$.

*Comment* 2.16. Using these two relations one may show that $\mathcal{A}^w(H;T)$ is isomorphic to the set of arrow diagrams in which only arrow heads land on the head strands (obvious, by the first relation) and in which only arrow tails meet the tail strands (use $\overrightarrow{STU}_2$ to slide any arrow head on a tail strand until it's near the bottom, then use the second relation; see also Comment 2.1), still modulo $\overrightarrow{AS}$, $\overrightarrow{IHX}$, $\overrightarrow{STU}_1$ and $TC$. Thus a typical element of $\mathcal{A}^w(H;T)$ is shown on the right.




In topology (see [BN4]), head strands correspond to "hoops", or based knotted circles, and tail strands correspond to balloons, or based knotted spheres. The two relations and the isomorphism above are also meaningful [BN4].

In Lie theory head strands represent $\mathcal{U}(\mathfrak{g})$ and tail strands represent the (right) Verma module $\mathcal{U}(I\mathfrak{g})/\mathfrak{g}\mathcal{U}(I\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}^*) \simeq \mathcal{S}(\mathfrak{g}^*)$. The evaluation $\mathfrak{g}^* \to 0$ induces a surjection of $\mathcal{U}(I\mathfrak{g})$ onto the first of these spaces whose kernel is "any word containing a letter in $\mathfrak{g}^*$", explaining the first relation above. The second relation is the definition of the Verma module.

2.4.2. *Operations on* $\{\mathcal{A}^w(H;T)\}$.

**Definition 2.17.** Just as in Definition 2.2, there are several operations that are defined on $\mathcal{A}^w(H;T)$. In brief, these are:

(1) A union operation $\sqcup$: $\mathcal{A}^w(H_1;T_1)\otimes\mathcal{A}^w(H_2;T_2) \to \mathcal{A}^w(H_1\sqcup H_2; T_1\sqcup T_2)$, defined when $H_1\cap H_2 = T_1\cap T_2 = \varnothing$, with obvious topological (compare with "$*$" of [BN4, Figure 5]) and Lie theoretic meanings. (The symbol $\sqcup$ is sometimes omitted: $D_1D_2 := D_1\sqcup D_2$).

(2) A "stacking" product # can be defined on $\mathcal{A}^w(H;T)$ by stitching all pairs of equally-labelled head strands and then merging all pairs of equally-labelled tail strands in a pair of diagrams $D_1, D_2 \in \mathcal{A}^w(H;T)$. The "merging" of tail strands is described in more detail as the operation $tm$ below. In fact, it may be better to define # using a formula similar to Equation (2) and the operations $hm$, $tm$, $h\sigma$, and $t\sigma$ defined below:

$$D_1\#D_2 = \left(D_1 \sqcup \left(D_2/\!\!/\prod_{x\in H} h\sigma^x_{\bar{x}}/\!\!/\prod_{u\in T} t\sigma^u_{\bar{u}}\right)\right)/\!\!/\prod_{x\in H} hm^{x\bar{x}}_x/\!\!/\prod_{u\in T} tm^{u\bar{u}}_u. \tag{18}$$

In topology, # is the stitching of hoops followed by the merging of balloons; this is not the same as the stitching of knotted tubes. In Lie theory, # corresponds to the componentwise product of $\mathcal{U}(\mathfrak{g})^{\otimes H}\otimes\mathcal{S}(\mathfrak{g}^*)^{\otimes T}$. Even when $H$ and $T$ are both singletons, this is not the same as the product of $\mathcal{U}(I\mathfrak{g})$, even though linearly $\mathcal{U}(I\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g})\otimes\mathcal{S}(\mathfrak{g}^*)$.

(3) If $x\in H$ and $u\in T$, the operations $h\eta^x$ and $t\eta^u$ drop the head-strand $x$ or the tail-strand $u$ similarly to the operation $d\eta^a$ of Definition 2.2.

(4) $hA^x$ reverses the head-strand $x$ while multiplying by a $(-1)$ factor for every arrow head on $x$. $tA^u$ is the identity.

(5) $hS^x = hA^x$ while $tS^u$ multiplies by a factor of $(-1)$ for every arrow tail on $u$ (by $TC$, there's no need to reverse $u$).

(6) The operation $hm^{xy}_z$ is defined similarly to $dm^{ab}_c$ of Definition 2.2. Likewise for $tm^{uv}_w$, except in this case, the tail-strands $u$ and $v$ must first be cleared of all arrow-heads using the process of Comment 2.16. Once $u$ and $v$ carry only arrow-tails, all these tail can be

put on a new tail-strand $w$ in some arbitrary order (which doesn't matter, by $TC$). Note that $tm^{uv}_w = tm^{vu}_w$, so $tm$ is "meta-commutative".

In topology, $tm^{uv}_w$ is the "merging of balloons" operation of [BN4, Section 3.1], which in itself is analogues to the (commutative) multiplication of $\pi_2$.

In Lie theory, $tm^{uv}_w$ is the product of $\mathcal{S}(\mathfrak{g}^*)$. Note that tail strands more closely represent the Verma module $\mathcal{U}(I\mathfrak{g})/\mathfrak{g}\mathcal{U}(I\mathfrak{g})$ whose isomorphism with $\mathcal{S}(\mathfrak{g}^*)$ involves "sliding all $\mathfrak{g}$-letters in a $\mathcal{U}(I\mathfrak{g})$-word to the left and then cancelling them". This is analogous to the process of cancelling arrow-heads which is a pre-requisite to the definition of $tm^{uv}_w$.

(7) $h\Delta^x_{yz}$ and $t\Delta^u_{vw}$ are defined similarly to $d\Delta^a_{bc}$.
(8) $h\sigma^x_y$ and $t\sigma^u_v$ are defined similarly to $d\sigma^a_b$.
(9) **New!** Given a tail $u \in T$, a "new" tail label $v \notin T\backslash u$ and a head $x \in H$ the operation $thm^{ux}_v\colon \mathcal{A}^w(H;T) \to \mathcal{A}^w(H\backslash x; (T\backslash u) \sqcup \{v\})$ is the obvious "tail-strand head-strand stitching" — similarly to $dm^{ab}_c$, stitch the strand $u$ to the strand $x$ putting $u$ before $x$, and call the resulting "new" strand $v$. Note that for this to be well defined, $v$ must be a tail strand.[14]

In practise, $thm^{ux}_v$ is never used on its own, but the combination $h\Delta^x_{xx'} /\!\!/ thm^{ux'}_u$ (where $x'$ is a temporary label) is very useful. Hence we set $tha^{ux}\colon \mathcal{A}^w(H;T) \to \mathcal{A}^w(H;T)$ ("tail by head action on $u$ by $x$") to be that combination. In words, this is "double the strand $x$ and put one of the copies on top of $u$".[15]

In topology, $tha$ is the action of hoops on balloons as in [BN4, Section 3.1], which is similar to the action of $\pi_1$ on $\pi_2$. In Lie theory, it is the right action of $\mathcal{U}(\mathfrak{g})$ on the Verma module $\mathcal{U}(I\mathfrak{g})/\mathfrak{g}\mathcal{U}(I\mathfrak{g})$, or better, the action of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{S}(\mathfrak{g}^*)$ induced from the co-adjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. 2.17

*Exercise 2.18.* In the cases when we did not state the topological or Lie theoretical meaning of an operation in Definition 2.17, find what it is.

2.4.3. *Group-like elements in* $\{\mathcal{A}^w(H;T)\}$. For any fixed finite sets $H$ and $T$ there is a co-product $\square\colon \mathcal{A}^w(H;T) \otimes \mathcal{A}^w(H;T)$ defined just as in the case of $\mathcal{A}^w(S)$ (Definition 2.3), and along with the product $\#$ (and obvious units and co-units), $\mathcal{A}^w(H;T)$ is a graded connected co-commutative bi-algebra. Hence it makes sense to speak of the group-like elements $\mathcal{A}^w_{\exp}(H;T)$ within $\mathcal{A}^w(H;T)$, and they are all $\#$-exponentials of primitives in $\mathcal{A}^w(H;T)$. The primitives $\mathcal{A}^w_{\text{prim}}(H;T)$ in $\mathcal{A}^w(H;T)$ are connected diagrams and hence they are trees and wheels. As in Comment 2.16, the trees must have their roots on head strands and their leafs on tail strands, and the wheels must have all their "legs" on tail strands. As tails commute, we may think of the trees as abstract trees with leafs labelled by labels in $T$ and roots in $H$, and the wheels are abstract cyclic words with letters in $T$. Hence canonically $\mathcal{A}^w_{\text{prim}}(H;T) \simeq FL(T)^H \oplus CW(T)$ and hence there is a bijection (called "the split presentation $E_s$")

$$E_s\colon\ TW_s(H;T) := FL(T)^H \oplus CW(T) \stackrel{\sim}{\longrightarrow} \mathcal{A}^w_{\exp}(H;T) \tag{19}$$

[14] Note also that the analogous operation $htm^{xu}_v$ "put $x$ before $u$ to get a tail $v$" is 0 and hence we can safely ignore it, and that $thm^{xu}_y$ and $htm^{xu}_y$, defined in the same way as $thm^{ux}_v$ and $htm^{xu}_v$ except to produce a head strand $y$, are not well defined because they do not respect the $CP$ relation.

[15] Note that $thm^{ux}_v = tha^{ux} /\!\!/ h\eta^x /\!\!/ t\sigma^u_v$ so we lose no generality by considering $tha^{ux}$ instead of $thm^{ux}_v$.

defined on an ordered pair $(\lambda;\,\omega)_s$ in $TW_s(H;T)$ by

$$(\lambda;\,\omega)_s \mapsto \exp_{\#}\left(e_s(\lambda;\omega)\right), \tag{20}$$

where $e_s(\lambda;\omega)_s$ is the sum over $x \in H$ of planting $\lambda_x$ with its root on strand $x$ and its leafs on the strands in $T$ so that the labels match but at an arbitrary order on any $T$ strand, plus the result of planting $\omega$ on just the $T$ strands so that the labels match but at an arbitrary order on any $T$ strand. A pictorial representation of $E_s(\lambda;\,\omega)_s$, using the same visual language as in Figure 2.13, appears on the right.

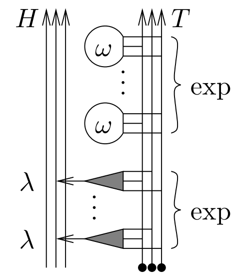



**Figure 2.19.** $E_s(\lambda;\,\omega)_s$.

It is easy to verify that the operations in Definition 2.17 intertwine $\square$ and hence map group-like elements to group-like elements and hence they induce operations on $TW_s(H;T)$. These are summarized within the following definition-proposition.

**Definition-Proposition 2.20.** *The bijection $E_s$ intertwines the operations defined below with the operations in Definition 2.17:*[16] [C24]

---

[16] Here we no longer state conditions such as $H_1 \cap H_2 = \varnothing$, $u \in T$, $x \in H$. They are the same as in Definition 2.17, and more importantly, they are "what makes sense".

*computations below*

[C24] We quote from `AwCalculus.m` only the most interesting implementations — of $\sqcup$ (21), of *hm* (29), of *tm* (30), and of *tha* (35). Then we set the values of two "sample" elements in the $E_s$ presentation (on the computer we represent $(\lambda;\,\omega)_s$ as `Es[λ,ω]`):

```
Es /: Es[λ1_, ω1_] Es[λ2_, ω2_] /; Support[λ1] ⋂ Support[λ2] == {} := Es[λ1 ⋃ λ2, ω1 + ω2];
Es[λ_, ω_] // hm[x_, y_, z_] := Es[λ // hm[x, y, z], ω];
Es[λ_, ω_] // tm[u_, v_, w_] := LieMorphism[LW@u → LW@w, LW@v → LW@w] /@ Es[λ, ω];
Es[λ_, ω_] // tha[u_, x_] := Es[λ // RCu[λx], (ω + Ju[λx]) // RCu[λx]];
```

```
ξa = Es[⟨1 → LS[u + b[u, v]], 2 → LS[v - b[u, b[u, v]]], 3 → LS[u - b[b[u, v], b[u, v]]]⟩,
  CWS[cw[u] - 3 cw[u, v, u]]]
```

$$\mathrm{Es}\left[\left\langle 1 \to \mathrm{LS}[\overline{u}, \overline{uv}, 0, 0, \ldots], 2 \to \mathrm{LS}[\overline{v}, 0, -\overline{u\,\overline{uv}}, 0, \ldots], 3 \to \mathrm{LS}[\overline{u}, 0, 0, 0, \ldots]\right\rangle, \mathrm{CWS}[\widehat{u}, 0, -3\,\widehat{uuv}, 0, \ldots]\right]$$

```
ξb = RandomEsSeries[0, {1, 2, 3, 4}];
ξb@{2}
```

$$\begin{aligned}\mathrm{Es}\Big[\Big\langle\, & 1 \to \mathrm{LS}\Big[-\overline{1} - 2\,\overline{2} + 2\,\overline{3} - 2\,\overline{4},\ 2\,\overline{12} + \frac{\overline{13}}{2} + \overline{14} - \frac{\overline{23}}{2} - \frac{\overline{24}}{2} + 2\,\overline{34},\ \ldots\Big],\\ & 2 \to \mathrm{LS}\Big[2\,\overline{1} - \overline{2} - 2\,\overline{3} + \overline{4},\ 2\,\overline{12} + \frac{3\,\overline{13}}{2} - 2\,\overline{14} - \overline{23} - \overline{24} - \frac{\overline{34}}{2},\ \ldots\Big],\\ & 3 \to \mathrm{LS}\Big[-\overline{1} + \overline{2} + 2\,\overline{4},\ -2\,\overline{12} + 2\,\overline{13} - \overline{14} - \frac{3\,\overline{23}}{2} + 2\,\overline{24} - 2\,\overline{34},\ \ldots\Big],\\ & 4 \to \mathrm{LS}\Big[-2\,\overline{1} + 2\,\overline{2} + 2\,\overline{3} + \overline{4},\ -\frac{\overline{12}}{2} + \frac{3\,\overline{13}}{2} - 2\,\overline{24} + \overline{34},\ \ldots\Big]\Big\rangle,\\ & \mathrm{CWS}\Big[\widehat{3} - \widehat{4},\ \frac{3\,\widehat{11}}{2} + \frac{3\,\widehat{12}}{2} - 2\,\widehat{13} + \widehat{14} + \widehat{22} + 2\,\widehat{23} - \frac{\widehat{24}}{2} - 2\,\widehat{33} - \widehat{34} + \widehat{44},\ \ldots\Big]\Big]\end{aligned}$$

(Note that the second of sample elements was set to be a random series, with a seed of 0. It is printed only to degree 2, but it extends indefinitely as a random series.)

*(1)* $(\lambda_1;\omega_1)_s(\lambda_2;\omega_2)_s = (\lambda_1;\omega_1)_s \sqcup (\lambda_2;\omega_2)_s := (\lambda_1 \sqcup \lambda_2;\omega_1+\omega_2)_s$ (21)

*(2)* $(\lambda_1;\omega_1)_s \# (\lambda_2;\omega_2)_s := ((x \to \mathrm{BCH}(\lambda_{1x},\lambda_{2x}))_{x\in H};\omega_1+\omega_2)_s$ (22)

*(3)* $(\lambda;\omega)_s /\!\!/ h\eta^x := (\lambda \backslash x;\omega)_s$ (23)

$(\lambda;\omega)_s /\!\!/ t\eta^u := (\lambda /\!\!/ (u\to 0);\omega /\!\!/ (u \to 0))_s$ (24)

*(4)* $(\lambda;\omega)_s /\!\!/ hA^x := ((\lambda\backslash x) \sqcup (x \to -\lambda_x);\omega)_s$ (25)

$tA^u := I$ (26)

*(5)* $hS^x := hA^x,$ (27)

$(\lambda;\omega)_s /\!\!/ tS^u := (\lambda /\!\!/ (u \to -u);\omega /\!\!/ (u\to -u))_s$ (28)

*(6)* $(\lambda;\omega)_s /\!\!/ hm_z^{xy} := ((\lambda\backslash\{x,y\}) \sqcup (z \to \mathrm{BCH}(\lambda_x,\lambda_y));\omega)_s$ (29)

$(\lambda;\omega)_s /\!\!/ tm_w^{uv} := (\lambda /\!\!/ (u,v\to w);\omega /\!\!/ (u,v \to w))_s$ (30)

*(7)* $(\lambda;\omega)_s /\!\!/ h\Delta_{yz}^x := ((\lambda\backslash x) \sqcup (y \to \lambda_x, z\to\lambda_x);\omega)_s$ (31)

$(\lambda;\omega)_s /\!\!/ t\Delta_{vw}^u := (\lambda /\!\!/ (u \to v+w);\omega /\!\!/ (u\to v+w))_s$ (32)

*(8)* $(\lambda;\omega)_s /\!\!/ h\sigma_y^x := ((\lambda\backslash x) \sqcup (y\to\lambda_x);\omega)_s$ (33)

$(\lambda;\omega)_s /\!\!/ t\sigma_v^u := (\lambda /\!\!/ (u\to v);\omega /\!\!/ (u\to v))_s$ (34)

*(9)* $(\lambda;\omega)_s /\!\!/ tha^{ux} := \left(\lambda /\!\!/ RC_u^{\lambda_x};(\omega + J_u(\lambda_x)) /\!\!/ RC_u^{\lambda_x}\right)_s.$ (35)

*Proof.* The first 8 assertions (14 operations) are very easy. The main challenge to the reader should be to gather her concentration for the 14-times repetitive task of unwrapping definitions. If you are ready to cut corners, only go over (21), (29), (30), (31), and (32). Let us turn to the proof of the last assertion, Equation (35). That proof is in fact in [BN4], or at least can be assembled from pieces already in [BN4]. Yet the assembly would be a bit delicate, and hence a proof is reproduced below which refers back to [BN4] only at one technical point.

By inspecting the definition of $tha^{ux}$, it is clear that there is *some* assignment $\gamma \mapsto R_u^\gamma$ that assigns an operator $R_u^\gamma\colon FL(T) \to FL(T)$ to every $\gamma \in FL(T)$ and that there is *some* functional $K_u\colon FL(T) \to CW(T)$, for which a version of Equation (35) holds:

$$E_s(\lambda;\omega)_s /\!\!/ tha^{ux} = E_s\left(\lambda /\!\!/ R_u^{\lambda_x};(\omega+K_u(\lambda_x)) /\!\!/ R_u^{\lambda_x}\right)_s \tag{36}$$

Indeed, $tha^{ux}$ acts on $E_s(\lambda;\omega)_s$ by placing a copy of $\exp(\lambda_x)$ at the top of the tail strand $u$, and then re-writing the result without having any heads on strand $u$ so as to invert $E_s$ back again. The re-writing is done by sliding the heads of $\exp(\lambda_x)$ down to the bottom of strand $u$, where they cancel by $CP$. Every time a head slides past a tail we get a contribution from $\overrightarrow{STU}_2$. Sometimes a head of a $\lambda_x$ will slide against a tail of another $\lambda_x$, whose head will have to slide down too, leading to a rather complicated iterative process. Nevertheless, these contributions are the same for every tail on strand $u$, namely for every occurrence of the variable $u$ in $FL(T)^H$ and/or in $CW(T)$. This explains the terms $\lambda /\!\!/ R_u^{\lambda_x}$ and $\omega /\!\!/ R_u^{\lambda_x}$ in Equation (36). We note that the degree 0 part of the operator $R_u^{\lambda_x}$ is the identity, and hence it is invertible.

But yet another type of term arises in the process — sometimes a head of some tree will slide against a tail of its own, and then the contribution arising from $\overrightarrow{STU}_2$ will be a wheel. Hence there is an additional contribution to the output, some $L_u(\lambda_x)$ which clearly can depend only on $u$ and $\lambda_x$. Using the invertibility of $R_u^{\lambda_x}$ to write $L_u(\lambda_x) = K_u(\lambda_x) /\!\!/ R_u^{\lambda_x}$ we completely reproduce Equation (36).

We now need to show that $R_u^\gamma$ and $K_u(\gamma)$ are $RC_u^\gamma$ and $J_u(\gamma)$ of Definitions 2.9 and 2.11. Tracing again through the discussion in the previous two paragraphs, we see that at any fixed degree, $R_u^\gamma$ and $K_u(\gamma)$ depend polynomially on the coefficients of $\gamma$, and hence it is legitimate to study their variation with respect to $\gamma$. It is also easy to verify that $R_u^0 = RC_u^0 = I$ and that $K_u(0) = J_u(0) = 0$, and hence it is enough to show that, with an indeterminate scalar $\tau$,

$$\frac{d}{d\tau} R_u^{\tau\gamma} = \frac{d}{d\tau} RC_u^{\tau\gamma} \qquad \text{and} \qquad \frac{d}{d\tau} K_u(\tau\gamma) = \frac{d}{d\tau} J_u(\tau\gamma). \tag{37}$$

Let us compute the left-hand-sides of the above equations. If $\tau$ is an infinitesimal (so $\tau^2 = 0$), or more precisely, computing the above left-hand-sides at $\tau = 0$, we can re-trace the process described in the two paragraphs following Equation (36) keeping in mind that with $\lambda_x = \tau\gamma$ the $\overrightarrow{STU}_2$ relation can only by applied once (or else terms proportional to $\tau^2$ will arise). The result is

$$\left.\frac{d}{d\tau} R_u^{\tau\gamma}\right|_{\tau=0} = \mathrm{ad}_u^\gamma \qquad \text{and} \qquad \left.\frac{d}{d\tau} K_u(\tau\gamma)\right|_{\tau=0} = \mathrm{div}_u(\gamma), \tag{38}$$

where $\mathrm{ad}_u^\gamma \colon FL(T) \to FL(T)$ is the derivation which maps the generator $u$ of $FL(T)$ to $[\gamma, u]$ and annihilates all other generators of $FL(T)$ (compare [BN4, Definition 10.5]) and where $\mathrm{div}_u(\gamma)$ is the same as in Definition 2.11.

Moving on to general $\tau$, we note that the operations $hm$ and $tha$ satisfy[C25]

$$hm_z^{xy} /\!\!/ tha^{uz} = tha^{ux} /\!\!/ tha^{uy} /\!\!/ hm_z^{xy} \tag{39}$$

(stitching strands $x$ and $y$ and then stitching a copy of the result to $u$ is the same as stitching a copy of $x$ to $u$, then a copy of $y$, and then stitching $x$ to $y$; compare [BN4, Equation (6)]). Applying the operators on the two sides of Equation (39) to $E_s(\lambda; \omega)$ (assuming $H$ and $T$ are such that it makes sense), then expanding using (29) and (36), and then ignoring the wheels in the resulting equality, we find that $R_u$ satisfies

$$R_u^{\mathrm{BCH}(\lambda_x, \lambda_y)} = R_u^{\lambda_x} /\!\!/ R_u^{\lambda_y /\!\!/ R_u^{\lambda_x}} \tag{40}$$

(compare [BN4, Equation (16)]). Similarly, looking only at the wheel part of (39) we get

$$K_u(\mathrm{BCH}(\lambda_x, \lambda_y)) /\!\!/ R_u^{\mathrm{BCH}(\lambda_x, \lambda_y)} = K_u(\lambda_x) /\!\!/ R_u^{\lambda_x} /\!\!/ R_u^{\lambda_y /\!\!/ R_u^{\lambda_x}} + K_u(\lambda_y /\!\!/ R_u^{\lambda_x}) /\!\!/ R_u^{\lambda_y /\!\!/ R_u^{\lambda_x}},$$

which, composing on the right with $R_u^{\mathrm{BCH}(\lambda_x, \lambda_y)}$ and using (40), is equivalent to

$$K_u(\mathrm{BCH}(\lambda_x, \lambda_y)) = K_u(\lambda_x) /\!\!/ R_u^{\lambda_x} + K_u(\lambda_y /\!\!/ R_u^{\lambda_x}) /\!\!/ C_u^{-\lambda_x} \tag{41}$$

(compare [BN4, Equation (19)]).

Equations (40) and (41) hold for any $\lambda$, and hence for any $\lambda_x$ and $\lambda_y$. Specializing to $\lambda_x = \tau\gamma$ and $\lambda_y = \epsilon\gamma$, where $\epsilon$ is some new indeterminate scalar, and using the fact that

*computations below*

[C25] None should believe without a verification:

```
lhs = ξa // hm[1, 2, 4] // tha[u, 4]; rhs = ξa // tha[u, 1] // tha[u, 2] // hm[1, 2, 4];
{lhs, (lhs ≡ rhs)@{8}}
```

$$\Big\{\mathrm{Es}\Big[\Big\langle 3 \to \mathrm{LS}\Big[\overline{u},\ -\overline{uv},\ -\overline{u\,\overline{uv}} + \frac{1}{2}\overline{\overline{uv}\,v},\ \frac{3}{2}\overline{u\,\overline{u\,\overline{uv}}} + \overline{u\,\overline{\overline{uv}\,v}} - \frac{1}{6}\overline{\overline{\overline{uv}\,v}\,v},\ \ldots\Big],$$
$$4 \to \mathrm{LS}\Big[\overline{u} + \overline{v},\ \frac{\overline{uv}}{2},\ -\frac{23}{12}\overline{u\,\overline{uv}} - \frac{5}{12}\overline{\overline{uv}\,v},\ \overline{u\,\overline{u\,\overline{uv}}} + \frac{13}{24}\overline{u\,\overline{\overline{uv}\,v}} + \frac{1}{12}\overline{\overline{\overline{uv}\,v}\,v},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[2\,\widehat{u},\ -\widehat{uv},\ -\frac{3\,\widehat{uuv}}{2},\ -\frac{\widehat{uuuv}}{6} + \widehat{uuvv} - \widehat{uvuv},\ \ldots\Big]\Big],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\Big\}$$

$\mathrm{BCH}(\tau\gamma, \epsilon\gamma) = (\tau+\epsilon)\gamma$, Equations (40) and (41) become

$$R_u^{(\tau+\epsilon)\gamma} = R_u^{\tau\gamma} /\!\!/ R_u^{\epsilon\gamma /\!\!/ R_u^{\tau\gamma}} \qquad \text{and} \qquad K_u((\tau+\epsilon)\gamma) = K_u(\tau\gamma) /\!\!/ R_u^{\tau\gamma} + K_u(\epsilon\gamma /\!\!/ R_u^{\tau\gamma}) /\!\!/ C_u^{-\tau\gamma}.$$

Now differentiating with respect to $\epsilon$ at $\epsilon = 0$ and using Equation (38) with $\tau$ replaced with $\epsilon$, we get

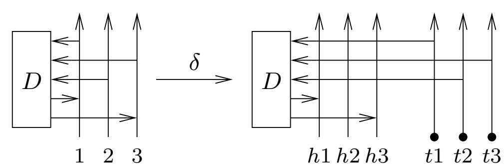
$$\frac{d}{d\tau} R_u^{\tau\gamma} = R_u^{\tau\gamma} /\!\!/ \operatorname{ad}_u^{\gamma /\!\!/ R_u^{\tau\gamma}} \qquad \text{and} \qquad \frac{d}{d\tau} K_u(\tau\gamma) = \operatorname{div}_u(\gamma /\!\!/ R_u^{\tau\gamma}) /\!\!/ C_u^{-\tau\gamma}.$$

The first of these equations is the same equation that is satisfied by $RC_u$ (see [BN4, Lemma 10.7], with $\delta\gamma$ proportional to $\gamma$), and hence $R_u = RC_u$. By a simple change of variables, $J_u(\tau\gamma) = \int_0^\tau dt\, \operatorname{div}_u(\gamma /\!\!/ RC_u^{t\gamma}) /\!\!/ C_u^{-t\gamma}$, and hence $\frac{d}{d\tau} J_u(\tau\gamma) = \operatorname{div}_u(\gamma /\!\!/ RC_u^{\tau\gamma}) /\!\!/ C_u^{-\tau\gamma}$ (compare with the formula for the full differential of $J$, [BN4, Proposition 10.10]). Comparing with the above formula for the derivative of $K_u$, we find that $K_u = J_u$. □

2.4.4. *The inclusion* $\{\mathcal{A}^w(S)\} \hookrightarrow \{\mathcal{A}^w(H;T)\}$. The following definition and proposition imply that there is no loss in studying the spaces $\mathcal{A}^w(H;T)$ rather than the spaces $\mathcal{A}^w(S)$.

**Definition 2.21.** Let $\delta\colon \mathcal{A}^w(S) \to \mathcal{A}^w(S;S)$ be the composition of the "double every strand" map $\prod_{a\in S} \Delta^a_{ha,ta}\colon \mathcal{A}^w(S) \to \mathcal{A}^w(hS \sqcup tS)$ with the projection $\mathcal{A}^w(hS \sqcup tS) \to \mathcal{A}^w(S;S)$ (as an exception to the rule of Footnote 13 we temporarily highlight the distinction between head and tail labels by affixing them with the prefixes $h$ and $t$).

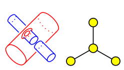
*Comment* 2.22. If $D \in \mathcal{A}^w(S)$ is sorted "heads below tails" as in Comment 2.1, then $\delta D$ is $D$ with its arrow heads placed on the head strands and its arrow tails placed on the tail strands, as shown on the right.




**Proposition 2.23.** *$\delta$ is a (non-multiplicative) vector space isomorphism. The inverse of $\delta$ on $D \in \mathcal{A}^w(S;S)$ is given by the process*

1. *Write $D$ with only arrow heads on the head strands and only arrow tails on the tail strands. By Comment 2.16 this produces a well-defined element $D'$ of $\mathcal{A}^w(hS \sqcup tS)$.*
2. *Stitch all the head-tail pairs of strands in $D'$ by putting each head ahead of its corresponding tail: $\delta^{-1}D = D' /\!\!/ \prod_a dm_a^{ha,ta}$.*

*Proof.* $\delta^{-1} /\!\!/ \delta = I$ by inspection, and $\delta /\!\!/ \delta^{-1}$ is clearly the identity on diagrams sorted to have heads ahead of tails as in Comment 2.1. □

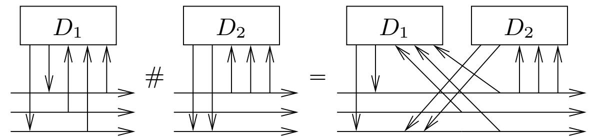


In topology, $\delta$ agrees with the $\delta$ of [BN4, Section 2.2]. In Lie theory, it agrees with the linear (non-multiplicative) isomorphism $\mathcal{U}(I\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g}^*)$ and with similar isomorphisms considered by Etingof and Kazhdan within their work on the quantization of Lie bialgebras [EK] (albeit only when the Lie bialgebras in question are cocommutative).

**Definition 2.24.** The product # of $\mathcal{A}^w(S;S)$ induces a new product, also denoted #, on $\mathcal{A}^w(S)$. If $D_1$ and $D_2$ are in $\mathcal{A}^w(S)$, set

$$D_1 \# D_2 := (\delta(D_1) \# \delta(D_2)) /\!\!/ \delta^{-1}. \tag{42}$$

*Comment* 2.25. With Comment 2.22 in mind, we see that if $D_1$ and $D_2$ are sorted as in Comment 2.1, then $D_1 \# D_2$ is "heads of $D_1$, then of $D_2$, then tails of $D_1$, then of $D_2$" (with the last two parts interchangeable, by $TC$). The picture is nicer when rotated, as on the right.




See the comments following Discussion 2.5.

The next proposition shows how the operations of defined on the $\mathcal{A}^w(S)$-spaces in Definition 2.2 can be written in terms of the "head and tail" operations of Definition 2.17, thus completing the description of the $E_s$ presentation.

**Proposition 2.26.** *(1) If $S_1$ and $S_2$ are disjoint and $D_1 \in \mathcal{A}^w(S_1)$ and $D_2 \in \mathcal{A}^w(S_2)$, then $\delta(D_1 \sqcup D_2) = \delta(D_1) \sqcup \delta(D_2)$.*

*(2) Let $D_1, D_2 \in \mathcal{A}^w(S)$. Then $\delta(D_1D_2)$ can be written in terms of $\delta(D_1)$ and $\delta(D_2)$ using its description in terms of $\sqcup$, $d\sigma$, and $dm$ in Equation (2) and using the formulas for $\sqcup$, $d\sigma$, and $dm$ that appear in parts (1), (8), and (6) of this proposition.*[C26]

*(3) $d\eta^a /\!\!/ \delta = \delta /\!\!/ h\eta^a /\!\!/ t\eta^a$.*

*(4) $dA^a /\!\!/ \delta = \delta /\!\!/ hA^a /\!\!/ tA^a /\!\!/ tha^{aa}$.*

*(5) $dS^a /\!\!/ \delta = \delta /\!\!/ hS^a /\!\!/ tS^a /\!\!/ tha^{aa}$.*

*(6) $dm^{ab}_c /\!\!/ \delta = \delta /\!\!/ tha^{ab} /\!\!/ hm^{ab}_c /\!\!/ tm^{ab}_c$.*[C26]

*(7) $d\Delta^a_{bc} /\!\!/ \delta = \delta /\!\!/ h\Delta^a_{bc} /\!\!/ t\Delta^a_{bc}$.*

*(8) $d\sigma^a_b /\!\!/ \delta = \delta /\!\!/ h\sigma^a_b /\!\!/ t\sigma^a_b$.*

*Proof.* The only difficulty is with items *(4)*–*(6)*. Item *(4)* is easier to understand in the form $\delta^{-1} /\!\!/ dA^a = hA^a /\!\!/ tA^a /\!\!/ tha^{aa} /\!\!/ \delta^{-1}$. Indeed, $\delta^{-1}$ plants heads ahead of tails on strand $a$. Applying $dA^a$ reverses that strand (and adds some signs). This reversal can be achieved by reversing the head part (with signs), then the tail part (with signs), and then by swapping the two parts across each other. The first reversal is $hA^a$, the second is $tA^a$, and the swap

*computations below*

[C26] As a sample for the whole proposition, we quote the implementation of $dm$ and verify its meta-associativity $dm^{ab}_a /\!\!/ dm^{ac}_a = dm^{bc}_b /\!\!/ dm^{ab}_a$ (compare [BN4, Equation (32)]). We then include our implementation of the stacking product (item *(2)* above) without further explanations:

```
AC ξ_Es // dm[a_, b_, c_] := ξ // tha[a, b] // tm[a, b, c] // hm[a, b, c];
```

```
lhs = ξb // dm[1, 2, 1] // dm[1, 3, 1]; rhs = ξb // dm[2, 3, 2] // dm[1, 2, 1];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

$$\Big\{\mathrm{Es}\Big[\Big\langle 1 \to \mathrm{LS}\Big[-2\,\overline{1} + \overline{4},\ -\frac{3\,\overline{14}}{2},\ 20\,\overline{1\overline{14}} - \frac{19}{3}\,\overline{\overline{14}4},\ \ldots\Big],$$
$$4 \to \mathrm{LS}\Big[2\,\overline{1} + \overline{4},\ \overline{14},\ -\frac{31}{2}\,\overline{1\overline{14}} - \frac{13}{6}\,\overline{\overline{14}4},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[3\,\widehat{1} - \widehat{4},\ -3\,\widehat{11} + \frac{\widehat{14}}{2} + \widehat{44},\ \frac{71\,\widehat{111}}{4} + \frac{19\,\widehat{114}}{4} - \frac{7\,\widehat{144}}{6} - \frac{2\,\widehat{444}}{3},\ \ldots\Big]\Big],\ \mathrm{BS}[6\ \mathrm{True},\ \ldots]\Big\}$$

```
AC Es /: Es[λ1_, ω1_] ** Es[λ2_, ω2_] /; Support[λ1] == Support[λ2] := Module[
    {S = Support[λ1], ξ, a},
    ξ = Es[λ1, ω1] * (Es[λ2, ω2] // dσ[S → (bar /@ S)]);
    Table[ξ = ξ // dm[a, bar[a], a], {a, S}] // Last
  ];
```

is $tha^{aa}$ followed by $\delta^{-1}$. Item *(5)* is proven in exactly the same way, and item *(6)* is proven in a similar way, where the right hand side traces the schematics $(ha\,ta\,hb\,tb) \xrightarrow{tha} (ha\,hb\,ta\,tb) \xrightarrow{hm/\!\!/tm} ((ha\,hb)(ta\,tb))$. $\square$

*Discussion* 2.27. It is easy to verify that $\delta\colon \mathcal{A}^w(S) \to \mathcal{A}^w(S;\, S)$ is a co-algebra morphism, and hence it restricts to an isomorphism $\delta\colon \mathcal{A}^w_{\exp}(S) \to \mathcal{A}^w_{\exp}(S;\, S)$. Therefore $E_s/\!\!/\delta^{-1}$ is a bijection between $TW_s(S) := TW_s(S; S)$ and $\mathcal{A}^w_{\exp}(S)$. Proposition 2.26 now tells us how to write all the "*d*" operations of Definition 2.2 as compositions of "*h*" and "*t*" operations, and Definition-Proposition 2.20 tells us how to write these as operations on $TW_s(H;T)$ (the $H$ and $T$ label sets that occur here are always $S$ with one or two labels added or removed). Hence overall $E_s/\!\!/\delta^{-1}$, acting on $TW_s(S)$, is a complete presentation of $\mathcal{A}^w_{\exp}(S)$.

**Definition 2.29.** The "factored" presentation $E_f$ of $\mathcal{A}^w_{\exp}$ is the composition $E_f := E_s/\!\!/\delta^{-1}$. Namely, for a set $S$ of strands, we define $E_f\colon TW_s(S) \xrightarrow{\sim} \mathcal{A}^w_{\exp}(S)$ by $(\lambda;\, \omega)_s \mapsto E_s(\lambda;\, \omega)_s/\!\!/\delta^{-1} = \exp_{\#}(l\lambda + \iota\omega)$. See the illustration on the right.

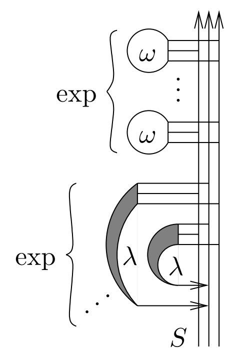



**Figure 2.28.** $E_f(\lambda;\, \omega)_s$.

2.5. **Converting between the $E_l$ and the $E_f$ presentations.** We now have two presentations for elements of $\mathcal{A}^w_{\exp}(S)$, and we wish to be able to convert between the two. This turns out to involve the maps $\Gamma$ and $\Lambda$ of Propositions 2.6 and 2.8.

**Definition 2.30.** Define a pair of inverse maps $\Gamma\colon TW_l(S) \to TW_s(S)$ and $\Lambda\colon TW_s(S) \to TW_l(S)$ by

$$\Gamma\colon (\lambda;\, \omega)_l \mapsto (\Gamma(\lambda);\, \omega)_s \quad \text{and} \quad \Lambda\colon (\lambda;\, \omega)_s \mapsto (\Lambda(\lambda);\, \omega)_l.$$

**Theorem 2.31.** *The left-most triangle in Figure 1.2 commutes. Namely,*

$$E_l = \Gamma/\!\!/E_f \quad \textit{and} \quad E_f = \Lambda/\!\!/E_l. \tag{43}$$

*(All other parts of Figure 1.2 commute by definition).*

Before we can prove this theorem we need a few preliminaries. For an element $D \in \mathcal{A}^w_{\exp}(S)$, we can define three associated quantities:

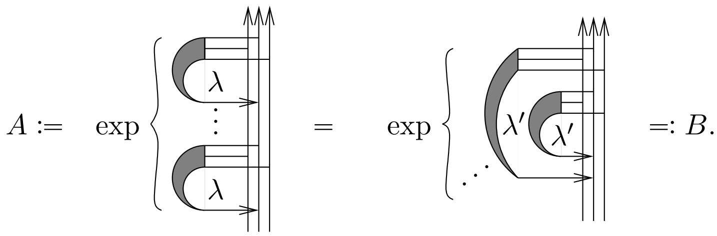
- The projection of $D$ to the degree 1 part of $\mathcal{A}^w(S)$, and especially, the projection $\pi_A(D)$ of the degree 1 part to its "framing" part $A_S$ (consisting of self-arrows, that begin and end on the same strand and point, say, up).
- A conjugation automorphism $C_D$ of $FL(S)$, defined as follows. First, embed $FL(S)$ into $\mathcal{A}^w(S \sqcup \{\infty\})$ by mapping any generator $a \in S$ to a degree 1 diagram in $\mathcal{A}^w(S \sqcup \{\infty\})$, the arrow whose tail is on strand $a$ and whose head is on the new "$\infty$" strand and extending in a bracket-preserving way, using the commutator of the stacking product as the bracket on $\mathcal{A}^w(S \sqcup \{\infty\})$. Then note that $FL(S) \subset \mathcal{A}^w(S \sqcup \{\infty\})$ is invariant under conjugation by $D$ and let $C_D$ denote this conjugation action.

  This is a direct analog of the Artin action of the pure braid groups $PuB_n$ / $PwB_n$ on the free group $FG(n)$.
- $\pi_{\bullet}(D)$ is the result of adding a bullet at the bottom of every strand of $D$, in the same sense as in Section 2.4.1. Equivalently, $\pi_{\bullet} = \delta/\!\!/\prod_{a\in S} h\eta^a$ is the composition of $\delta$ with "delete all head strands". The target space of $\pi_{\bullet}$ is $\mathcal{A}^w(\varnothing; S)$, which is the symmetric algebra $\mathcal{S}(CW(S))$ generated by wheels.

**Proposition 2.32.** *$D$ is determined by the above three quantities $\pi_A(D)$, $C_D$, and $\pi_\downarrow(D)$.*

*Proof.* As in Section 2.3, every $D \in \mathcal{A}^w_{\exp}(S)$ can be written uniquely in the form $D = e^{l\lambda}e^{\iota\omega}$, where $\lambda \in FL(S)^S$ and $\omega \in CW(S)$. One may easily verify that $\pi_\downarrow(D)$ is $\omega$, that $C_D$ is the exponential of the derivation in $\mathrm{tder}_S$ corresponding to $\lambda$, and that $\pi_A(D)$ determines the part of $\lambda$ lost by the projection $FL(S)^S \to \mathrm{tder}_S$. □

*Proof of Theorem 2.31.* For $\lambda \in FL(S)^S$ let $\lambda' = \Gamma(\lambda)$. Comparing Figures 2.13 and 2.28, we find that the $\omega$ parts drop out and we need to prove, schematically, that in $\mathcal{A}^w_{\exp}(S)$,




A simple degree 1 calculation shows that $\pi_A(A) = \pi_A(B) = 0$. The CP relation of Section 2.4.1 shows that $\pi_\downarrow(A) = \pi_\downarrow(B) = 0$. Finally, it is easy to verify that $C_A = e^{-\partial_\lambda}$ while $C_B = C^{\lambda'}$, and hence $C_A = C_B$ follows from Proposition 2.6. □

## 3. Some Computations

### 3.1. Tangle Invariants.

3.1.1. *The General Framework.* Recall from [WKO2] that the assignment $Z^w$: ⤱ ↦ exp(↦)⤱ defined on $S$-component tangles and taking values in $\mathcal{A}^w_{\exp}(S)$ (where ↦ denotes an arrow connecting the upper strand to the lower strand and exponentiation is in a formal sense) defines an invariant of tangles with values in $\mathcal{A}^w_{\exp}(S)$. We'd like to compute $Z^w$ (more precisely, its logarithm), in as much as possible, using both the $TW_l(S)$-valued [AT]-presentation $E_l$ or using the $TW_s(S)$-valued factored presentation $E_f$ (recall Figure 1.2).

We let $R_l^+(a,b)$ and $R_s^+(a,b)$ denote the value $R(a,b) = Z^w\left(\nearrow\!\!\!\!\nwarrow_{a\ b}\right)$ of the positive crossing in $TW_l$ and $TW_s$, respectively, and similarly, let $R_l^-(a,b)$ and $R_s^-(a,b)$ denote the value $R^{-1}(a,b) = Z^w\left(\nwarrow\!\!\!\!\nearrow_{b\ a}\right)$ of the negative crossing in $TW_l$ and $TW_s$, respectively (for both signs we label the upper strand $a$ and the lower strand $b$). That is,

$$Z^w\left(\nearrow\!\!\!\!\nwarrow_{a\ b}\right) = R_l^+(a,b) /\!\!/ E_l = R_s^+(a,b) /\!\!/ E_s \qquad \text{and} \qquad Z^w\left(\nwarrow\!\!\!\!\nearrow_{b\ a}\right) = R_l^-(a,b) /\!\!/ E_l = R_s^-(a,b) /\!\!/ E_s.$$

One may easily verify that $R^\pm_{l,s}(a,b) = (a \to 0,\ b \to \pm a;\ 0)_{l,s}$[C27], and it is a simple exercise to verify that $R$ satisfies the Yang-Baxter / Reidemeister 3 relation $R^+_{l,s}(1,2) * R^+_{l,s}(1,3) * R^+_{l,s}(2,3) = R^+_{l,s}(2,3) * R^+_{l,s}(1,3) * R^+_{l,s}(1,2)$[C28].

*computations below*

[C27] In computer talk, this is

```
Rl[a_, b_] := El[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRl[a_, b_] := El[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
```

[C28] Indeed, here's a computer verification in $E_l$, to degree 5:

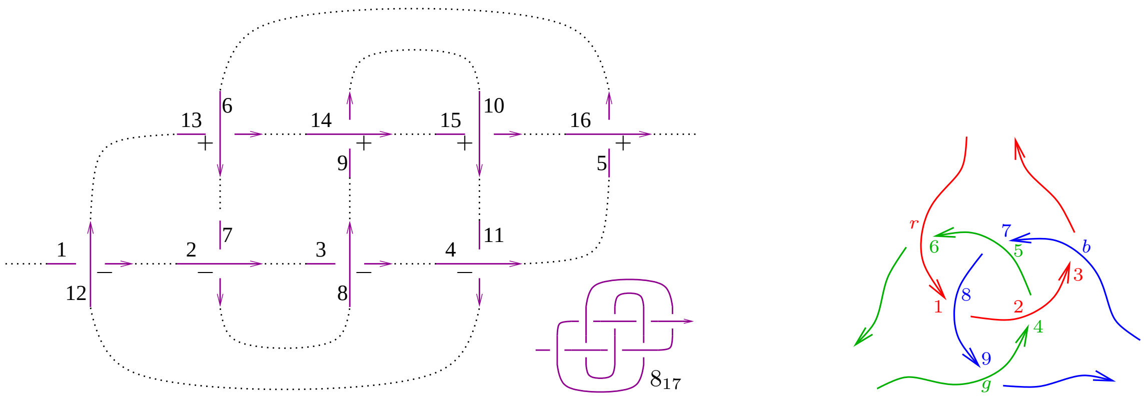



**Figure 3.1.** The knot $8_{17}$ and the Borromean tangle.

3.1.2. *The Knot* $8_{17}$ *and the Borromean Tangle.* In this short section we evaluate $Z^w$ on the knot $8_{17}$ and on the Borromean tangle, both shown in Figure 3.1. An expanded version of this section appears as [BN4, Sections 6.3 and 6.4].

For the 8-crossing knot $8_{17}$ we need to take 8 copies of $R_s^\pm$ with strands labelled 1 through 16 as in Figure 3.1, and then stitch strands 1 to 2, 2 to 3, etc[C29]. This is done using $dm$ operations, and hence we cannot use the $E_l$ presentation.

Similarly for the 6-crossings Borromean tangle we need 6 copies of $R_s^\pm$ followed by some stitching[C30]. A colourful evaluation of the Borromean tangle appears in [BN4, Section 6.4].

### 3.2. Solutions of the Kashiwara-Vergne Equations.

In [WKO2, Section 4.1] we found that in order to construct a homomorphic expansion $Z^w$ for the class $uTF^o$ of orientable w-tangled foams, defined there, we need to find elements $V = Z^w(\text{⤒})\in \mathcal{A}^w_{\exp}(x,y)$[C31] and

*computations below*

```
lhs = Rl[1, 2] ** Rl[1, 3] ** Rl[2, 3]; rhs = Rl[2, 3] ** Rl[1, 3] ** Rl[1, 2];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

```
{El[⟨1 → LS[0, 0, 0, ...], 2 → LS[1̅, 0, 0, ...], 3 → LS[1̅ + 2̅, 0, 0, ...]⟩, CWS[0, 0, 0, ...]],
 BS[6 True, ...]}
```

[C29] Here it is, to degree 6:

```
t1 = iRs[12, 1] iRs[2, 7] iRs[8, 3] iRs[4, 11] Rs[16, 5] Rs[6, 13] Rs[14, 9] Rs[10, 15];
Do[t1 = t1 // dm[1, k, 1], {k, 2, 16}];
t1@{6}
```

Fuller output: [WKO4]/817.nb

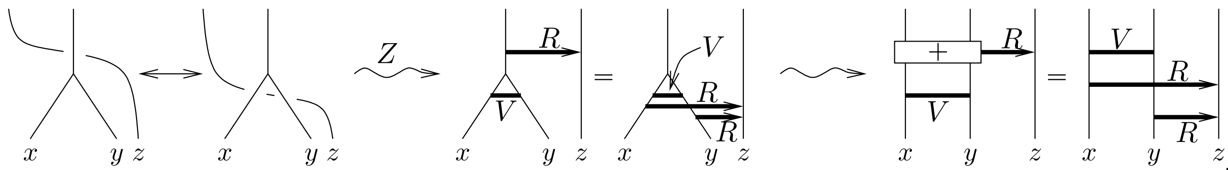
```
Es[⟨1 → LS[0, 0, 0, 0, 0, 0, ...]⟩,
  CWS[0, -11̅, 0, -(31 1111̅)/12, 0, -(1351 111111̅)/360, ...]]
```

[C30] To degree 4, we get

```
t2 = iRs[r, 6] Rs[2, 4] iRs[g, 9] Rs[5, 7] iRs[b, 3] Rs[8, 1];
(Do[t2 = t2 // dm[r, k, r], {k, 1, 3}]; Do[t2 = t2 // dm[g, k, g], {k, 4, 6}];
 Do[t2 = t2 // dm[b, k, b], {k, 7, 9}]; t2)
```

*Cap* $= Z^w(\text{⍿}) \in \mathcal{A}^w_{\exp}(\uparrow_x)$[17] [C32] that are required to satisfy the three equations in (44) and (45) below. Recall from [WKO2, Section 4.4] that these equations are equivalent to equations considered by Alekseev and Torossian in [AT] (see [WKO2, Equation 14] and [AT, Section 5.3]), and that the latter equations were shown in [AT, Section 5.2] to be equivalent to the Kashiwara-Vergne equations of [KV].

The purpose of this section is to trace through all that at the level of actual computations. Let us start by recalling from [WKO2] the equations for $V$ and for *Cap*. The first of those is the $R4$ equation [WKO2, (11)], $V^{12}R^{(12)3} = R^{23}R^{13}V^{12}$, coming from the picture




In the language of this paper, and denoting the three strands $x$, $y$, and $z$, this equation becomes

$$V * (R(x,z) /\!\!/ d\Delta^x_{xy}) = R(y,z) * R(x,z) * V \quad (44)$$ [C33]

[17] *Cap* is called $C$ in [WKO2] and we trust that the other minor notational differences with [WKO2] will cause no difficulty to the reader. Note that $\mathcal{A}^w(\uparrow_S)$ is $\mathcal{A}^w(S)$ with $CP$ relations imposed at the tops of the strands; compare with Section 2.4.1.

*computations below*

Fuller output: [WKO4]/Borromean.nb

```
Es[⟨b → LS[0, gr, 1/2 ggr + brg + 1/2 grr,
    -1/2 bbrg + 1/6 gggr + 1/4 ggrr - 1/2 bgbr - 1/2 brgg - 1/2 brrg + 1/6 grrr, ...],
  g → LS[0, -br, 1/2 bbr - bgr - brg + 1/2 brr, -1/6 bbbr - 1/2 bbgr - 1/2 bggr - 1/2 bbrg -
    1/4 bbrr + 1/2 bgrr + 1/2 bgbr + brgr - bgrg - 1/2 brgg + 1/2 brrg - 1/6 brrr, ...],
  r → LS[0, bg, 1/2 bbg + bgr + 1/2 bgg, 1/6 bbbg + 1/2 bbgr + 1/2 bggr +
    1/4 bbgg + 1/2 bgrr + 1/6 bggg, ...]⟩,
  CWS[0, 0, 2 bgr, bbgr - bgbr + bggr - bgrg + bgrr - brgr, ...]]
```

[C31] For computations, we use the $E_s$ presentation for $V$. As $V$ is presented in $TW_s(\{x,y\})$, it is of the form $V = ((x \to \alpha, y \to \beta); \gamma)_s$, where $\alpha, \beta \in FL(x,y)$ and $\gamma \in CW(x,y)$, and where the coefficients of $\alpha$, $\beta$, and $\gamma$, what we call the $\alpha$s, the $\beta$s, and the $\gamma$s, will be determined later. The first line below sets $\alpha$, $\beta$, and $\gamma$ to be series with yet-unknown coefficients, and the second line sets $V$ to be the appropriate combination of $\alpha$, $\beta$, and $\gamma$:

```
x = LW["x"]; y = LW["y"]; z = LW["z"];
α = LS[{x, y}, αs]; β = LS[{x, y}, βs]; γ = CWS[{x, y}, γs];
V0 = Es[⟨x → α, y → β⟩, γ];
```

(for a technical reason, in computations we use the symbol $\mathtt{V}_0$ to denote $V$).

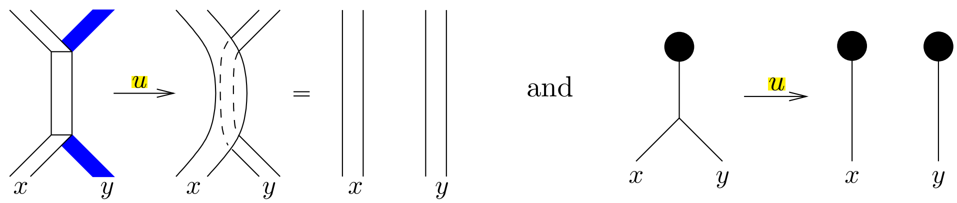
[C32] Similarly, *Cap* is presented in $TW_s(x)$. As it is made only of wheels, its tree part is 0, or the Lie series `LS[0]`. The wheels part of *Cap* is a series $\kappa \in CW(x)$ whose coefficients are the yet-unknown $\kappa$s:

```
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
```

The second and the third, "unitarity" and the "cap equation", [WKO2, (12)] and [WKO2, (13)], are the equations

$$V * (V /\!\!/ dA) = 1 \quad \text{in } \mathcal{A}^w(x,y) \quad \text{and} \quad V * (\mathit{Cap} /\!\!/ d\Delta^x_{xy}) = \mathit{Cap}(\mathit{Cap} /\!\!/ d\sigma^x_y) \quad \text{in } \mathcal{A}^w(\uparrow_{x,y}),^{\text{C33}} \qquad (45)$$

which come from the two unzip operations,




Solving Equations (44) and (45) degree by degree with the initial condition $\alpha = -y/2 + \ldots$ we find that one possible solution, given in the factored presentation, is

$$V = E_f\Bigg(x \to -\frac{\overline{xy}}{24} + \frac{7\,\overline{x\,\overline{x\,\overline{xy}}}}{5760} - \frac{7\,\overline{x\,\overline{\overline{xy}\,y}}}{5760} + \frac{\overline{\overline{\overline{xy}\,y}\,y}}{1440} + \ldots,$$

$$y \to \frac{\overline{x}}{2} - \frac{\overline{xy}}{12} + \frac{\overline{x\,\overline{x\,\overline{xy}}}}{5760} - \frac{\overline{x\,\overline{\overline{xy}\,y}}}{720} + \frac{\overline{\overline{\overline{xy}\,y}\,y}}{720} + \ldots;$$

$$-\frac{\widehat{xy}}{48} + \frac{\widehat{xxxy}}{2880} + \frac{\widehat{xxyy}}{2880} + \frac{\widehat{xyxy}}{5760} + \frac{\widehat{xyyy}}{2880} + \ldots\Bigg)_s,$$

---

*computations below*

C33 The three equations in (44) and (45) are coded as follows:

```
R4Eqn = V0 ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V0 ;
UnitarityEqn = V0 ** (V0 // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]];
CapEqn = (V0 ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
   (Cap * (Cap // dσ[x, y]) // dc[x] // dc[y]);
```

and $\mathcal{C}ap = -\widehat{xx}/96+\widehat{xxxx}/11,520-\widehat{xxxxxx}/725,760+\ldots$[C34]. Note that according to [WKO3], $\mathcal{C}ap$ is always $\sum a_n\widehat{x^n}$, where $\sum a_n\hbar^n = \frac{1}{4}\log\left(\frac{\hbar/2}{\sinh\hbar/2}\right)$[C35].

We can also write $V$ in the lower-interlaced presentation:

$$V = E_l\Bigg(x \to -\frac{\overline{xy}}{24}+\frac{\overline{x\,\overline{xy}}}{96}+\frac{\overline{x\,\overline{x\,\overline{xy}}}}{2880}-\frac{\overline{x\,\overline{\overline{xy}\,y}}}{480}+\frac{\overline{\overline{\overline{xy}\,y}\,y}}{1440}+\ldots,$$

$$y \to \frac{\overline{x}}{2}-\frac{\overline{xy}}{12}+\frac{\overline{x\,\overline{xy}}}{96}+\frac{\overline{x\,\overline{x\,\overline{xy}}}}{960}-\frac{\overline{x\,\overline{\overline{xy}\,y}}}{320}+\frac{\overline{\overline{\overline{xy}\,y}\,y}}{720}+\ldots;$$

$$-\frac{\widehat{xy}}{48}+\frac{\widehat{xxxy}}{2880}+\frac{\widehat{xxyy}}{2880}+\frac{\widehat{xyxy}}{5760}+\frac{\widehat{xyyy}}{2880}+\ldots\Bigg)_s,$$ [C36]

($\mathcal{C}ap$ is the same in both presentations).

Recall from [WKO2, Section 4.4] and from Comment 2.15 that the tree part of "our" $V$, taken in the lower-interlaced presentation, is $\log F^{21}$, where $F$ is the solution of "generalized

*computations below*

[C34]We set the initial condition for $\alpha$ in degree 1, then declare that $\alpha$, $\beta$, $\gamma$, and $\kappa$ are the series which solve equations `R4Eqn`, `UnitarityEqn`, and `CapEqn`, and then print the values of $V$ and $\kappa$ (note the $\hbar^{-1}$ that comes with `R4Eqn` — it indicates a degree shift — `R4Eqn` in degree $k$ only puts conditions on our unknowns at degree $k-1$):

```
βs["x"] = 1 / 2; βs["y"] = 0;
SeriesSolve[{α, β, γ, κ}, (ℏ^-1 R4Eqn) ∧ UnitarityEqn ∧ CapEqn];
{V0@{4}, κ@{6}}
```

Fuller output: [WKO4]/`VCapSolution.nb`

```
SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {κs[x] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {αs[x, y, y] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.
General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. >>
```

$$\Big\{\mathrm{Es}\Big[\Big\langle\overline{x}\to\mathrm{LS}\Big[0,\ -\frac{\overline{xy}}{24},\ 0,\ \frac{7\,\overline{x\,\overline{x\,\overline{xy}}}}{5760}-\frac{7\,\overline{x\,\overline{\overline{xy}\,y}}}{5760}+\frac{\overline{\overline{\overline{xy}\,y}\,y}}{1440},\ \ldots\Big],$$

$$\overline{y}\to\mathrm{LS}\Big[\frac{\overline{x}}{2},\ -\frac{\overline{xy}}{12},\ 0,\ \frac{\overline{x\,\overline{x\,\overline{xy}}}}{5760}-\frac{1}{720}\,\overline{x\,\overline{\overline{xy}\,y}}+\frac{1}{720}\,\overline{\overline{\overline{xy}\,y}\,y},\ \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[0,\ -\frac{\widehat{xy}}{48},\ 0,\ \frac{\widehat{xxxy}}{2880}+\frac{\widehat{xxyy}}{2880}+\frac{\widehat{xyxy}}{5760}+\frac{\widehat{xyyy}}{2880},\ \ldots\Big]\Big],\ \mathrm{CWS}\Big[0,\ -\frac{\widehat{xx}}{96},\ 0,\ \frac{\widehat{xxxx}}{11\,520},\ 0,\ -\frac{\widehat{xxxxxx}}{725\,760},\ \ldots\Big]\Big\}$$

The solutions of (44) and (45) are not unique, and hence occasionally `SeriesSolve` encounters a coefficient whose value is not determined by the equations. When this happens its default action is to set the missing coefficient to 0. In the computation this happened to the coefficient of $\widehat{x}$ in $\kappa$ and to the coefficient of $\overline{\overline{xy}\,y}$ in $\alpha$.

[C35]Indeed, the series below matches with the computation of $\kappa$, above.

```
Series[1/4 Log[(ℏ/2)/Sinh[ℏ/2]], {ℏ, 0, 12}]
```

$$-\frac{\hbar^2}{96}+\frac{\hbar^4}{11\,520}-\frac{\hbar^6}{725\,760}+\frac{\hbar^8}{38\,707\,200}-\frac{\hbar^{10}}{1\,916\,006\,400}+\frac{691\,\hbar^{12}}{62\,768\,369\,664\,000}+O[\hbar]^{13}$$

[C36]We could re-compute $V$ in $E_l$ by making some simple modifications to the input lines in C33, but it is easier to use our tools and convert between the two presentations:

```
Λ[V0]
```

KV problem" of [AT, Section 5.3] and where the superscript 21 means "interchange the role of $x$ and $y$". Thus using the notation of [AT] a solution to degree 4 of the generalized KV problem is[C37]

$$\log F = \left(\frac{\overline{y}}{2}+\frac{\overline{xy}}{12}+\frac{\overline{\overline{xy}\,y}}{96}-\frac{\overline{x\,\overline{x\,\overline{xy}}}}{720}+\frac{\overline{x\,\overline{\overline{xy}\,y}}}{320}-\frac{\overline{\overline{\overline{xy}\,y}\,y}}{960},\ \frac{\overline{xy}}{24}+\frac{\overline{\overline{xy}\,y}}{96}-\frac{\overline{x\,\overline{x\,\overline{xy}}}}{1440}+\frac{\overline{x\,\overline{\overline{xy}\,y}}}{480}-\frac{\overline{\overline{\overline{xy}\,y}\,y}}{2880}\right).$$

Next, we'd like to compute a solution of the original Kashiwara-Vergne equations of [KV]. These are the two equations below, written for unknowns $f, g \in FL(x, y)$:

$$x + y - \log e^y e^x = (1 - e^{-\operatorname{ad} x})f + (e^{\operatorname{ad} y} - 1)g, \tag{46}$$

$$\operatorname{div}_x f + \operatorname{div}_y g = \frac{1}{2}\operatorname{tr}_u\left(\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1}+\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1}-\frac{\operatorname{ad}\operatorname{BCH}(x,y)}{e^{\operatorname{ad}\operatorname{BCH}(x,y)}-1}\right)(u)\right). \tag{47}$$

By tracing the definitions of the comparison map $\kappa$ which appears in [AT, Theorem 5.8], we find that a solution $(f, g)$ of the Kashiwara-Vergne equations can be computed from $\log F$ via the formula

$$(f,g) = \frac{e^{\operatorname{ad}(\log F)}-1}{\operatorname{ad}(\log F)}(\mathcal{E}(\log F)),$$

where $\mathcal{E}$ denotes the Euler operator, which multiplies every homogeneous element by its degree. To degree 4, we find[C38] that

$$(f,g) = \left(\frac{\overline{y}}{2}+\frac{\overline{xy}}{6}+\frac{\overline{\overline{xy}\,y}}{24}-\frac{\overline{x\,\overline{x\,\overline{xy}}}}{180}+\frac{\overline{x\,\overline{\overline{xy}\,y}}}{80}+\frac{\overline{\overline{\overline{xy}\,y}\,y}}{360},\ \frac{\overline{xy}}{12}+\frac{\overline{\overline{xy}\,y}}{24}-\frac{\overline{x\,\overline{x\,\overline{xy}}}}{360}+\frac{\overline{x\,\overline{\overline{xy}\,y}}}{120}+\frac{\overline{\overline{\overline{xy}\,y}\,y}}{180}\right).$$

*computations below*

$$\mathrm{El}\Big[\Big\langle \overline{x}\to \mathrm{LS}\Big[0,\ -\frac{\overline{xy}}{24},\ \frac{1}{96}\,\overline{x\,\overline{xy}},\ \frac{\overline{x\,\overline{x\,\overline{xy}}}}{2880}-\frac{1}{480}\,\overline{x\,\overline{\overline{xy}\,y}}+\frac{\overline{\overline{\overline{xy}\,y}\,y}}{1440},\ \ldots\Big],$$
$$\overline{y}\to \mathrm{LS}\Big[\frac{\overline{x}}{2},\ -\frac{\overline{xy}}{12},\ \frac{1}{96}\,\overline{x\,\overline{xy}},\ \frac{1}{960}\,\overline{x\,\overline{x\,\overline{xy}}}-\frac{1}{320}\,\overline{x\,\overline{\overline{xy}\,y}}+\frac{1}{720}\,\overline{\overline{\overline{xy}\,y}\,y},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[0,\ -\frac{\widehat{xy}}{48},\ 0,\ \frac{\widehat{xxxy}}{2880}+\frac{\widehat{xxyy}}{2880}+\frac{\widehat{xyxy}}{5760}+\frac{\widehat{xyyy}}{2880},\ \ldots\Big]\Big]$$

[C37]The more authoritative version, of course, is the one printed directly by the computer:

```
logF = Λ[V₀]⟦1⟧ // dσ[{x, y} → {y, x}]
```

$$\Big\langle \overline{x}\to \mathrm{LS}\Big[\frac{\overline{y}}{2},\ \frac{\overline{xy}}{12},\ \frac{1}{96}\,\overline{\overline{xy}\,y},\ -\frac{1}{720}\,\overline{x\,\overline{x\,\overline{xy}}}+\frac{1}{320}\,\overline{x\,\overline{\overline{xy}\,y}}-\frac{1}{960}\,\overline{\overline{\overline{xy}\,y}\,y},\ \ldots\Big],$$
$$\overline{y}\to \mathrm{LS}\Big[0,\ \frac{\overline{xy}}{24},\ \frac{1}{96}\,\overline{\overline{xy}\,y},\ -\frac{\overline{x\,\overline{x\,\overline{xy}}}}{1440}+\frac{1}{480}\,\overline{x\,\overline{\overline{xy}\,y}}-\frac{\overline{\overline{\overline{xy}\,y}\,y}}{2880},\ \ldots\Big]\Big\rangle$$

[C38]With higher authority:

```
atkv = logF // EulerE // adSeries[(e^ad - 1)/ad, logF, tb];
{f = atkvₓ, g = atkvᵧ}
```

$$\Big\{\mathrm{LS}\Big[\frac{\overline{y}}{2},\ \frac{\overline{xy}}{6},\ \frac{1}{24}\,\overline{\overline{xy}\,y},\ -\frac{1}{180}\,\overline{x\,\overline{x\,\overline{xy}}}+\frac{1}{80}\,\overline{x\,\overline{\overline{xy}\,y}}+\frac{1}{360}\,\overline{\overline{\overline{xy}\,y}\,y},\ \ldots\Big],$$
$$\mathrm{LS}\Big[0,\ \frac{\overline{xy}}{12},\ \frac{1}{24}\,\overline{\overline{xy}\,y},\ -\frac{1}{360}\,\overline{x\,\overline{x\,\overline{xy}}}+\frac{1}{120}\,\overline{x\,\overline{\overline{xy}\,y}}+\frac{1}{180}\,\overline{\overline{\overline{xy}\,y}\,y},\ \ldots\Big]\Big\}$$

3.3. **The involution $\tau$ and the Twist Equation.** Alekseev and Torossian [AT, Section 8.2] construct an involution $\tau$ on the set SolKV of solutions of the Kashiwara-Vergne equations. Phrased using the language of [WKO2], Alekseev and Torossian define a map $\tau\colon \mathcal{A}^w(\uparrow_2) \to \mathcal{A}^w(\uparrow_2)$ by $\tau(V) := R(1,2)V^{21}\Theta^{-1/2}$, where $\Theta^s = e^{st}$ and $t = ⋈ + ⋈ \in \mathcal{A}^w(\uparrow_2)$. They then prove that $\tau$ restricts to an involution of the set of solutions Equations (44) and (45). It is not known if $\tau$ is different from the identity; in other words, it is not known if every $V$ satisfying (44) and (45) also satisfies the "Twist Equation"

$$V = \tau(V). \tag{48}$$

In topology, the Twist Equation is essential for the compatibility between $Z^u$ and $Z^w$; see [WKO2, Section 4.7]. So it is not known if "every $Z^w$ is compatible with some $Z^u$". Below the dark line we verify that to degree 6, "our" $V$ satisfies the Twist Equation (48)[C39].

computations below

We can then verify that $(f, g)$ indeed satisfy Equations (46) and (47), at least to degree 9:

```
(ħ⁻¹ (LS[x + y] - BCH[y, x] ≡ f - g - Ad[-x][f] + Ad[y][g]) ⋀
    divₓ[f] + div_y[g] ≡
     1/2 trᵤ[adSeries[ad/(e^ad-1), x][u] + adSeries[ad/(e^ad-1), y][u] - adSeries[ad/(e^ad-1), BCH[x, y]][u]]) @
  {6} // Timing
```

```
SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.
{13.8281, BS[7 True, ...]}
```

Of course, we could have simply solved Equations (46) and (47) directly:

```
{F = LS[{x, y}, Fs], G = LS[{x, y}, Gs]}; Fs["y"] = 1/2;
SeriesSolve[{F, G},
  ħ⁻¹ (LS[x + y] - BCH[y, x] ≡ F - G - Ad[-x][F] + Ad[y][G]) ⋀
   divₓ[F] + div_y[G] ≡
    1/2 trᵤ[adSeries[ad/(e^ad-1), x][u] + adSeries[ad/(e^ad-1), y][u] - adSeries[ad/(e^ad-1), BCH[x, y]][u]]];
{F, G}
```

Fuller output: [WKO4]/KVDirect.nb

$$\Big\{\mathrm{LS}\Big[\frac{\overline{y}}{2}, \frac{\overline{xy}}{6}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} + \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \ldots\Big],$$
$$\mathrm{LS}\Big[0, \frac{\overline{xy}}{12}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} + \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \ldots\Big]\Big\}$$

(To the degree shown, the results are the same. But starting at degree 8 they diverge as the solutions are non-unique.)

[C39]We define `Θl[x,y,s]` to be $e^{st}$ in the $E_l$ presentation in a straightforward manner, then convert it to the $E_s$ presentation, and then print its value in both the $E_l$ and $E_s$ presentations:

```
Θl[x_, y_, s_] := El[⟨x → LS[s LW@y], y → LS[s LW@x]⟩, CWS[0]];
Θs[x_, y_, s_] := Θl[x, y, s] // Γ;
{Θl[x, y, 1], Θs[x, y, 1]}
```

$$\Big\{\mathrm{El}[\langle \overline{x} \to \mathrm{LS}[\overline{y}, 0, 0, 0, \ldots], \overline{y} \to \mathrm{LS}[\overline{x}, 0, 0, 0, \ldots]\rangle, \mathrm{CWS}[0, 0, 0, 0, \ldots]],$$
$$\mathrm{Es}\Big[\Big\langle \overline{x} \to \mathrm{LS}\Big[\overline{y}, \frac{\overline{xy}}{2}, \frac{1}{6}\overline{x\overline{xy}} - \frac{1}{12}\overline{\overline{xy}y}, \frac{1}{24}\overline{x\overline{x\overline{xy}}} - \frac{1}{24}\overline{x\overline{\overline{xy}y}}, \ldots\Big],$$
$$\overline{y} \to \mathrm{LS}\Big[\overline{x}, -\frac{\overline{xy}}{2}, -\frac{1}{12}\overline{x\overline{xy}} + \frac{1}{6}\overline{\overline{xy}y}, \frac{1}{24}\overline{x\overline{\overline{xy}y}} - \frac{1}{24}\overline{\overline{\overline{xy}y}y}, \ldots\Big]\Big\rangle, \mathrm{CWS}[0, 0, 0, 0, \ldots]\Big]\Big\}$$

Following that, we reproduce the results of Albert, Harinck, and Torossian [AHT], who studied the linearizations

$$[x, A] + [y, B] = 0 \qquad \text{and} \qquad \operatorname{div}_x A + \operatorname{div}_y B = 0 \qquad \text{with } A, B \in FL(x, y) \tag{49}$$

of Equations (46) and (47) (which are equivalent to (44) and (45)), and the linearization of Equation (48),

$$A(x, y) = B(y, x). \tag{50}$$

We find[C40] that up to degree 16, the dimensions of the spaces of solutions of (49) and of (49)∧(50) are the same and are given by the following table:

| $\deg A, B$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| dimension | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 5 |

(51)

Assuming that every solution of the KV equations to degree $k$ can be extended to a solution at all degrees (and similarly for KV∧Twist)[18], the above table shows the number of degrees of freedom for the solutions of KV (and/or KV∧Twist), in each degree.

[18] I am not aware that this was ever proven for KV (and/or KV∧Twist), yet a similar result holds for Drinfel'd associators; see [Dr1, Dr2, BN2, BN3].

*computations below*

This done, the computation of $\tau(V_0)$ and the verification that it is equal to $V_0$ to degree 6 s routine:

```
τV = Rs[x, y] ** (V₀ // dσ[{x, y} → {y, x}]) ** Θs[x, y, -1 / 2];
(V₀ ≡ τV) @{6}
```

```
BS[7 True, ...]
```

[C40] We solve for series $A$ and $B$ satisfying (49). These equations are linear, so the printed solution is 0. Yet we store messages produced by `LinearSolve` in a stream called `msgs`. As `LinearSolve` progresses, it outputs messages detailing which coefficients were set in an arbitrary manner in each degree, and the dimension of the space of solutions in each degree can be read from that information:

```
{A = LS[{x, y}, As], B = LS[{x, y}, Bs]};
msgs = SeriesSolve[{A, B},
   ℏ⁻¹ (b[x, A] + b[y, B] ≡ LS[0]) ⋀ (divₓ[A] + div_y[B] ≡ CWS[0])];
{A, B}
```

```
SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {As[y] → 0}.
{LS[0, 0, 0, 0, ...], LS[0, 0, 0, 0, ...]}
```

Next, we read the stream `msgs`, just to explore its format:

```
Read[msgs]
```

```
{{ArbitrarilySetting, 1, {Hold[As[y]] → 0}}, {ArbitrarilySetting, 2, {}},
 {ArbitrarilySetting, 3, {}}, {ArbitrarilySetting, 4, {}}}
```

Next we compute $A$ to degree 12, and read only the dimensions information contained in `msgs`:

```
A@12; Length[Last[#]] & /@ Read[msgs]
```

3.4. **Drinfel'd Associators.** It pains me to say so little about Drinfel'd associators, but this is a computational paper and everything we need about associators was already said elsewhere; e.g., in Drinfel'd's original papers [Dr1, Dr2], in my [BN2, BN3], and in earlier papers in this series [WKO2, WKO3]. Hence here I will only recall the few things that are necessary in order to understand the computations below.

Recall that the Drinfel'd-Kohno algebra $\mathfrak{t}_n$ is the completed graded Lie algebra with degree 1 generators $\{t_{ij} = t_{ji} \colon 1 \leqslant i \neq j \leqslant n\}$ and relations $[t_{ij}, t_{kl}] = 0$ when $i, j, k, l$ are distinct ("locality relations") and $[t_{ij} + t_{ik}, t_{jk}] = 0$ when $i, j, k$ are distinct ("4T relations")[C41]. For any fixed $2 \leqslant k \leqslant n$ the $k-1$ elements $\{t_{ik} \colon 1 \leqslant i < k\}$ form a free subalgebra $FL_{k-1}$ of $\mathfrak{t}_n$, and $\mathfrak{t}_n$ is an iterated semi-direct product of these subalgebras:

$$\mathfrak{t}_n \cong ((\ldots(FL_1 \ltimes FL_2) \ltimes \ldots) \ltimes FL_{n-2}) \ltimes FL_{n-1}. \tag{52}$$

Hence as a vector space, $\mathfrak{t}_n$ has a basis with elements ordered pairs $(k, w)$, where $2 \leqslant k \leqslant n$ and $w$ is a Lyndon word in the letters $\{1, \ldots, k-1\}$ (which really stand for $\{t_{1k}, \ldots, t_{k-1,k}\}$)[C42].

computations below

Fuller output: [WKO4]/`dims.nb`

```
SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {As[x, x, x, x, y, x, y, y] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 10 arbitrarily setting {As[x, x, x, x, x, x, y, x, y, y] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 11 arbitrarily setting {As[x, x, x, x, x, x, y, x, y, y, y] → 0}.
General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫
{1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 2}
```

Finally we do the same, but now adding Equation (50):

```
{A1 = LS[{x, y}, A1s], B1 = LS[{x, y}, B1s]};
msgs1 = SeriesSolve[{A1, B1},
   ℏ^-1 (b[x, A1] + b[y, B1] ≡ LS[0]) ⋀ (divx[A1] + divy[B1] ≡ CWS[0]) ⋀
    (A1 ≡ (B1 // LieMorphism[x → y, y → x]))];
A1@12; Length[Last[#]] & /@ Read[msgs1]
```

Fuller output: [WKO4]/`dims1.nb`

```
SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {A1s[y] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {A1s[x, x, x, x, y, x, y, y] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 10 arbitrarily setting {A1s[x, x, x, x, x, x, y, x, y, y] → 0}.
General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫
{1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 2}
```

[C41]We verify these relations, using obvious notation:

```
{b[t[1, 3], t[4, 2]], b[t[1, 2] + t[1, 3], t[2, 3]]}
```

```
{0, 0}
```

[C42]Hence for example, $[t_{13}, t_{12}] = -[t_{13}, t_{23}]$ (the bracket of a generator of $FL_3$ with the generator of $FL_2$ is an element of $FL_3$). In computer speak, this is

```
b[t[1, 3], t[1, 2]]
```

```
DK[3, -12]
```

Note that the head `DK` represents "a basis element in a Drinfel'd-Kohno algebra", and that the Lyndon word 12 becomes $[t_{13}, t_{23}]$ when interpreted in $FL_3 \subset \mathfrak{t}_3$.
We could make the last output a bit friendlier by turning it into a "Drinfel'd-Kohno Series" (`DKS`):

The collection $\{\mathfrak{t}_n\}$ of all Drinfel'd-Kohno algebras forms an "operad" (e.g. [Fr]). We only need to mention a part of that structure here: that for any $n$ and $m$, there are many maps $\mathfrak{t}_n \to \mathfrak{t}_m$. Namely, whenever $\{s_i\}_{i=1}^n$ is a collection of disjoint subsets of $\{1,\dots,m\}$ (some of which may be empty), we have a morphism of Lie algebras $\Psi \mapsto \Psi^{s_1,\dots,s_n}$ mapping $\mathfrak{t}_n$ to $\mathfrak{t}_m$, and defined by its values on the generators of $\mathfrak{t}_n$ as follows:

$$(t_{ij})^{s_1,\dots,s_n} := \sum_{\alpha\in s_i,\,\beta\in s_j} t_{\alpha\beta}.$$ [C43]

Note also that by regarding elements of $\mathfrak{t}_n$ as formal exponentials and using the BCH product each $\mathfrak{t}_n$ also acquires a (non-commutative) group structure.[C44] By convention, when we think of $\mathfrak{t}_n$ as a group, we refer to it as "$\exp\mathfrak{t}_n$".

We are finally in position to recall the definition of a Drinfel'd associator. With $R = e^{t_{12}/2} \in \exp\mathfrak{t}_2$, a Drinfel'd associator is an element $\Phi \in \exp\mathfrak{t}_3$ which satisfies the "unitarity condition" (53), the pentagon equation (54), and the hexagon equations (55):

$$\text{Unitarity:} \qquad \Phi^{321} = \Phi^{-1}, \tag{53}$$

$$\pentagon: \qquad \Phi\cdot\Phi^{1,23,4}\cdot\Phi^{2,3,4} = \Phi^{12,3,4}\cdot\Phi^{1,2,34}, \tag{54}$$

$$\hexagon_\pm: \qquad (R^{\pm1})^{12,3} = \Phi\cdot(R^{\pm1})^{2,3}\cdot(\Phi^{-1})^{1,3,2}\cdot(R^{\pm1})^{1,3}\cdot\Phi^{3,1,2}. \tag{55}$$

computations below

```
b[t[1, 3], t[1, 2]] // DKS
```

$$\mathrm{DKS}\left[0,\ -\overline{t_{13}\,t_{23}},\ 0,\ 0,\ \dots\right]$$

[C43] As an example we repeat a single evaluation of a map $\mathfrak{t}_4 \to \mathfrak{t}_9$ twice. First using a complete and somewhat cumbersome notation, and then using a shortened notation that works only if all indices are single-digit:

$$\{t[2,3]^{\sigma[\{2,4\},\{1,5\},\{3,7,8\},\{9\}]}\ //\ \mathrm{DKS},\ t[2,3]^{\sigma[24,15,378,9]}\ //\ \mathrm{DKS}\}$$

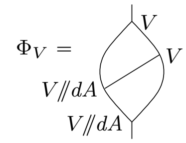
$$\left\{\mathrm{DKS}\left[\overline{t_{13}}+\overline{t_{17}}+\overline{t_{18}}+\overline{t_{35}}+\overline{t_{57}}+\overline{t_{58}},\ 0,\ 0,\ 0,\ \dots\right],\ \mathrm{DKS}\left[\overline{t_{13}}+\overline{t_{17}}+\overline{t_{18}}+\overline{t_{35}}+\overline{t_{57}}+\overline{t_{58}},\ 0,\ 0,\ 0,\ \dots\right]\right\}$$

[C44] For example, in $\mathfrak{t}_3$ the elements $t_{12}$ and $t_{23}$ do not commute, and hence the product $e^{t_{12}/2}e^{t_{23}/2}$ is messy. Yet by a 4T relation the elements $t_{12}$ and $(t_{12})^{12,3} = t_{13} + t_{23}$ do commute, and hence the product $e^{t_{12}/2}\left(e^{t_{12}/2}\right)^{12,3}$ is much simpler:

$$R = \mathrm{DKS}[t[1,2]/2];\quad \{R ** R^{\sigma[2,3]},\ R ** R^{\sigma[12,3]}\}$$

$$\left\{\mathrm{DKS}\left[\frac{\overline{t_{12}}}{2}+\frac{\overline{t_{23}}}{2},\ -\frac{1}{8}\overline{t_{13}\,t_{23}},\ -\frac{1}{48}\overline{\overline{t_{13}\,t_{23}}\,t_{23}}+\frac{1}{96}\overline{t_{13}\,\overline{t_{13}\,t_{23}}},\ -\frac{1}{384}\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}}+\frac{1}{384}\overline{t_{13}\,\overline{\overline{t_{13}\,t_{23}}\,t_{23}}},\ \dots\right],\ \mathrm{DKS}\left[\frac{\overline{t_{12}}}{2}+\frac{\overline{t_{13}}}{2}+\frac{\overline{t_{23}}}{2},\ 0,\ 0,\ 0,\ \dots\right]\right\}$$

A surprising result by Furusho [Fu] (see also [BND1]) states that in the context of $\exp \mathfrak{t}_n$ the hexagon equations follow from unitarity and the pentagon, provided $\Phi$ is initialized to degree 2 by $\Phi = \exp\left([t_{13}, t_{23}]/24 + \text{higher terms}\right)$.[C45]




3.5. **Associators in $\mathcal{A}^w$.** We know from [AT, Section 1] that a certain combination of four copies of $V$ makes a solution of the pentagon equation, with values in $\mathrm{tder}_3$. In the language of [WKO2], this is the statement that $V$ is the $Z^w$-value of a vertex, that four vertices can make a tetrahedron, and that the $Z^w$-value $\Phi_V$ of a tetrahedron is an associator in $\mathcal{A}^w$ (see the figure on the right). Specifically,

$$\Phi_V = (V/\!\!/dA)^{12,3}(V/\!\!/dA)^{1,2}V^{2,3}V^{1,23},$$ [C46]

where we use standard notation: $V^{2,3}$, for example, means "$V$ with its $x$ strand renamed 2 and its $y$ strand renamed 3" and $V^{1,23}$ means "V with its $x$ strand renamed 1 and its $y$ strand doubled to become strands 2 and 3". With the language of Definition 2.2, this is $V^{2,3} = V/\!\!/d\sigma_2^x/\!\!/d\sigma_3^y$ and $V^{1,23} = V/\!\!/d\sigma_1^x/\!\!/d\Delta_{23}^y$.

*computations below*

[C45]Here's an associator $\Phi_0$, computed to degree 6. The data file [WKO4]/`Phi.nb` contains a computation of an associator to degree 10, higher than was previously computed [BN2, Br].

```
Φs[2, 1] = Φs[3, 1] = Φs[3, 2] = 0; Φs[3, 1, 2] = 1 / 24; Φ0 = DKS[3, Φs];
SeriesSolve[Φ0, (Φ0^σ[3,2,1] ≡ -Φ0) ∧ (Φ0 ** Φ0^σ[1,23,4] ** Φ0^σ[2,3,4] ≡ Φ0^σ[12,3,4] ** Φ0^σ[1,2,34])];
Φ0@{6}
```

Fuller output: [WKO4]/`Phi.nb`

```
SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {Φs[3, 1, 1, 2] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {Φs[3, 1, 1, 1, 1, 2] → 0}.

DKS[0, 1/24 t13 t23, 0, -7 t13 t23 t23 t23/5760 + 7 t13 t13 t23 t23/5760 - t13 t13 t13 t23/1440,
 0, 31 t13 t23 t23 t23 t23 t23/967680 - 157 t13 t13 t23 t23 t13 t23/1935360 - 31 t13 t23 t13 t23 t23 t23/387072 -
 31 t13 t13 t23 t23 t23 t23/483840 + 11 t13 t13 t13 t23 t13 t23/290304 + 31 t13 t13 t23 t13 t23 t23/725760 +
 83 t13 t13 t13 t23 t23 t23/967680 - 13 t13 t13 t13 t13 t23 t23/241920 + t13 t13 t13 t13 t13 t23/60480, ...]
```

To be on the safe side, we verify that $\Phi_0$ satisfies the hexagon equations to degree 6:

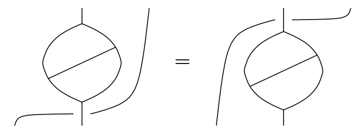
```
R = DKS[t[1, 2] / 2];
(R^σ[12,3] ≡ Φ0 ** R^σ[2,3] ** (-Φ0)^σ[1,3,2] ** R^σ[1,3] ** Φ0^σ[3,1,2] ∧
   (-R)^σ[12,3] ≡ Φ0 ** (-R)^σ[2,3] ** (-Φ0)^σ[1,3,2] ** (-R)^σ[1,3] ** Φ0^σ[3,1,2])@{6}
```

```
BS[7 True, ...]
```

[C46]And here is $\Phi_V$, to degree 4:

```
V12 = V0 // dσ[{x, y} → {1, 2}];
ΦV = (V12 // dA)^σ[12,3] ** (V12 // dA)^σ[1,2] ** V12^σ[2,3] ** V12^σ[1,23]
```

$\Phi_V$ satisfies the pentagon equation.[C47] If our $V$ also satisfies the Twist Equation, then $\Phi_V$ also satisfies the hexagon equations (though we do not test that here). Finally, Alekseev and Torossian [AT] prove that if the tree part of $\Phi_V$ is written as an exponential $\exp(l\phi)$ of an element $\phi$ of $\mathrm{tder}_3$, then in fact $\phi \in \mathrm{sder}_3$, where as in [AT], $\mathrm{sder}_n$ is the space of "special derivations in $\mathrm{tder}_n$", the derivations which annihilate the sum of all generators on $FL_n$[C48].



The topological meaning of "$\phi \in \mathrm{sder}_3$" is that one may perform a sequence of four $R4$ moves to slide a strand underneath a tetrahedron, as shown on the right.

Recall that there is a map $\alpha\colon \mathfrak{t}_n \to \mathcal{A}^w_{\mathrm{prim}}(\uparrow_n)$ (equivalently, $\alpha\colon \mathcal{U}(\mathfrak{t}_n) \to \mathcal{A}^w(\uparrow_n)$), defined by its values on the generators by sending $t_{ij}$ to a sum of a single arrow from strand $i$ to strand $j$ plus a single arrow from strand $j$ to strand $i$: $t_{ij} \mapsto {}_i[\text{arrow from } i \text{ to } j]_j + {}_i[\text{arrow from } j \text{ to } i]_j$. Using the map $\alpha$, every Drinfel'd associator becomes an associator in $\mathcal{A}^w$.[C49]

In topology, $\alpha$ is the associated graded of the "do nothing" map $a$ which maps ordinary knots to virtual knots. [chord] $\mapsto$ [arrow] + [arrow] because [chord] $\sim$ [crossing difference] $\sim$ [positive crossing] $-$ [negative crossing] $\mapsto$ ([positive crossing] $-$ [virtual crossing]) + ([virtual crossing] $-$ [negative crossing]) $\sim$ [semi-virtual crossing] + [semi-virtual crossing] $\sim$ [arrow] + [arrow]. See [WKO1, Section 2.5.5] and [WKO2, Section 3.3].

*computations below*

```
Es[⟨1 → LS[0, 2 3/24, 0, -1 1 2 3/1440 + 7 1 2 2 3/5760 + 1 2 3 3/5760 - 7 2 2 2 3/5760 +
      7 2 2 3 3/5760 + 1/480 1 2 1 3 - 1 3 2 3/1920 + 1/640 1 2 3 2 - 1 3 2 2/1152 - 1 3 3 2/1152 - 2 3 3 3/1440, ...],
   2 → LS[0, -1 3/24, 0, 1 1 1 3/1440 - 1 1 2 3/1152 + 7 1 2 2 3/1920 - 1/480 1 1 3 2 - 1 1 3 3/5760 + 1 2 3 3/1152 +
      7 1 2 1 3/5760 + 19 1 3 2 3/5760 + 7 1 2 3 2/1920 + 7 1 3 2 2/5760 + 7 1 3 3 2/5760 + 1 3 3 3/1440, ...],
   3 → LS[0, 1 2/24, 0, -1 1 1 2/1440 + 1 1 2 3/5760 + 7 1 2 2 3/5760 + 7 1 1 2 2/5760 - 1 1 3 2/1440 - 1 2 3 3/1440 + 1 2 1 3/5760 +
      1 3 2 3/1440 - 1 2 3 2/1152 - 7 1 2 2 2/5760 - 7 1 3 2 2/5760 - 1 3 3 2/1440, ...]⟩, CWS[0, 0, 0, 0, ...]]
```

[C47]Indeed,

```
ΦV ** ΦV^σ[1,23,4] ** ΦV^σ[2,3,4] ≡ ΦV^σ[12,3,4] ** ΦV^σ[1,2,34]
```

```
BS[5 True, ...]
```

[C48]We convert $\Phi_V$ to the $E_l$ presentation and take its first (tree) part and call it $\phi$, and then we verify that $[x_1, \phi_1] + [x_2, \phi_2] + [x_3, \phi_3] = 0$:

```
ϕ = (ΦV // Λ)⟦1⟧;
(b[LW@1, ϕ1] + b[LW@2, ϕ2] + b[LW@3, ϕ3])@{6}
```

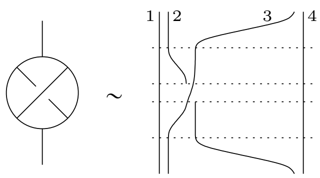
```
LS[0, 0, 0, 0, 0, 0, ...]
```

[C49]Indeed, we define a map `DK2Es` which takes Drinfel'd-Kohno series to elements of $\mathcal{A}^w$ given in the $E_s$ presentation by applying the built-in $\alpha$`Map`, adding 0 wheels, and applying the $E_l$ to $E_s$ conversion $\Gamma$. Applying this map to the Drinfel'd associator $\Phi_0$ computed before, we get and associators in $\mathcal{A}^w$:

In Lie theory, the existence of $\alpha$ corresponds to the fact that the invariant metric on $I\mathfrak{g} = \mathfrak{g} \ltimes \mathfrak{g}^*$ (represented by an undirected chord) is the sum of the two possible contractions of a space with its dual in $(\mathfrak{g} \ltimes \mathfrak{g}^*) \otimes (\mathfrak{g} \ltimes \mathfrak{g}^*)$ (the two arrows).

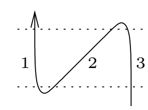
[AT] The [AT, Proposition 3.11] version of $\alpha$ is the map $\mathfrak{t}_n \to \mathfrak{sder}_n \subset \mathfrak{tder}_n$ taking $t_{ij}$ to $\partial\,(i \to x_j,\ j \to x_i,\ (k \neq i,j) \to 0)$.

3.6. **Solving the Kashiwara-Vergne Equations Using a Drinfel'd Associator.** Following [WKO3] (in a deeper sense, following [AET]), we know that an element $V$ solving the KV equations (44) and (45) can be computed from a Drinfel'd associator $\Phi$ by first computing the invariant $Z_B = Z^u(B)$ of the "buckle" $B$, shown below both as a knotted trivalent graph and as a product of associators, then puncturing strands 1 and 3 and capping strands 2 and 4 from below, and then regarding the result in $\mathcal{A}^w(\uparrow_2)$ by applying an "Etingof-Kazhdan (EK) isomorphism":C50

$B =$ 


$$\mapsto Z_B = (\Phi^{-1})^{13,2,4}\Phi^{1,3,2}R^{23}\Phi^{-1}\Phi^{12,3,4} \xrightarrow{\text{puncture, cap, EK}} V.$$

Likewise following [WKO3], we know that $\mathit{Cap} = \alpha(\nu^{1/4})$, where $\nu$ is the Kontsevich integral of the unknot, or the inverse of the associator-combination shown on the right and given by the formula $\alpha(\nu^{-1}) = \Phi /\!\!/ \alpha /\!\!/ dS^2 /\!\!/ dm_2^{32} /\!\!/ dm_1^{21}$.C51 (Note




computations below

```
DK2Es[s___][ξ_] := El[ξ // αMap[s], CWS[0]] // Γ;
DK2Es[1, 2, 3][Φ0]
```

$$\text{Es}\Big[\Big\langle 1 \to \text{LS}\Big[0,\ \frac{\overline{23}}{24},\ 0,\ -\frac{\overline{1\overline{1\overline{23}}}}{1440} + \frac{7\,\overline{1\overline{2\overline{23}}}}{5760} + \frac{\overline{1\overline{\overline{23}3}}}{5760} - \frac{7\,\overline{2\overline{2\overline{23}}}}{5760} + \frac{7\,\overline{2\overline{\overline{23}3}}}{5760} + \frac{1}{480}\overline{\overline{12}\,\overline{13}} - \frac{\overline{\overline{13}\,\overline{23}}}{1920} + \frac{1}{640}\overline{\overline{1\overline{23}}2} - \frac{\overline{\overline{\overline{13}2}2}}{1152} - \frac{\overline{\overline{\overline{13}3}2}}{1152} - \frac{\overline{\overline{\overline{23}3}3}}{1440},\ \ldots\Big],$$

$$2 \to \text{LS}\Big[0,\ -\frac{\overline{13}}{24},\ 0,\ \frac{\overline{1\overline{1\overline{13}}}}{1440} - \frac{\overline{1\overline{1\overline{23}}}}{1152} + \frac{7\,\overline{1\overline{2\overline{23}}}}{1920} - \frac{1}{480}\overline{1\overline{\overline{13}2}} - \frac{\overline{1\overline{\overline{13}3}}}{5760} + \frac{\overline{1\overline{\overline{23}3}}}{1152} + \frac{7\,\overline{\overline{12}\,\overline{13}}}{5760} + \frac{19\,\overline{\overline{13}\,\overline{23}}}{5760} + \frac{7\,\overline{\overline{1\overline{23}}2}}{1920} + \frac{7\,\overline{\overline{\overline{13}2}2}}{5760} + \frac{7\,\overline{\overline{\overline{13}3}2}}{5760} + \frac{\overline{\overline{\overline{13}3}3}}{1440},\ \ldots\Big],$$

$$3 \to \text{LS}\Big[0,\ \frac{\overline{12}}{24},\ 0,\ -\frac{\overline{1\overline{1\overline{12}}}}{1440} + \frac{\overline{1\overline{1\overline{23}}}}{5760} + \frac{7\,\overline{1\overline{2\overline{23}}}}{5760} + \frac{7\,\overline{1\overline{\overline{12}2}}}{5760} - \frac{\overline{1\overline{\overline{13}2}}}{1440} - \frac{\overline{1\overline{\overline{23}3}}}{1440} + \frac{\overline{\overline{12}\,\overline{13}}}{5760} + \frac{\overline{\overline{13}\,\overline{23}}}{1440} - \frac{\overline{\overline{1\overline{23}}2}}{1152} - \frac{7\,\overline{\overline{\overline{12}2}2}}{5760} - \frac{7\,\overline{\overline{\overline{13}2}2}}{5760} - \frac{\overline{\overline{\overline{13}3}2}}{1440},\ \ldots\Big]\Big\rangle,\ \text{CWS}[0,\ 0,\ 0,\ 0,\ \ldots]\Big]$$

The result matches $\Phi_V$, computed before, to the degree shown. But this is only because both associators are supported in even degrees, and there's a unique even associator in $\mathcal{A}^w$ up to degree 4. In degree 8 these two associators diverge.

C50We start with a straightforward computation of $Z_B$:

```
R = DKS[t[1, 2] / 2];
ZB = (-Φ0)^σ[13,2,4] ** Φ0^σ[1,3,2] ** R^σ[2,3] ** (-Φ0)^σ[1,2,3] ** Φ0^σ[12,3,4]
```

that this computation uses the operation $dS^a$, which is not easily available in the $E_l$ presentation).

An alternative (yet equivalent) formula for $V$ in terms of $\Phi$ follows [AET] more closely. Indeed according to [AET, Theorem 4] and [WKO3] $V$ generates the tangential automorphism of $FL(x, y)$ given explicitly by $(x \mapsto F_x x F_x^{-1}, y \mapsto F_y y F_y^{-1})$, where

$$F = (F_x, F_y) = \left(\Phi^{-1}(x, -x-y),\ e^{(x+y)/2}\Phi^{-1}(y, -x-y)e^{-y/2}\right) \tag{56}$$

(though note that our conventions here agree with the conventions of [WKO3] but slightly differ from the conventions of [AET]).

*computations below*

```
DKS[t23/2, -1/12 t13 t23 - 1/24 t14 t24 + 1/24 t14 t34 + 1/12 t24 t34, 0,
  t13 t23 t23 t23/5760 + 7 t14 t24 t24 t24/5760 + t14 t34 t24 t24/1920 - t14 t34 t34 t24/1920 - 7 t14 t34 t34 t34/5760 -
  t24 t34 t34 t34/5760 + t14 t24 t34 t24/1920 + t14 t24 t14 t34/1920 - t14 t34 t24 t34/1920 - 1/720 t13 t13 t23 t23 +
  1/720 t13 t13 t13 t23 - 7 t14 t14 t24 t24/5760 + 7 t14 t14 t34 t34/5760 - t14 t24 t34 t34/5760 + t14 t14 t14 t24/1440 -
  t14 t14 t14 t34/1440 - 1/960 t14 t14 t24 t34 + t14 t24 t24 t34/5760 - 1/960 t24 t24 t34 t34 - t24 t24 t24 t34/5760, ...]
```

In the $E_s$ presentation, "puncture" is $t\eta$. So we puncture strands 1 and 3:

```
ZB // DK2Es[1, 2, 3, 4] // tη¹ // tη³
```

```
Es[⟨1 → LS[0, -24/24, 0, 7 2 2 2 4/5760 - 7 2 2 4 4/5760 + 2 4 4 4/1440, ...],
  2 → LS[0, 0, 0, 0, ...], 3 → LS[2/2, -2 4/12, 0, 2 2 2 4/5760 - 1/720 2 2 4 4 + 1/720 2 4 4 4, ...],
  4 → LS[0, 0, 0, 0, ...]⟩, CWS[0, 0, 0, 0, ...]]
```

At this point we would normally need to cap and apply EK. But fortunately, strands 2 and 4 carry no arrow heads (as can be seen in the above output), so there is no need to cap them and the EK isomorphisms act by doing nothing. Hence apart from some obvious renaming, the above is already a solution of the KV equations. It matches with the previously-computed $V$ to degree 4 but diverges from it in degree 8 (not shown here). This is consistent with the result in (51), which shows that non-uniqueness starts only in degree 8.
C51 Indeed here is $\nu^{-1}$, followed by a verification that $\nu^{-1}\,Cap^4$ is trivial:

```
vinv = Φ0 // DK2Es[1, 2, 3] // dS[2] // dm[3, 2, 2] // dm[2, 1, x]
```

```
Es[⟨x → LS[0, 0, 0, 0, ...]⟩, CWS[0, xx/24, 0, -xxxx/2880, ...]]
```

```
(vinv ** Cap ** Cap ** Cap ** Cap) @ {6}
```

```
Es[⟨x → LS[0, 0, 0, 0, 0, 0, ...]⟩, CWS[0, 0, 0, 0, 0, 0, ...]]
```

Below the line we verify Equation (56).[C52]

**3.7. A Potential $S_4$ Action on Solutions of KV.** In [BND2], Z. Dancso and I discussed how "the expansion of a tetrahedron" can be interpreted as an associator valued in the appropriate space $\mathcal{A}^u(\triangle) \cong \mathcal{A}^u(\uparrow_3)$ (see also [Th]). The symmetry group of an oriented tetrahedron is the alternating group $A_4$, and hence $A_4$ acts on the set of all associators in $\mathcal{A}^u(\uparrow_3)$ (note that while the action of the permutation group $S_3$ on $\mathcal{A}^u(\uparrow_3)$ is obvious, its extension to an action of $S_4$ is non-obvious and is best understood using the isomorphism $\mathcal{A}^u(\triangle) \cong \mathcal{A}^u(\uparrow_3)$). The unitarity equation (53) means that odd permutations map associators to objects whose inverses are associators; with some abuse of language we simply say that "$S_4$ acts on the set of associators" (really, it acts on "associators and inverse-associators"). As there are bi-directional relations between associators and solutions of the KV equations, we can expect an action of $S_4$ on the set of solutions of the KV equations and their inverses.

As mathematicians, Z. Dancso and I only lightly explored this potential action of $S_4$; we wrote down what we think are the formulas inherited from the action on associators, but on the formal level, we've verified almost nothing. Yet computer experiments, described below, suggest that our formulas are correct and that they have the properties described below.

**The first $\mathbb{Z}/2$ action** is the involution $\tau$ discussed in Section 3.3. We have nothing further to add.

**The second $\mathbb{Z}/2$ action** is the involution $\rho_2$ of $\mathcal{A}^w$ which multiplies every degree $d$ element by $(-1)^d$. Solutions $V$ of the KV equations are not invariant under $\rho_2$. Yet if $V_0$ is the solution computed in this paper then $V_1 := R^{-1/2}V_0$ is invariant under $\rho_2$, at least experimentally. Alternatively, $V_0$ is (experimentally) invariant under $\rho_2' := R\rho_2$.[C53]

*computations below*

[C52] We first have to rewrite $\Phi$ in terms of $x = t_{12}$ and $y = t_{23}$. To do this we "3" term in $\Phi$, the one involving $t_{13}$ and $t_{23}$ in the factorization (52) (it is the only non trivial term), and apply the appropriate change of variables $t_{13} \to -x - y$, $t_{23} \to y$. It is smooth sailing afterwards:

```
Φ1 = (Φ0)3 // LieMorphism[LW@1 → -x - y, LW@2 → y]
```

$\mathrm{LS}\left[0, -\frac{\overline{xy}}{24}, 0, \frac{\overline{x\overline{x\overline{xy}}}}{1440} - \frac{\overline{x\overline{\overline{xy}y}}}{5760} + \frac{\overline{\overline{\overline{xy}y}y}}{1440}, \ldots\right]$

```
F = ⟨x → LieMorphism[y → -x - y][-Φ1],
  y → LS[(x + y) / 2] ~BCH~ LieMorphism[x → y, y → -x - y][-Φ1] ~BCH~ LS[-y / 2]⟩
```

$\Big\langle \overline{x} \to \mathrm{LS}\left[0, -\frac{\overline{xy}}{24}, 0, \frac{7\,\overline{x\overline{x\overline{xy}}}}{5760} - \frac{7\,\overline{x\overline{\overline{xy}y}}}{5760} + \frac{\overline{\overline{\overline{xy}y}y}}{1440}, \ldots\right],$

$\overline{y} \to \mathrm{LS}\left[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, 0, \frac{\overline{x\overline{x\overline{xy}}}}{5760} - \frac{1}{720}\,\overline{x\overline{\overline{xy}y}} + \frac{1}{720}\,\overline{\overline{\overline{xy}y}y}, \ldots\right]\Big\rangle$

```
(F ≡ V0⟦1⟧)@{7}
```

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {Φs[3, 1, 1, 1, 1, 1, 1, 2] → 0}.

```
BS[8 True, ...]
```

[C53] Indeed,

**A $\mathbb{Z}/3$ action.** For $\xi \in \mathcal{A}^w(x,y)$ let $\rho_3(\xi) := \xi /\!\!/ dS^y /\!\!/ d\Delta^y_{yz} /\!\!/ dm^{xz}_x /\!\!/ d\sigma^{xy}_{yx}$, where $d\sigma^{xy}_{yx}$ simply means "swap the labels $x$ and $y$". Then $\rho_3$ is a trivolution ($(\rho_3)^3 = 1$)[C54], and a renormalized version of $V_0$, namely $V_2 := V_0 * \Theta^{-1/4} * \exp\left(\frac{\hat{x}-\hat{y}}{12}\right) * d\Delta^x_{xy}(Cap^2)$ is, at least experimentally, invariant under the action of $\rho_3$.[C55]

*computations below*

```
ρ2[V_] := V // (-1)^deg;
V1 = Es[⟨x → LS[0], y → LS[-x / 2]⟩, CWS[0]] ** V0;
{(V1 ≡ ρ2[V1])@{8}, (V0 ≡ Rs[x, y] ** ρ2[V0])@{8}}
```

```
SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {αs[x, x, x, x, y, x, y, y] → 0}.
{BS[9 True, ...], BS[9 True, ...]}
```

[C54] Indeed for a random $\xi_c$, $\xi_c /\!\!/ \rho_3 /\!\!/ \rho_3 /\!\!/ \rho_3 = \xi_c$:

```
ρ3[ξ_Es] := ξ // dS[y] // dΔ[y, y, z] // dm[x, z, x] // dσ[{x, y} → {y, x}];
ξc = RandomEsSeries[1, {x, y}];
ξc ≡ (ξc // ρ3 // ρ3 // ρ3)
```

```
BS[5 True, ...]
```

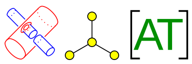
[C55] Indeed,

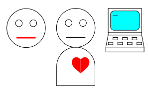
```
V2 = V0 ** Θs[x, y, -1 / 4] **
    Es[⟨x → LS@0, y → LS@0⟩, CWS[cw[x] / 12 - cw[y] / 12] - (2 Cap⟦2⟧ // tΔ[x, x, y])];
(V2 ≡ ρ3[V2])@{6}
```

```
BS[7 True, ...]
```

# 4. Glossary of notation

Icons, then Greek letters, then Latin, and then symbols:


Links with topology, finite-dimensional Lie theory, and the Alekseev-Torossian paper [AT].


Human input, multi-line human input, and computer output.


Source code quotes from the Mathematica packages `FreeLie.m` and `AwCalculus.m` [WKO4].

| | | |
|---|---|---|
| $\alpha$ | a map $\mathfrak{t}_n \to \mathcal{A}^w_{\text{prim}}$ / $\mathcal{A}^u \to \mathcal{A}^w$ | 41 |
| $\Gamma$ | the conversion $TW_l \to TW_s$ | 29 |
| $\Gamma(\lambda)$ | $\Gamma_1(\lambda)$ | 14 |
| $\Gamma_t(\lambda)$ | solution of $e^{-t\partial_\lambda} = C^{\Gamma_t(\lambda)}$ | 13 |
| $\Delta$ | a co-product | 8 |
| $\delta$ | double all strands $\mathcal{A}^w(S) \to \mathcal{A}^w(S;S)$ | 27 |
| $\eta$ | a co-unit | 7 |
| $\Theta$ | $\exp(\mapsto + \mapsfrom)$ | 36 |
| $\iota$ | the embedding $CW \to \mathcal{A}^w$ | 17 |
| $\lambda$ | generic element of $FL(S)^S$ | 9 |
| $\Lambda$ | the conversion $TW_s \to TW_l$ | 29 |
| $\Lambda(\lambda)$ | $\Lambda_1(\lambda)$ | 14 |
| $\Lambda_t(\lambda)$ | solution of $C^{t\lambda} = e^{-\partial_{\Lambda_t(\lambda)}}$ | 14 |
| $\nu$ | Kontsevich integral of the unknot | 42 |
| $\pi_A$ | projection on "framing part" | 29 |
| $\pi_T$ | projection on trees | 20 |
| $\pi_\downarrow$ | a projection on wheels | 29 |
| $\rho_2$ | an involution on $\mathcal{A}^w$ | 44 |
| $\rho_3$ | a trivolution on $\mathcal{A}^w(x,y)$ | 45 |
| $\tau$ | an involution on SolKV | 36 |
| $\Phi$ | a Drinfel'd associator | 39 |
| $\Phi_V$ | an associator in $\mathcal{A}^w$ | 40 |
| $\omega$ | generic element of $CW(S)$ | 9 |
| $a, \bar{a}, a_i, b, \dots$ | generic strand labels | 6 |
| $a$ | the inclusion usual$\hookrightarrow$virtual | 41 |
| $A$ | Abelian lie algebra | 11 |
| $\mathfrak{a}$ | [AT] notation for $A$ | 11 |
| $\operatorname{ad}^\gamma_u$ | a derivation on $FL(T)$ | 26 |
| $\overrightarrow{AS}$ | the directed AS relation | 6 |
| $\mathcal{A}^w$ | arrow-diagram spaces | 6 |
| $\mathcal{A}^w_{\exp}$ | exponentials in $\mathcal{A}^w$ | 8 |
| $\mathcal{A}^w(H;T)$ | arrow-diagram space on heads-tails skeleton | 21 |
| $\mathcal{A}^w_{\exp}(H;T)$ | exponentials in $\mathcal{A}^w(H;T)$ | 23 |
| $B$ | the "buckle" KTG | 42 |
| BCH | the Baker-Campbell-Hausdorff series | 11 |
| $\text{BCH}_{tb}$ | BCH relative to $tb$ | 16 |
| $C^\lambda$ | conjugating generators by exponentials | 13 |
| $\mathit{Cap}$ | $Z^w$ of a knot-theoretic cap | 32 |
| $CP$ | the $CP$ relation | 21 |
| $C^\gamma_u$ | $C^{(u\to\gamma)}$ | 15 |
| $CW$ | cyclic words | 9 |
| $D$ | a diagram in $\mathcal{A}^w$ | 7 |
| $d\Delta$ | strand doubling in $\mathcal{A}^w(S)$ | 8 |
| $d\Delta$ | "strand doubling" in $TW_l$ | 19 |
| $d\eta$ | strand deletion in $\mathcal{A}^w(S)$ | 7 |
| $d\eta$ | "strand deletion" in $TW_l$ | 18 |
| $d\sigma$ | strand renaming in $\mathcal{A}^w(S)$ | 8 |
| $d\sigma$ | "strand renaming" in $TW_l$ | 20 |
| $dA$ | strand adjoint in $\mathcal{A}^w(S)$ | 7 |
| $dA$, $dA^S$ | "strand adjoint" in $TW_l$ | 18 |
| der | derivations of $FL$ | 11 |
| $\mathfrak{der}$ | [AT] notation for der | 11 |
| div | $\sum_u \operatorname{div}_u$ | 16 |
| $\operatorname{div}_u$ | a "self-action" map $FL(S) \to CW(S)$ | 15 |
| $dm$ | strand stitching in $\mathcal{A}^w(S)$ | 8 |
| $dS$ | strand antipode in $\mathcal{A}^w(S)$ | 8 |
| $dS$, $dS^S$ | "strand antipode" in $TW_l$ | 19 |
| $\mathcal{E}$ | the Euler operator | 35 |
| $E_f$ | the factored presentation | 29 |
| $E_l$ | the lower-interlaced presentation | 17 |
| $E_s$ | the split presentation | 23 |
| $E_u$ | the upper-interlaced presentation | 17 |
| $e_s$ | a map $FL(T)^H \to \mathcal{A}^w_{\exp}(H;T)$ | 24 |
| $\exp \mathfrak{t}_n$ | the exponential group of $\mathfrak{t}_n$ | 39 |
| $F$ | solution of the generalized KV equations | 34 |
| $f, g$ | solution of the original KV equations | 35 |
| $FL$ | free Lie algebra | 9 |
| $\mathfrak{g}$ | a finite-dimensional Lie algebra | 6 |
| $H$ | a set of head labels | 21 |
| $h_i$ | head labels | 21 |
| $h^{\deg}$ | degree-scaling | 11 |
| $h\Delta$ | head-strand doubling in $\mathcal{A}^w(H;T)$ | 23 |
| $h\Delta$ | "head-strand doubling" in $TW_s$ | 25 |
| $h\eta$ | deleting a head-strand in $\mathcal{A}^w(H;T)$ | 22 |
| $h\eta$ | "deleting a head-strand" in $TW_s$ | 25 |
| $h\sigma$ | head-strand renaming in $\mathcal{A}^w(H;T)$ | 23 |

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