Pensieve header: Calculations appearing in the WKO4 paper.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
```

Section 1 - Introduction

Initialization

```
<< FreeLie.m;
<< AwCalculus.m;
$SeriesShowDegree = 4;
```

Initialization

```
FreeLie` implements / extends
 {⋆, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, CW,
  CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp,
  Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
  Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
  t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ℏ, ⌐, ⌢}.
```

Initialization

```
FreeLie` is in the public domain. Dror Bar-Natan
  is committed to support it within reason until July 15, 2022.
```

Initialization

```
AwCalculus` implements / extends {⋆, **, ≡, dA, dc, deg, dm, dS,
  dΔ, dη, dσ, El, Es, hA, hm, hS, hΔ, hη, hσ, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.
```

Initialization

```
AwCalculus` is in the public domain. Dror Bar-Natan
  is committed to support it within reason until July 15, 2022.
```

Section 2.2 - Some Preliminaries on Lie Algebras and Cyclic Words

alphabetagamma

```
x1 = LW[1]; x2 = LW[2];
{α, β, γ} = LS /@ {x1 + b[x1, x2], x2 - b[x1, b[x1, x2]], x1 + x2 - 2 b[x1, x2]}
```

alphabetagamma

$\{LS[\overline{1}, \overline{1\,2}, 0, 0, \ldots], LS[\overline{2}, 0, -\overline{1\,\overline{1\,2}}, 0, \ldots], LS[\overline{1}+\overline{2}, -2\,\overline{1\,2}, 0, 0, \ldots]\}$

BracketExample

```
{b[α, β], b[α, b[β, γ]] + b[β, b[γ, α]] + b[γ, b[α, β]]}
```

BracketExample

$\{LS[0, \overline{1\,2}, \overline{\overline{1\,2}\,2}, -\overline{1\,\overline{1\,\overline{1\,2}}}, \ldots], LS[0, 0, 0, 0, \ldots]\}$

bch

```
bch = BCH[LW@x, LW@y]
```

bch

LS[$\overline{x}+\overline{y}$, $\frac{\overline{x\,y}}{2}$, $\frac{1}{12}\overline{x\,\overline{x\,y}}+\frac{1}{12}\overline{\overline{x\,y}\,y}$, $\frac{1}{24}\overline{x\,\overline{\overline{x\,y}\,y}}$, ...]

bch16

```
Timing@{Length@(bch@16), (bch@16)[[1090 ;; 1092]] // TopBracketForm}
```

bch16

{38.282645, {2181, $\frac{53\,x\,x\,x\,x\,y\,x\,y\,x\,x\,y\,x\,x\,y\,x\,y\,y}{1\,089\,728\,640}$ − $\frac{17\,x\,x\,x\,x\,y\,x\,y\,x\,x\,y\,x\,x\,x\,y\,y\,y}{179\,625\,600}$ + $\frac{389\,x\,x\,x\,x\,y\,x\,y\,x\,x\,y\,x\,x\,y\,x\,x\,y}{1\,320\,883\,200}$}}

omegas

```
{ω1, ω2} = CWS /@ {CW[1] - 3 CW[2, 1, 1], CW[2] + CW[2, 2]}
```

omegas

{CWS[$\widehat{1}$, 0, $-3\,\widehat{112}$, 0, ...], CWS[$\widehat{2}$, $\widehat{22}$, 0, 0, ...]}

DegreeScale

```
DegreeScale[h] /@ {ω1, ω2}
```

DegreeScale

{CWS[$h\,\widehat{1}$, 0, $-3\,h^3\,\widehat{112}$, 0, ...], CWS[$h\,\widehat{2}$, $h^2\,\widehat{22}$, 0, 0, ...]}

TangentialDerivative

```
{λ = ⟨1 → α, 2 → β⟩, γ // Dλ}
```

TangentialDerivative

{⟨1 → LS[$\overline{1}$, $\overline{12}$, 0, 0, ...], 2 → LS[$\overline{2}$, 0, $-\overline{1\,\overline{12}}$, 0, ...]⟩, LS[0, 0, $\overline{1\,\overline{12}}$, $-\overline{1\,\overline{\overline{12}\,2}}$, ...]}

tb

```
λ1 = λ; λ2 = ⟨1 → β, 2 → γ⟩; tb[λ1, λ2]
```

tb

⟨1 → LS[0, 0, $\overline{1\,\overline{12}}$, $-\overline{1\,\overline{\overline{12}\,2}}$, ...], 2 → LS[0, 0, $\overline{1\,\overline{12}}$, $-\overline{1\,\overline{\overline{12}\,2}}$, ...]⟩

tb2

```
lhs = D_tb[λ1,λ2][ω1]; rhs = b[D_λ1, D_λ2][ω1];
{lhs@{8}, (lhs ≡ rhs)@{8}}
```

tb2

{CWS[0, 0, 0, 0, 0, 0, 0, $18\,\widehat{11112122} - 18\,\widehat{11112212} - 36\,\widehat{11121122} + 36\,\widehat{11122112}$, ...],
BS[9 True, ...]}

TestingGammaODE

$$\text{lhs} = \partial_t \Gamma_t[\lambda];\ \text{rhs} = \lambda\ //\ e^{-t D_\lambda}\ //\ \text{adSeries}\left[\frac{\text{ad}}{e^{\text{ad}}-1},\ \Gamma_t[\lambda]\right];$$

$$\{\Gamma_0[\lambda],\ \text{lhs},\ (\text{lhs} \equiv \text{rhs})\,@\,\{6\}\}$$

TestingGammaODE

$$\Big\{\langle 1 \to \text{LS}[0, 0, 0, 0, \ldots],\ 2 \to \text{LS}[0, 0, 0, 0, \ldots]\rangle,$$
$$\Big\langle 1 \to \text{LS}\Big[\overline{1},\ \overline{12},\ -t\ \overline{1\overline{12}},\ \frac{1}{4} t^2\ \overline{1\overline{1\overline{12}}} - t\ \overline{1\overline{\overline{12}2}},\ \ldots\Big],$$
$$2 \to \text{LS}\Big[\overline{2},\ 0,\ -\overline{1\overline{12}},\ -t\ \overline{1\overline{\overline{12}2}},\ \ldots\Big]\Big\rangle,\ \text{BS}[7\ \text{True}, \ldots]\Big\}$$

TestingGamma

$$\{\gamma\ //\ e^{-t D_\lambda},\ \gamma\ //\ \text{CC}[\Gamma_t[\lambda]]\}$$

TestingGamma

$$\Big\{\text{LS}\Big[\overline{1}+\overline{2},\ -2\,\overline{12},\ -t\ \overline{1\overline{12}},\ t\ \overline{1\overline{\overline{12}2}},\ \ldots\Big],\ \text{LS}\Big[\overline{1}+\overline{2},\ -2\,\overline{12},\ -t\ \overline{1\overline{12}},\ t\ \overline{1\overline{\overline{12}2}},\ \ldots\Big]\Big\}$$

TestingLambdaODE

$$\text{lhs} = \partial_t \Lambda_t[\lambda];\ \text{rhs} = \lambda\ //\ e^{D_{\Lambda_t[\lambda]}}\ //\ \text{adSeries}\left[\frac{\text{ad}}{e^{\text{ad}}-1},\ \Lambda_t[\lambda],\ \text{tb}\right];$$

$$\{\Lambda_0[\lambda],\ \text{lhs},\ (\text{lhs} \equiv \text{rhs})\,@\,\{6\}\}$$

TestingLambdaODE

$$\Big\{\langle 1 \to \text{LS}[0, 0, 0, 0, \ldots],\ 2 \to \text{LS}[0, 0, 0, 0, \ldots]\rangle,$$
$$\Big\langle 1 \to \text{LS}\Big[\overline{1},\ \overline{12},\ t\ \overline{1\overline{12}},\ \frac{1}{2} t^2\ \overline{1\overline{1\overline{12}}} + t\ \overline{1\overline{\overline{12}2}},\ \ldots\Big],\ 2 \to \text{LS}\Big[\overline{2},\ 0,\ -\overline{1\overline{12}},\ t\ \overline{1\overline{\overline{12}2}},\ \ldots\Big]\Big\rangle,$$
$$\text{BS}[7\ \text{True}, \ldots]\Big\}$$

TestingLambda

$$\{\gamma\ //\ \text{CC}[t\,\lambda],\ \gamma\ //\ e^{-D_{\Lambda_t[\lambda]}}\}$$

TestingLambda

$$\Big\{\text{LS}\Big[\overline{1}+\overline{2},\ -2\,\overline{12},\ -t\ \overline{1\overline{12}},\ -\frac{1}{2} t^2\ \overline{1\overline{1\overline{12}}} + t\ \overline{1\overline{\overline{12}2}},\ \ldots\Big],$$
$$\text{LS}\Big[\overline{1}+\overline{2},\ -2\,\overline{12},\ -t\ \overline{1\overline{12}},\ -\frac{1}{2} t^2\ \overline{1\overline{1\overline{12}}} + t\ \overline{1\overline{\overline{12}2}},\ \ldots\Big]\Big\}$$

Unclassified aside: an alternative formulation of Λ (on March 1, 2015, this took 61 Seconds):

$$\lambda 2 = \langle 1 \to \text{RandomLieSeries}[\{1, 2\}],\ 2 \to \text{RandomLieSeries}[\{1, 2\}]\rangle;$$

$$\Big\{\text{lhs} = \lambda 2\ //\ \text{EulerE}\ //\ \text{adSeries}\left[\frac{e^{\text{ad}}-1}{\text{ad}},\ \lambda 2\right]\ //\ \text{RC}[-\lambda 2],$$
$$\text{rhs} = \Lambda[\lambda 2]\ //\ \text{EulerE}\ //\ \text{adSeries}\left[\frac{e^{\text{ad}}-1}{\text{ad}},\ \Lambda[\lambda 2],\ \text{tb}\right];\ (\text{lhs} \equiv \text{rhs})\,@\,\{8\}\Big\}\ //\ \text{Timing}$$

$$\Big\{54.491149,$$
$$\Big\{\Big\langle 1 \to \text{LS}\Big[-\overline{1}+2\,\overline{2},\ -4\,\overline{12},\ \frac{11}{2}\ \overline{1\overline{12}} - \frac{11}{2}\ \overline{\overline{12}2},\ -12\ \overline{1\overline{1\overline{12}}} + \frac{121}{6}\ \overline{1\overline{\overline{12}2}} - \frac{47}{6}\ \overline{\overline{\overline{12}2}2},\ \ldots\Big],$$
$$2 \to \text{LS}\Big[\overline{1}-2\,\overline{2},\ 6\,\overline{12},\ -8\ \overline{1\overline{12}} + \frac{33}{2}\ \overline{\overline{12}2},\ -\frac{1}{3}\ \overline{1\overline{1\overline{12}}} - \frac{271}{6}\ \overline{1\overline{\overline{12}2}} + \frac{209}{6}\ \overline{\overline{\overline{12}2}2},\ \ldots\Big]\Big\rangle,$$
$$\text{BS}[9\ \text{True}, \ldots]\Big\}\Big\}$$

CCAndRC

```
{α // CC1[-γ], α // CC1[-γ] // RC1[γ], α // CC1[-γ] // CC1[γ]}
```

CCAndRC

$\{LS[\overline{1}, 2\overline{\overline{1}2}, -\frac{5}{2}\overline{1\overline{\overline{1}2}} + \frac{3}{2}\overline{\overline{\overline{1}2}2}, \frac{7}{6}\overline{1\overline{1\overline{\overline{1}2}}} - \frac{23}{6}\overline{1\overline{\overline{\overline{1}2}2}} + \frac{2}{3}\overline{\overline{\overline{\overline{1}2}2}2}, \ldots],$
$LS[\overline{1}, \overline{\overline{1}2}, 0, 0, \ldots], LS[\overline{1}, \overline{\overline{1}2}, -\overline{1\overline{\overline{1}2}}, 2\overline{1\overline{1\overline{\overline{1}2}}} + \overline{1\overline{\overline{\overline{1}2}2}}, \ldots]\}$

tru

```
With[{γ = b[b[LW@v, LW@u], LW@u]}, tr_u[γ]] // TopBracketForm
```

tru

$-\widehat{uv}$

divu

```
With[{γ = LW@u + b[b[LW@v, LW@u], LW@u]}, div_u[γ]] // TopBracketForm
```

divu

$\widehat{u} - \widehat{uuv}$

Ju

```
J1[γ]
```

Ju

$CWS[\widehat{1}, \frac{5\widehat{12}}{2}, -\frac{7\widehat{112}}{6} + \frac{7\widehat{122}}{6}, \frac{3\widehat{1112}}{8} - \frac{11\widehat{1122}}{4} - \frac{3\widehat{1212}}{4} + \frac{3\widehat{1222}}{8}, \ldots]$

j

```
{div[λ]@{5}, j[λ]@{5}}
```

j

$\{CWS[\widehat{1} + \widehat{2}, -\widehat{12}, -\widehat{112}, 0, 0, \ldots],$
$CWS[\widehat{1} + \widehat{2}, -\widehat{12}, -\widehat{112}, -\widehat{1122} + \widehat{1212}, -\widehat{11122} + \widehat{11212}, \ldots]\}$

cocycle4j

```
lhs = j[BCH_tb[λ1, λ2]]; rhs = j[λ1] + e^(D_λ1)[j[λ2]];
{lhs, (lhs ≡ rhs)@{8}}
```

cocycle4j

$\{CWS[-\widehat{2}, 0, -\frac{7\widehat{112}}{3} + \widehat{122}, \frac{\widehat{1112}}{4} + \frac{7\widehat{1122}}{6} - \frac{7\widehat{1212}}{3} + \frac{2\widehat{1222}}{3}, \ldots], BS[9\ True, \ldots]\}$

```
lhs = j[BCH_b[λ1, λ2]]; rhs = j[λ1] + e^(D_λ1)[j[λ2]];
{lhs, (lhs ≡ rhs)}
```

$\{CWS[-\widehat{2}, -\frac{3\widehat{12}}{2}, -\frac{37\widehat{112}}{12} + \frac{31\widehat{122}}{12}, \frac{\widehat{1112}}{4} - \frac{25\widehat{1122}}{24} - \frac{\widehat{1212}}{6} + \frac{\widehat{1222}}{4}, \ldots],$
$BS[2\ True, -\frac{3\widehat{12}}{2} == 0, -\frac{3\widehat{12}}{2} == 0\ \&\&\ -\frac{37\widehat{112}}{12} + \frac{31\widehat{122}}{12} == -\frac{7\widehat{112}}{3} + \widehat{122},$
$-\frac{3\widehat{12}}{2} == 0\ \&\&\ -\frac{37\widehat{112}}{12} + \frac{31\widehat{122}}{12} == -\frac{7\widehat{112}}{3} + \widehat{122}\ \&\&$
$\frac{\widehat{1112}}{4} - \frac{25\widehat{1122}}{24} - \frac{\widehat{1212}}{6} + \frac{\widehat{1222}}{4} == \frac{\widehat{1112}}{4} + \frac{7\widehat{1122}}{6} - \frac{7\widehat{1212}}{3} + \frac{2\widehat{1222}}{3}, \ldots]\}$

dj

```
ϵ /: ϵ^2 = 0;
{j[ϵ λ], j[ϵ λ] ≡ ϵ div[λ]}
```

dj

$$\{\text{CWS}[\epsilon\,\widehat{1}+\epsilon\,\widehat{2},\ -\epsilon\,\widehat{12},\ -\epsilon\,\widehat{112},\ 0,\ \ldots],\ \text{BS}[5\ \text{True},\ \ldots]\}$$

Section 2.3 - The [AT]-inspired presentation El of $A^{w}{}_{exp}$

ElSetup

```
x1 = LW[1]; x2 = LW[2];
{ζa =
   El[⟨1 → LS[x1 + b[x1, x2]], 2 → LS[x2 - b[x1, b[x1, x2]]]⟩, CWS[CW[1] - 3 CW[1, 2, 1]]],
  ζb = El[⟨1 → LS[x2 - b[x1, x2]], 2 → LS[x1 + x2 + b[x2, b[x1, x2]]]⟩,
    CWS[CW[2] - 2 CW[1, 2]]],
  ζc = El[⟨1 → LS[x1 - b[b[x1, x2], b[x1, x2]]], 2 → LS[x2 + 3 b[x1, b[x1, x2]]]⟩,
    CWS[CW[1] - 2 CW[1, 2] + CW[1, 2, 1]]]}
```

ElSetup

$$\{\text{El}[\langle 1\to\text{LS}[\overline{1},\ \overline{12},\ 0,\ 0,\ \ldots],\ 2\to\text{LS}[\overline{2},\ 0,\ -\overline{1\overline{12}},\ 0,\ \ldots]\rangle,\ \text{CWS}[\widehat{1},\ 0,\ -3\,\widehat{112},\ 0,\ \ldots]],$$
$$\text{El}[\langle 1\to\text{LS}[\overline{2},\ -\overline{12},\ 0,\ 0,\ \ldots],\ 2\to\text{LS}[\overline{1}+\overline{2},\ 0,\ -\overline{\overline{12}\,2},\ 0,\ \ldots]\rangle,$$
$$\text{CWS}[\widehat{2},\ -2\,\widehat{12},\ 0,\ 0,\ \ldots]],\ \text{El}[$$
$$\langle 1\to\text{LS}[\overline{1},\ 0,\ 0,\ 0,\ \ldots],\ 2\to\text{LS}[\overline{2},\ 0,\ 3\,\overline{1\overline{12}},\ 0,\ \ldots]\rangle,\ \text{CWS}[\widehat{1},\ -2\,\widehat{12},\ \widehat{112},\ 0,\ \ldots]]\}$$

ElAssociativity

```
lhs = ζa ** (ζb ** ζc); rhs = (ζa ** ζb) ** ζc;
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

ElAssociativity

$$\{\text{El}[\langle 1\to\text{LS}[2\,\overline{1}+\overline{2},\ 0,\ \frac{1}{2}\,\overline{1\overline{12}},\ \ldots],\ 2\to\text{LS}[\overline{1}+3\,\overline{2},\ 0,\ \frac{5}{2}\,\overline{1\overline{12}}-\overline{\overline{12}\,2},\ \ldots]\rangle,$$
$$\text{CWS}[2\,\widehat{1}+\widehat{2},\ -4\,\widehat{12},\ -2\,\widehat{112},\ \ldots]],\ \text{BS}[9\ \text{True},\ \ldots]\}$$

detaExample

```
{ζa // dη^1, ζa // dη^2}
```

detaExample

$$\{\text{El}[\langle 2\to\text{LS}[\overline{2},\ 0,\ 0,\ 0,\ \ldots]\rangle,\ \text{CWS}[0,\ 0,\ 0,\ 0,\ \ldots]],$$
$$\text{El}[\langle 1\to\text{LS}[\overline{1},\ 0,\ 0,\ 0,\ \ldots]\rangle,\ \text{CWS}[\widehat{1},\ 0,\ 0,\ 0,\ \ldots]]\}$$

dA1

```
{ζd = El[λ, CWS[0]], ζd // dA}
```

dA1

$$\{\text{El}[\langle 1\to\text{LS}[\overline{1},\ \overline{12},\ 0,\ 0,\ \ldots],\ 2\to\text{LS}[\overline{2},\ 0,\ -\overline{1\overline{12}},\ 0,\ \ldots]\rangle,\ \text{CWS}[0,\ 0,\ 0,\ 0,\ \ldots]],$$
$$\text{El}[\langle 1\to\text{LS}[-\overline{1},\ -\overline{12},\ 0,\ 0,\ \ldots],\ 2\to\text{LS}[-\overline{2},\ 0,\ \overline{1\overline{12}},\ 0,\ \ldots]\rangle,$$
$$\text{CWS}[-\widehat{1}-\widehat{2},\ \widehat{12},\ \widehat{112},\ \widehat{1122}-\widehat{1212},\ \ldots]]\}$$

dA2

(ζd ≡ (ζd // dA // dA))@{8}

dA2

```
BS[9 True, ...]
```

dA3

lhs = (ζa ** ζb) // dA; rhs = (ζb // dA) ** (ζa // dA);
{lhs@{3}, (lhs ≡ rhs)@{8}}

dA3

$\{\text{El}[\langle 1 \to \text{LS}[-\overline{1} - \overline{2},\ 0,\ -\frac{1}{2}\overline{1\overline{1}\overline{2}},\ \ldots],\ 2 \to \text{LS}[-\overline{1} - 2\,\overline{2},\ 0,\ \frac{1}{2}\overline{1\overline{1}\overline{2}} + \overline{\overline{1}\overline{2}2},\ \ldots]\rangle,$
$\quad \text{CWS}[-\widehat{2},\ -2\,\widehat{12},\ -2\,\widehat{112} - \widehat{122},\ \ldots]],\ \text{BS}[9\ \text{True},\ \ldots]\}$

dS

ζd // dS

dS

$\text{El}[\langle 1 \to \text{LS}[\overline{1},\ -\overline{\overline{1}2},\ 0,\ 0,\ \ldots],\ 2 \to \text{LS}[\overline{2},\ 0,\ -\overline{1\overline{1}\overline{2}},\ 0,\ \ldots]\rangle,$
$\quad \text{CWS}[\widehat{1} + \widehat{2},\ \widehat{12},\ -\widehat{112},\ \widehat{1122} - \widehat{1212},\ \ldots]]$

dD1

{ζa, ζa // dΔ[2, 2, 3]}

dD1

$\{\text{El}[\langle 1 \to \text{LS}[\overline{1},\ \overline{\overline{1}2},\ 0,\ 0,\ \ldots],\ 2 \to \text{LS}[\overline{2},\ 0,\ -\overline{1\overline{1}\overline{2}},\ 0,\ \ldots]\rangle,\ \text{CWS}[\widehat{1},\ 0,\ -3\,\widehat{112},\ 0,\ \ldots]],$
$\quad \text{El}[\langle 1 \to \text{LS}[\overline{1},\ \overline{\overline{1}2} + \overline{\overline{1}3},\ 0,\ 0,\ \ldots],\ 2 \to \text{LS}[\overline{2} + \overline{3},\ 0,\ -\overline{1\overline{1}\overline{2}} - \overline{1\overline{1}\overline{3}},\ 0,\ \ldots],$
$\quad\quad 3 \to \text{LS}[\overline{2} + \overline{3},\ 0,\ -\overline{1\overline{1}\overline{2}} - \overline{1\overline{1}\overline{3}},\ 0,\ \ldots]\rangle,\ \text{CWS}[\widehat{1},\ 0,\ -3\,\widehat{112} - 3\,\widehat{113},\ 0,\ \ldots]]\}$

dD2

lhs = (ζa ** ζb) // dΔ[2, 2, 3]; rhs = (ζa // dΔ[2, 2, 3]) ** (ζb // dΔ[2, 2, 3]);
{lhs@{3}, (lhs ≡ rhs)@{8}}

dD2

$\{\text{El}[\langle 1 \to \text{LS}[\overline{1} + \overline{2} + \overline{3},\ 0,\ \frac{1}{2}\overline{1\overline{1}\overline{2}} + \frac{1}{2}\overline{1\overline{1}\overline{3}},\ \ldots],$
$\quad 2 \to \text{LS}[\overline{1} + 2\,\overline{2} + 2\,\overline{3},\ 0,\ -\frac{1}{2}\overline{1\overline{1}\overline{2}} - \frac{1}{2}\overline{1\overline{1}\overline{3}} - \overline{1\overline{2}\overline{3}} - \overline{\overline{1}\overline{2}2} - 2\,\overline{\overline{1}\overline{3}2} - \overline{\overline{1}\overline{3}3},\ \ldots],$
$\quad 3 \to \text{LS}[\overline{1} + 2\,\overline{2} + 2\,\overline{3},\ 0,\ -\frac{1}{2}\overline{1\overline{1}\overline{2}} - \frac{1}{2}\overline{1\overline{1}\overline{3}} - \overline{1\overline{2}\overline{3}} - \overline{\overline{1}\overline{2}2} - 2\,\overline{\overline{1}\overline{3}2} - \overline{\overline{1}\overline{3}3},\ \ldots]\rangle,$
$\quad \text{CWS}[\widehat{1} + \widehat{2} + \widehat{3},\ -2\,\widehat{12} - 2\,\widehat{13},\ -3\,\widehat{112} - 3\,\widehat{113},\ \ldots]],\ \text{BS}[9\ \text{True},\ \ldots]\}$

Section 2.4 - The factored presentation Ef of $A^w{}_{exp}$ and its stronger precursor Es

EsSetup1

```
u = LW@"u"; v = LW@"v";
ξa = Es[⟨1 → LS[u + b[u, v]], 2 → LS[v - b[u, b[u, v]]], 3 → LS[u - b[b[u, v], b[u, v]]]⟩,
   CWS[CW["u"] - 3 CW["u", "v", "u"]]]
```

EsSetup1

```
Es[⟨1 → LS[u, uv, 0, 0, ...], 2 → LS[v, 0, -u uv, 0, ...], 3 → LS[u, 0, 0, 0, ...]⟩,
 CWS[u, 0, -3 uuv, 0, ...]]
```

EsSetup2

```
SeedRandom[0]; ξb = Es[
   ⟨Table[i → RandomLieSeries[{1, 2, 3, 4}], {i, 4}]⟩, RandomCWSeries[{1, 2, 3, 4}]];
ξb@
 {2}
```

EsSetup2

$$\text{Es}\Big[\Big\langle 1\to\text{LS}\Big[-\overline{1}-2\,\overline{2}+2\,\overline{3}-2\,\overline{4},\ 2\,\overline{1\,2}+\frac{\overline{1\,3}}{2}+\overline{1\,4}-\frac{\overline{2\,3}}{2}-\frac{\overline{2\,4}}{2}+2\,\overline{3\,4},\ \ldots\Big],$$
$$2\to\text{LS}\Big[2\,\overline{1}-\overline{2}-2\,\overline{3}+\overline{4},\ 2\,\overline{1\,2}+\frac{3\,\overline{1\,3}}{2}-2\,\overline{1\,4}-\overline{2\,3}-\overline{2\,4}-\frac{\overline{3\,4}}{2},\ \ldots\Big],$$
$$3\to\text{LS}\Big[-\overline{1}+\overline{2}+2\,\overline{4},\ -2\,\overline{1\,2}+2\,\overline{1\,3}-\overline{1\,4}-\frac{3\,\overline{2\,3}}{2}+2\,\overline{2\,4}-2\,\overline{3\,4},\ \ldots\Big],$$
$$4\to\text{LS}\Big[-2\,\overline{1}+2\,\overline{2}+2\,\overline{3}+\overline{4},\ -\frac{\overline{1\,2}}{2}+\frac{3\,\overline{1\,3}}{2}-2\,\overline{2\,4}+\overline{3\,4},\ \ldots\Big]\Big\rangle,$$
$$\text{CWS}\Big[\widehat{3}-\widehat{4},\ \frac{3\,\widehat{11}}{2}+\frac{3\,\widehat{12}}{2}-2\,\widehat{13}+\widehat{14}+\widehat{22}+2\,\widehat{23}-\frac{\widehat{24}}{2}-2\,\widehat{33}-\widehat{34}+\widehat{44},\ \ldots\Big]\Big]$$

haction

```
lhs = ξa // hm[1, 2, 4] // tha[u, 4];
rhs = ξa // tha[u, 1] // tha[u, 2] // hm[1, 2, 4];
{lhs, (lhs ≡ rhs)@{8}}
```

haction

$$\Big\{\text{Es}\Big[\Big\langle 3\to\text{LS}\Big[\overline{u},\ -\overline{u\,v},\ -\overline{u\,\overline{u\,v}}+\frac{1}{2}\,\overline{\overline{u\,v}\,v},\ \frac{3}{2}\,\overline{u\,\overline{u\,\overline{u\,v}}}+\overline{u\,\overline{\overline{u\,v}\,v}}-\frac{1}{6}\,\overline{\overline{\overline{u\,v}\,v}\,v},\ \ldots\Big],$$
$$4\to\text{LS}\Big[\overline{u}+\overline{v},\ \frac{\overline{u\,v}}{2},\ -\frac{23}{12}\,\overline{u\,\overline{u\,v}}-\frac{5}{12}\,\overline{\overline{u\,v}\,v},\ \overline{u\,\overline{u\,\overline{u\,v}}}+\frac{13}{24}\,\overline{u\,\overline{\overline{u\,v}\,v}}+\frac{1}{12}\,\overline{\overline{\overline{u\,v}\,v}\,v},\ \ldots\Big]\Big\rangle,$$
$$\text{CWS}\Big[2\,\widehat{u},\ -\widehat{uv},\ -\frac{3\,\widehat{uuv}}{2},\ -\frac{\widehat{uuuv}}{6}+\widehat{uuvv}-\widehat{uvuv},\ \ldots\Big]\Big],\ \text{BS}[9\ \text{True},\ \ldots]\Big\}$$

metaassoc

```
lhs = ξb // dm[1, 2, 1] // dm[1, 3, 1]; rhs = ξb // dm[2, 3, 2] // dm[1, 2, 1];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

metaassoc

```
{Es[⟨1 → LS[-2 1̄ + 4̄, -3/2 1̄4̄, 20 1̄1̄4̄ - 19/3 1̄4̄4̄, ...],
   4 → LS[2 1̄ + 4̄, 1̄4̄, -31/2 1̄1̄4̄ - 13/6 1̄4̄4̄, ...]⟩,
  CWS[3 1̂ - 4̂, -3 11̂ + 14̂/2 + 44̂, 71 111̂/4 + 19 114̂/4 - 7 144̂/6 - 2 444̂/3, ...]], BS[6 True, ...]}
```

Section 3.1 - Tangle Invariants

Section 3.1.1 - The General Framework

RDefs

```
Rl[a_, b_] := El[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRl[a_, b_] := El[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
```

R3

```
lhs = Rl[1, 2] ** Rl[1, 3] ** Rl[2, 3]; rhs = Rl[2, 3] ** Rl[1, 3] ** Rl[1, 2];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

R3

```
{El[⟨1 → LS[0, 0, 0, ...], 2 → LS[1̄, 0, 0, ...], 3 → LS[1̄ + 2̄, 0, 0, ...]⟩,
  CWS[0, 0, 0, ...]], BS[6 True, ...]}
```

Section 3.1.2 - The Knot 8_{17} and the Borromean Tangle

817

```
t1 = iRs[12, 1] iRs[2, 7] iRs[8, 3] iRs[4, 11] Rs[16, 5] Rs[6, 13] Rs[14, 9] Rs[10, 15];
Do[t1 = t1 // dm[1, k, 1], {k, 2, 16}];
t1@{6}
```

817

```
Es[⟨1 → LS[0, 0, 0, 0, 0, 0, ...]⟩, CWS[0, -11̂, 0, -31 1111̂/12, 0, -1351 111111̂/360, ...]]
```

Borromean

```
t2 = iRs[r, 6] Rs[2, 4] iRs[g, 9] Rs[5, 7] iRs[b, 3] Rs[8, 1];
(Do[t2 = t2 // dm[r, k, r], {k, 1, 3}]; Do[t2 = t2 // dm[g, k, g], {k, 4, 6}];
 Do[t2 = t2 // dm[b, k, b], {k, 7, 9}]; t2)
```

Borromean

```
Es[⟨b → LS[0, gr, 1/2 ggr + brg + 1/2 grr,
    -1/2 bbrg + 1/6 gggr + 1/4 ggrr - 1/2 bgbr - 1/2 brgg - 1/2 brrg + 1/6 grrr, ...], g →
   LS[0, -br, 1/2 bbr - bgr - brg + 1/2 brr, -1/6 bbbr - 1/2 bbgr - 1/2 bggr - 1/2 bbrg -
     1/4 bbrr + 1/2 bgrr + 1/2 bgbr + brgr - bgrg - 1/2 brgg + 1/2 brrg - 1/6 brrr, ...],
  r → LS[0, bg, 1/2 bbg + bgr + 1/2 bgg, 1/6 bbbg + 1/2 bbgr +
     1/2 bggr + 1/4 bbgg + 1/2 bgrr + 1/6 bggg, ...]⟩,
 CWS[0, 0, 2 bgr, bbgr - bgbr + bggr - bgrg + bgrr - brgr, ...]]
```

Section 3.2 - Solutions of the Kashiwara-Vergne Equations

Continues pensieve://2013-10/SolvingWKO.nb.

VSetup

```
α = LS[{x, y}, αs]; β = LS[{x, y}, βs]; γ = CWS[{x, y}, γs];
V0 = Es[⟨x → α, y → β⟩, γ];
```

CapSetup

```
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
```

VCapEqns

```
R4Eqn = V0 ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V0 ;
UnitarityEqn = (V0 ** (V0 // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]]);
CapEqn = ((V0 ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap (Cap // dσ[x, y]) // dc[x] // dc[y]));
```

VCapSolution

```
βs[x] = 1/2; βs[y] = 0;
SeriesSolve[{α, β, γ, κ}, (ℏ^-1 R4Eqn) ⋀ UnitarityEqn ⋀ CapEqn];
{V0@{4}, κ@{6}}
```

VCapSolution

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {κs[x] → 0}.

VCapSolution

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {αs[x, y, y] → 0}.

VCapSolution

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

VCapSolution

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

VCapSolution

$$\Big\{\mathrm{Es}\Big[\Big\langle x \to \mathrm{LS}\Big[0,\ -\frac{\overline{xy}}{24},\ 0,\ \frac{7\,\overline{x\,\overline{x\overline{xy}}}}{5760} - \frac{7\,\overline{x\,\overline{\overline{xy}\,y}}}{5760} + \frac{\overline{\overline{\overline{xy}\,y}\,y}}{1440},\ \ldots\Big],$$

$$y \to \mathrm{LS}\Big[\frac{\overline{x}}{2},\ -\frac{\overline{xy}}{12},\ 0,\ \frac{\overline{x\,\overline{x\overline{xy}}}}{5760} - \frac{1}{720}\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{1}{720}\,\overline{\overline{\overline{xy}\,y}\,y},\ \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[0,\ -\frac{\widehat{xy}}{48},\ 0,\ \frac{\widehat{xxxy}}{2880} + \frac{\widehat{xxyy}}{2880} + \frac{\widehat{xyxy}}{5760} + \frac{\widehat{xyyy}}{2880},\ \ldots\Big]\Big],$$

$$\mathrm{CWS}\Big[0,\ -\frac{\widehat{xx}}{96},\ 0,\ \frac{\widehat{xxxx}}{11\,520},\ 0,\ -\frac{\widehat{xxxxxx}}{725\,760},\ \ldots\Big]\Big\}$$

Sinh

```
Series[1/4 Log[(ℏ/2)/Sinh[ℏ/2]], {ℏ, 0, 12}]
```

Sinh

$$-\frac{\hbar^2}{96} + \frac{\hbar^4}{11\,520} - \frac{\hbar^6}{725\,760} + \frac{\hbar^8}{38\,707\,200} - \frac{\hbar^{10}}{1\,916\,006\,400} + \frac{691\,\hbar^{12}}{62\,768\,369\,664\,000} + O[\hbar]^{13}$$

LambdaV

```
Λ[V0]
```

LambdaV

$$\mathrm{El}\Big[\Big\langle x \to \mathrm{LS}\Big[0,\ -\frac{\overline{xy}}{24},\ \frac{1}{96}\,\overline{x\overline{xy}},\ \frac{\overline{x\,\overline{x\overline{xy}}}}{2880} - \frac{1}{480}\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{\overline{\overline{\overline{xy}\,y}\,y}}{1440},\ \ldots\Big],$$

$$y \to \mathrm{LS}\Big[\frac{\overline{x}}{2},\ -\frac{\overline{xy}}{12},\ \frac{1}{96}\,\overline{x\overline{xy}},\ \frac{1}{960}\,\overline{x\,\overline{x\overline{xy}}} - \frac{1}{320}\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{1}{720}\,\overline{\overline{\overline{xy}\,y}\,y},\ \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[0,\ -\frac{\widehat{xy}}{48},\ 0,\ \frac{\widehat{xxxy}}{2880} + \frac{\widehat{xxyy}}{2880} + \frac{\widehat{xyxy}}{5760} + \frac{\widehat{xyyy}}{2880},\ \ldots\Big]\Big]$$

logF

```
logF = Λ[V0]⟦1⟧ // dσ[{x, y} → {y, x}]
```

logF

$$\Big\langle x \to \mathrm{LS}\Big[\frac{\overline{y}}{2},\ \frac{\overline{xy}}{12},\ \frac{1}{96}\,\overline{\overline{xy}\,y},\ -\frac{1}{720}\,\overline{x\,\overline{x\overline{xy}}} + \frac{1}{320}\,\overline{x\,\overline{\overline{xy}\,y}} - \frac{1}{960}\,\overline{\overline{\overline{xy}\,y}\,y},\ \ldots\Big],$$

$$y \to \mathrm{LS}\Big[0,\ \frac{\overline{xy}}{24},\ \frac{1}{96}\,\overline{\overline{xy}\,y},\ -\frac{\overline{x\,\overline{x\overline{xy}}}}{1440} + \frac{1}{480}\,\overline{x\,\overline{\overline{xy}\,y}} - \frac{\overline{\overline{\overline{xy}\,y}\,y}}{2880},\ \ldots\Big]\Big\rangle$$

atkv

```
atkv = logF // EulerE // adSeries[(ℯ^ad - 1)/ad, logF, tb];
{f = atkv_x, g = atkv_y}
```

atkv

$$\left\{LS\left[\frac{\overline{y}}{2}, \frac{\overline{xy}}{6}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} + \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \ldots\right], LS\left[0, \frac{\overline{xy}}{12}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} + \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \ldots\right]\right\}$$

On March 1, 2015, the following took 379 seconds:

KVTest

```
(ℏ^-1 (LS[LW@x + LW@y] - BCH[LW@y, LW@x] ≡ f - g - Ad[-LW@x][f] + Ad[LW@y][g]) ⋀
    div_x[f] + div_y[g] ≡ 1/2 tr_u[adSeries[ad/(ℯ^ad - 1), LW@x][LW@u] + adSeries[ad/(ℯ^ad - 1), LW@y][
        LW@u] - adSeries[ad/(ℯ^ad - 1), BCH[LW@x, LW@y]][LW@u]]) @{8} // Timing
```

KVTest

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

KVTest

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.

KVTest

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {αs[x, x, x, x, y, x, y, y] → 0}.

KVTest

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

KVTest

```
{330.441318, BS[9 True, ...]}
```

KVDirect

```
{F = LS[{x, y}, Fs], G = LS[{x, y}, Gs]}; Fs[y] = 1/2;
SeriesSolve[{F, G},
  ℏ^-1 (LS[LW@x + LW@y] - BCH[LW@y, LW@x] ≡ F - G - Ad[-LW@x][F] + Ad[LW@y][G]) ⋀
   div_x[F] + div_y[G] ≡ 1/2 tr_u[adSeries[ad/(ℯ^ad - 1), LW@x][LW@u] +
      adSeries[ad/(ℯ^ad - 1), LW@y][LW@u] - adSeries[ad/(ℯ^ad - 1), BCH[LW@x, LW@y]][LW@u]]];
{F,
 G}
```

KVDirect

$$\left\{LS\left[\frac{\overline{y}}{2}, \frac{\overline{xy}}{6}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} + \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \ldots\right], LS\left[0, \frac{\overline{xy}}{12}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} + \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \ldots\right]\right\}$$

Section 3.3 - The involution τ and the twist equation

Theta

```
Θl[x_, y_, s_] := El[⟨x → LS[s LW@y], y → LS[s LW@x]⟩, CWS[0]];
Θs[x_, y_, s_] := Θl[x, y, s] // Γ;
{Θl[x, y, 1], Θs[x, y, 1]}
```

Theta

$$\{El[\langle x \to LS[\overline{y}, 0, 0, 0, \ldots], y \to LS[\overline{x}, 0, 0, 0, \ldots]\rangle, CWS[0, 0, 0, 0, \ldots]], Es\Big[\Big\langle x \to LS\Big[\overline{y}, \frac{\overline{x\,y}}{2}, \frac{1}{6}\overline{x\,\overline{x\,y}} - \frac{1}{12}\overline{\overline{x\,y}\,y}, \frac{1}{24}\,x\,\overline{x\,\overline{x\,y}} - \frac{1}{24}\,x\,\overline{\overline{x\,y}\,y}, \ldots\Big], y \to LS\Big[\overline{x}, -\frac{\overline{x\,y}}{2}, -\frac{1}{12}\overline{x\,\overline{x\,y}} + \frac{1}{6}\overline{\overline{x\,y}\,y}, \frac{1}{24}\,x\,\overline{\overline{x\,y}\,y} - \frac{1}{24}\overline{\overline{\overline{x\,y}\,y}\,y}, \ldots\Big]\Big\rangle, CWS[0, 0, 0, 0, \ldots]\Big]\}$$

Vtau

```
τV = Rs[x, y] ** (V0 // dσ[{x, y} → {y, x}]) ** Θs[x, y, -1/2];
(V0 ≡ τV) @ {6}
```

Vtau

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

Vtau

```
BS[7 True, ...]
```

Linearized

```
{A = LS[{x, y}, As], B = LS[{x, y}, Bs]};
msgs = SeriesSolve[{A, B},
   ℏ^-1 (b[LW@x, A] + b[LW@y, B] ≡ LS[0]) ⋀ (divx[A] + divy[B] ≡ CWS[0])];
{A, B}
```

Linearized

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {As[y] → 0}.

Linearized

```
{LS[0, 0, 0, 0, ...], LS[0, 0, 0, 0, ...]}
```

msgs

```
Read[msgs]
```

msgs

```
{{ArbitrarilySetting, 1, {Hold[As[y]] → 0}}, {ArbitrarilySetting, 2, {}},
 {ArbitrarilySetting, 3, {}}, {ArbitrarilySetting, 4, {}}}
```

dims

```
A@12; Length[Last[#]] & /@ Read[msgs]
```

dims

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {As[x, x, x, x, y, x, y, y] → 0}.

dims

SeriesSolve::ArbitrarilySetting : In degree 10 arbitrarily setting {As[x, x, x, x, x, x, y, x, y, y] → 0}.

dims

SeriesSolve::ArbitrarilySetting : In degree 11 arbitrarily setting {As[x, x, x, x, x, x, y, x, y, y, y] → 0}.

dims

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

dims

```
{1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 2}
```

dims1

```
{A1 = LS[{x, y}, A1s], B1 = LS[{x, y}, B1s]};
msgs1 = SeriesSolve[{A1, B1},
   ℏ^-1 (b[LW@x, A1] + b[LW@y, B1] ≡ LS[0]) ⋀
    (div_x[A1] + div_y[B1] ≡ CWS[0]) ⋀ (A1 ≡ (B1 // LieMorphism[x → y, y → x]))];
A1@12; Length[Last[#]] & /@ Read[msgs1]
```

dims1

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {A1s[y] → 0}.

dims1

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {A1s[x, x, x, x, y, y, x, y, y] → 0}.

dims1

SeriesSolve::ArbitrarilySetting : In degree 10 arbitrarily setting {A1s[x, x, x, x, x, x, y, x, y, y] → 0}.

dims1

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

dims1

```
{1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 2}
```

Section 3.4 - Drinfel'd Associators

4T

```
{b[t[1, 3], t[4, 2]], b[t[1, 2] + t[1, 3], t[2, 3]]}
```

4T

```
{0, 0}
```

DKExample

```
b[t[1, 3], t[1, 2]]
```

DKExample

```
DK[3, -LW[1, 2]]
```

DKSExample

```
b[t[1, 3], t[1, 2]] // DKS
```

DKSExample

$\mathrm{DKS}\left[0, -\overline{t_{13}\, t_{23}}, 0, 0, \ldots\right]$

sigmaExample

$\left\{t[2,3]^{\sigma[\{2,4\},\{1,5\},\{3,7,8\},\{9\}]} \mathbin{//} \mathrm{DKS},\ t[2,3]^{\sigma[24,15,378,9]} \mathbin{//} \mathrm{DKS}\right\}$

sigmaExample

$\{\mathrm{DKS}[\overline{t_{13}} + \overline{t_{17}} + \overline{t_{18}} + \overline{t_{35}} + \overline{t_{57}} + \overline{t_{58}}, 0, 0, 0, \ldots],$
$\mathrm{DKS}[\overline{t_{13}} + \overline{t_{17}} + \overline{t_{18}} + \overline{t_{35}} + \overline{t_{57}} + \overline{t_{58}}, 0, 0, 0, \ldots]\}$

BCH4DK

$R = \mathrm{DKS}[t[1, 2] / 2];$
$\{R ** R^{\sigma[2,3]},\ R ** R^{\sigma[12,3]}\}$

BCH4DK

$$\left\{\mathrm{DKS}\left[\frac{\overline{t_{12}}}{2} + \frac{\overline{t_{23}}}{2}, -\frac{1}{8}\overline{t_{13}\,t_{23}}, -\frac{1}{48}\overline{\overline{t_{13}\,t_{23}}\,t_{23}} + \frac{1}{96}\overline{t_{13}\,\overline{t_{13}\,t_{23}}}, -\frac{1}{384}\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}} + \frac{1}{384}\overline{t_{13}\,\overline{\overline{t_{13}\,t_{23}}\,t_{23}}}, \ldots\right], \mathrm{DKS}\left[\frac{\overline{t_{12}}}{2} + \frac{\overline{t_{13}}}{2} + \frac{\overline{t_{23}}}{2}, 0, 0, 0, \ldots\right]\right\}$$

Phi

```
Φs[2, 1] = Φs[3, 1] = Φs[3, 2] = 0; Φs[3, 1, 2] = 1/24; Φ0 = DKS[3, Φs];
SeriesSolve[Φ0,
  (Φ0^σ[3,2,1] ≡ -Φ0) ⋀ (Φ0 ** Φ0^σ[1,23,4] ** Φ0^σ[2,3,4] ≡ Φ0^σ[12,3,4] ** Φ0^σ[1,2,34])];
Φ0@
 {6}
```

Phi

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {Φs[3, 1, 1, 2] → 0}.

Phi

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {Φs[3, 1, 1, 1, 1, 2] → 0}.

Phi

$$\mathrm{DKS}\Big[0,\ \frac{1}{24} t_{13} t_{23},\ 0,\ -\frac{7\, t_{13} t_{23} t_{23}}{5760} + \frac{7\, t_{13} t_{13} t_{23} t_{23}}{5760} - \frac{t_{13} t_{13} t_{13} t_{23}}{1440},$$

$$0,\ \frac{31\, t_{13} t_{23} t_{23} t_{23} t_{23} t_{23}}{967\,680} - \frac{157\, t_{13} t_{13} t_{23} t_{23} t_{13} t_{23}}{1\,935\,360} - \frac{31\, t_{13} t_{23} t_{13} t_{23} t_{23} t_{23}}{387\,072} -$$

$$\frac{31\, t_{13} t_{13} t_{23} t_{23} t_{23} t_{23}}{483\,840} + \frac{11\, t_{13} t_{13} t_{13} t_{23} t_{13} t_{23}}{290\,304} + \frac{31\, t_{13} t_{13} t_{23} t_{13} t_{23} t_{23}}{725\,760} +$$

$$\frac{83\, t_{13} t_{13} t_{13} t_{23} t_{23} t_{23}}{967\,680} - \frac{13\, t_{13} t_{13} t_{13} t_{13} t_{23} t_{23}}{241\,920} + \frac{t_{13} t_{13} t_{13} t_{13} t_{13} t_{23}}{60\,480},\ \ldots\Big]$$

Hexagons

```
R = DKS[t[1, 2]/2];
(R^σ[12,3] ≡ Φ0 ** R^σ[2,3] ** (-Φ0)^σ[1,3,2] ** R^σ[1,3] ** Φ0^σ[3,1,2] ⋀
   (-R)^σ[12,3] ≡ Φ0 ** (-R)^σ[2,3] ** (-Φ0)^σ[1,3,2] ** (-R)^σ[1,3] ** Φ0^σ[3,1,2]) @ {6}
```

Hexagons

```
BS[7 True, ...]
```

Section 3.5 - Associators in $\mathcal{A}^w$

PhiV

```
V12 = V0 // dσ[{x, y} → {1, 2}];
ΦV = (V12 // dA)^σ[12,3] ** (V12 // dA)^σ[1,2] ** V12^σ[2,3] ** V12^σ[1,23]
```

PhiV

```
Es[
⟨1 → LS[0, 23/24, 0, -1123/1440 + 71223/5760 + 1233/5760 - 72223/5760 + 72233/5760 + 1/480 1213 - 1323/1920 +
    1/640 1232 - 1322/1152 - 1332/1152 - 2333/1440, ...],
 2 → LS[0, -13/24, 0, 1113/1440 - 1123/1152 + 71223/1920 - 1/480 1132 - 1133/5760 + 1233/1152 +
    71213/5760 + 191323/5760 + 71232/1920 + 71322/5760 + 71332/5760 + 1333/1440, ...],
 3 → LS[0, 12/24, 0, -1112/1440 + 1123/5760 + 71223/5760 + 71122/5760 - 1132/1440 - 1233/1440 +
    1213/5760 + 1323/1440 - 1232/1152 - 71222/5760 - 71322/5760 - 1332/1440, ...]⟩, CWS[0, 0, 0, 0, ...]]
```

PentPhiV

```
ΦV ** ΦV^σ[1,23,4] ** ΦV^σ[2,3,4] ≡ ΦV^σ[12,3,4] ** ΦV^σ[1,2,34]
```

PentPhiV

```
BS[5 True, ...]
```

Phi_is_sder

```
ϕ = (ΦV // Λ)⟦1⟧;
(b[LW[1], ϕ1] + b[LW[2], ϕ2] + b[LW[3], ϕ3]) @{6}
```

Phi_is_sder

```
LS[0, 0, 0, 0, 0, 0, ...]
```

DK2Es

```
DK2Es[s___][ξ_] := El[ξ // αMap[s], CWS[0]] // Γ;
DK2Es[1, 2, 3][Φ0]
```

DK2Es

```
Es[
 ⟨1 → LS[0, [2,3]/24, 0, -[1,[1,[2,3]]]/1440 + 7[1,[2,[2,3]]]/5760 + [1,[[2,3],3]]/5760 - 7[2,[2,[2,3]]]/5760 + 7[2,[[2,3],3]]/5760 + 1/480 [[1,2],[1,3]] - [[1,3],[2,3]]/1920 +
     1/640 [[1,[2,3]],2] - [[[1,3],2],2]/1152 - [[[1,3],3],2]/1152 - [[[2,3],3],3]/1440, ...],
  2 → LS[0, -[1,3]/24, 0, [1,[1,[1,3]]]/1440 - [1,[1,[2,3]]]/1152 + 7[1,[2,[2,3]]]/1920 - 1/480 [1,[[1,3],2]] - [1,[[1,3],3]]/5760 + [1,[[2,3],3]]/1152 +
     7[[1,2],[1,3]]/5760 + 19[[1,3],[2,3]]/5760 + 7[[1,[2,3]],2]/1920 + 7[[[1,3],2],2]/5760 + 7[[[1,3],3],2]/5760 + [[[1,3],3],3]/1440, ...],
  3 → LS[0, [1,2]/24, 0, -[1,[1,[1,2]]]/1440 + [1,[1,[2,3]]]/5760 + 7[1,[2,[2,3]]]/5760 + 7[1,[[1,2],2]]/5760 - [1,[[1,3],2]]/1440 - [1,[[2,3],3]]/1440 +
     [[1,2],[1,3]]/5760 + [[1,3],[2,3]]/1440 - [[1,[2,3]],2]/1152 - 7[[[1,2],2],2]/5760 - 7[[[1,3],2],2]/5760 - [[[1,3],3],2]/1440, ...]⟩, CWS[0, 0, 0, 0, ...]]
```

The computation below takes a a couple of hours and yields “BS[8 True,False,...]”:

```
TrueQ[DK2Es[1, 2, 3][Φ0] ≡ ΦV]@{8}
```

```
BS[8 True, False, ...]
```

Section 3.6 - The Relation between Drinfel'd Associators and the Kashiwara-Vergne Equations

ZB

```
R = DKS[t[1, 2] / 2];
ZB = (-Φ0)^σ[13,2,4] ** Φ0^σ[1,3,2] ** R^σ[2,3] ** (-Φ0)^σ[1,2,3] ** Φ0^σ[12,3,4]
```

ZB

$$\mathrm{DKS}\Big[\frac{\overline{t_{23}}}{2},\ -\frac{1}{12}\overline{t_{13}t_{23}}-\frac{1}{24}\overline{t_{14}t_{24}}+\frac{1}{24}\overline{t_{14}t_{34}}+\frac{1}{12}\overline{t_{24}t_{34}},\ 0,$$
$$\frac{\overline{\overline{\overline{t_{13}t_{23}}t_{23}}t_{23}}}{5760}+\frac{7\,\overline{\overline{\overline{t_{14}t_{24}}t_{24}}t_{24}}}{5760}+\frac{\overline{\overline{\overline{t_{14}t_{34}}t_{24}}t_{24}}}{1920}-\frac{\overline{\overline{\overline{t_{14}t_{34}}t_{34}}t_{24}}}{1920}-\frac{7\,\overline{\overline{\overline{t_{14}t_{34}}t_{34}}t_{34}}}{5760}-$$
$$\frac{\overline{\overline{\overline{t_{24}t_{34}}t_{34}}t_{34}}}{5760}+\frac{\overline{\overline{t_{14}\overline{t_{24}t_{34}}}t_{24}}}{1920}+\frac{\overline{\overline{t_{14}t_{24}}\,\overline{t_{14}t_{34}}}}{1920}-\frac{\overline{\overline{t_{14}t_{34}}\,\overline{t_{24}t_{34}}}}{1920}-\frac{1}{720}\overline{t_{13}\overline{\overline{t_{13}t_{23}}t_{23}}}+$$
$$\frac{1}{720}\overline{t_{13}\overline{t_{13}\overline{t_{13}t_{23}}}}-\frac{7\,\overline{t_{14}\overline{\overline{t_{14}t_{24}}t_{24}}}}{5760}+\frac{7\,\overline{t_{14}\overline{\overline{t_{14}t_{34}}t_{34}}}}{5760}-\frac{\overline{t_{14}\overline{\overline{t_{24}t_{34}}t_{34}}}}{5760}+\frac{\overline{t_{14}\overline{t_{14}\overline{t_{14}t_{24}}}}}{1440}-$$
$$\frac{\overline{t_{14}\overline{t_{14}\overline{t_{14}t_{34}}}}}{1440}-\frac{1}{960}\overline{t_{14}\overline{t_{14}\overline{t_{24}t_{34}}}}+\frac{\overline{t_{14}\overline{t_{24}\overline{t_{24}t_{34}}}}}{5760}-\frac{1}{960}\overline{t_{24}\overline{\overline{t_{24}t_{34}}t_{34}}}-\frac{\overline{t_{24}\overline{t_{24}\overline{t_{24}t_{34}}}}}{5760},\ \ldots\Big]$$

VfromPhi

```
ZB // DK2Es[1, 2, 3, 4] // tη^1 // tη^3
```

VfromPhi

$$\mathrm{Es}\Big[\Big\langle 1\to \mathrm{LS}\Big[0,\ -\frac{\overline{2\,4}}{24},\ 0,\ \frac{\overline{7\,2\,\overline{2\,\overline{2\,4}}}}{5760}-\frac{\overline{7\,2\,\overline{\overline{2\,4}\,4}}}{5760}+\frac{\overline{\overline{\overline{2\,4}\,4}\,4}}{1440},\ \ldots\Big],$$
$$2\to \mathrm{LS}[0,\ 0,\ 0,\ 0,\ \ldots],\ 3\to \mathrm{LS}\Big[\frac{\overline{2}}{2},\ -\frac{\overline{2\,4}}{12},\ 0,\ \frac{\overline{2\,\overline{2\,\overline{2\,4}}}}{5760}-\frac{1}{720}\overline{2\,\overline{\overline{2\,4}\,4}}+\frac{1}{720}\overline{\overline{\overline{2\,4}\,4}\,4},\ \ldots\Big],$$
$$4\to \mathrm{LS}[0,\ 0,\ 0,\ 0,\ \ldots]\Big\rangle,\ \mathrm{CWS}[0,\ 0,\ 0,\ 0,\ \ldots]\Big]$$

The computation below takes a few hours and yields "BS[8 True,False,...]":

```
VB = ZB // DK2Es[1, 2, 3, 4] // tη^1 // tη^3 // hη^2 // hη^4 // hσ[{1, 3} → {x, y}] //
  tσ[{2, 4} → {x, y}]; TrueQ[VB[[1]] ≡ V0[[1]]] @ {8}
```

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {Φs[3, 1, 1, 1, 1, 1, 1, 2] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

```
BS[8 True, False, ...]
```

nu

```
νinv = Φ0 // DK2Es[1, 2, 3] // dS[2] // dm[3, 2, 2] // dm[2, 1, x]
```

nu

$$\mathrm{Es}\Big[\langle x\to \mathrm{LS}[0,\ 0,\ 0,\ 0,\ \ldots]\rangle,\ \mathrm{CWS}\Big[0,\ \frac{\widehat{xx}}{24},\ 0,\ -\frac{\widehat{xxxx}}{2880},\ \ldots\Big]\Big]$$

nucap4

```
(vinv ** Cap ** Cap ** Cap ** Cap) @ {6}
```

nucap4

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

nucap4

```
Es[⟨x → LS[0, 0, 0, 0, 0, 0, ...]⟩, CWS[0, 0, 0, 0, 0, 0, ...]]
```

Section 3.7 - A Likely S_4 Action on Solutions of KV

rhoV

```
ρV = Rs[x, y] ** (V₀ // (-1)^deg);
(V₀ ≡ ρV) @ {8}
```

rhoV

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.

rhoV

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {αs[x, x, x, x, y, x, y, y] → 0}.

rhoV

```
BS[9 True, ...]
```

```
ρ₃[ℰ_Es] := ℰ // dS[y] // dΔ[y, y, z] // dm[x, z, x] // dσ[{x, y} → {y, x}];
{V₀@{2}, (V₀ // ρ₃)@{2}, (V₀ // ρ₃ // ρ₃ // ρ₃)@{2}}
```

$$\left\{\mathrm{Es}\left[\left\langle x \to \mathrm{LS}\left[0, -\frac{\overline{x\,y}}{24}, \ldots\right], y \to \mathrm{LS}\left[\frac{\overline{x}}{2}, -\frac{\overline{x\,y}}{12}, \ldots\right]\right\rangle, \mathrm{CWS}\left[0, -\frac{\widehat{xy}}{48}, \ldots\right]\right],\right.$$
$$\mathrm{Es}\left[\left\langle x \to \mathrm{LS}\left[-\frac{\overline{y}}{2}, \frac{\overline{x\,y}}{12}, \ldots\right], y \to \mathrm{LS}\left[-\frac{\overline{y}}{2}, \frac{\overline{x\,y}}{24}, \ldots\right]\right\rangle, \mathrm{CWS}\left[-\frac{\widehat{y}}{2}, \frac{\widehat{xy}}{48} + \frac{\widehat{yy}}{48}, \ldots\right]\right],$$
$$\left.\mathrm{Es}\left[\left\langle x \to \mathrm{LS}\left[0, -\frac{\overline{x\,y}}{24}, \ldots\right], y \to \mathrm{LS}\left[\frac{\overline{x}}{2}, -\frac{\overline{x\,y}}{12}, \ldots\right]\right\rangle, \mathrm{CWS}\left[0, -\frac{\widehat{xy}}{48}, \ldots\right]\right]\right\}$$

```
(V₀ ≡ (V₀ // ρ₃ // ρ₃ // ρ₃)) @ {6}
```

```
BS[7 True, ...]
```

```
θ[x_, s_] := Module[{y}, Θs[x, y, -s] // dS[y] // dm[x, y, x]];
θ[1, 1] @ {6}
```

$$\mathrm{Es}\left[\left\langle 1 \to \mathrm{LS}\left[2\,\overline{1}, 0, 0, 0, 0, 0, \ldots\right]\right\rangle, \mathrm{CWS}\left[\widehat{1}, 0, 0, 0, 0, 0, \ldots\right]\right]$$

```
V₁ = V₀ ** Θs[x, y, -1/4] **
    Es[⟨x → LS@0, y → LS@0⟩, CWS[CW[x]/12 - CW[y]/12] - (2 Cap[[2]] // tΔ[x, x, y])];
(V₁ ≡ ρ₃[V₁]) @ {6}
```

```
BS[7 True, ...]
```

```
(V₁ ≡ ρ₃[V₁]) @ {8}
```

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {αs[x, x, x, x, y, x, y, y] → 0}.

$\theta[z, 1/4]$ ** (Cap ** Cap // $d\sigma[x, z]$)

$$\mathrm{Es}\left[\left\langle z \to \mathrm{LS}\left[\frac{\overline{z}}{2}, 0, 0, 0, \ldots\right]\right\rangle, \mathrm{CWS}\left[\frac{\widehat{z}}{4}, -\frac{\widehat{zz}}{48}, 0, \frac{\widehat{zzzz}}{5760}, \ldots\right]\right]$$

Cap

$$\mathrm{Es}\left[\langle x \to \mathrm{LS}[0, 0, 0, 0, \ldots]\rangle, \mathrm{CWS}\left[0, -\frac{\widehat{xx}}{96}, 0, \frac{\widehat{xxxx}}{11520}, \ldots\right]\right]$$

Cm2 = Es[⟨x → LS[0]⟩, -2 Cap⟦2⟧]

$$\mathrm{Es}\left[\langle x \to \mathrm{LS}[0, 0, 0, 0, \ldots]\rangle, \mathrm{CWS}\left[0, \frac{\widehat{xx}}{48}, 0, -\frac{\widehat{xxxx}}{5760}, \ldots\right]\right]$$

((Cm2 // $d\sigma[x, y]$) Es[⟨x → LS[0]⟩, CWS[0]])

$$\mathrm{Es}\left[\langle x \to \mathrm{LS}[0, 0, 0, 0, \ldots], y \to \mathrm{LS}[0, 0, 0, 0, \ldots]\rangle, \mathrm{CWS}\left[0, \frac{\widehat{yy}}{48}, 0, -\frac{\widehat{yyyy}}{5760}, \ldots\right]\right]$$

$\theta[y, 1/4]$

$$\mathrm{Es}\left[\left\langle y \to \mathrm{LS}\left[\frac{\overline{y}}{2}, 0, 0, 0, \ldots\right]\right\rangle, \mathrm{CWS}\left[\frac{\widehat{y}}{4}, 0, 0, 0, \ldots\right]\right]$$

V_2 = (V_0 // dS[x]) ** ((Cm2 // $d\sigma[x, y]$) Es[⟨x → LS[0]⟩, CWS[0]]) **
(Cm2 // $d\Delta[x, x, y]$ // dS[y]) ** ($\theta[y, 1/4]$ Es[⟨x → LS[0]⟩, CWS[0]]) **
($\theta[z, -1/4]$ // $d\Delta[z, x, y]$ // dS[y])

$$\mathrm{Es}\Big[\Big\langle x \to \mathrm{LS}\Big[-\frac{\overline{x}}{2} + \frac{\overline{y}}{2}, \frac{5\,\overline{xy}}{24}, \frac{1}{32}\overline{x\,\overline{xy}} - \frac{5}{96}\overline{\overline{xy}\,y}, \frac{\overline{x\,\overline{x\,\overline{xy}}}}{2880} - \frac{193\,\overline{x\,\overline{\overline{xy}\,y}}}{5760} + \frac{59\,\overline{\overline{\overline{xy}\,y}\,y}}{5760}, \ldots\Big],$$

$$y \to \mathrm{LS}\Big[0, \frac{5\,\overline{xy}}{24}, -\frac{1}{48}\overline{x\,\overline{xy}} - \frac{1}{12}\overline{\overline{xy}\,y}, \frac{\overline{x\,\overline{x\,\overline{xy}}}}{1440} - \frac{11\,\overline{x\,\overline{\overline{xy}\,y}}}{1920} + \frac{97\,\overline{\overline{\overline{xy}\,y}\,y}}{5760}, \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[-\frac{\widehat{x}}{4}, \frac{\widehat{xx}}{48} - \frac{5\,\widehat{xy}}{48} + \frac{\widehat{yy}}{24}, -\frac{\widehat{xxy}}{48} - \frac{\widehat{xyy}}{48},$$

$$-\frac{\widehat{xxxx}}{5760} - \frac{\widehat{xxxy}}{288} + \frac{83\,\widehat{xxyy}}{2880} - \frac{227\,\widehat{xyxy}}{5760} - \frac{17\,\widehat{xyyy}}{5760} - \frac{\widehat{yyyy}}{2880}, \ldots\Big]\Big]$$

V_0 // ρ_3

$$\mathrm{Es}\Big[\Big\langle x \to \mathrm{LS}\Big[-\frac{\overline{y}}{2}, \frac{\overline{xy}}{12}, 0, -\frac{1}{720}\overline{x\,\overline{x\,\overline{xy}}} + \frac{1}{720}\overline{x\,\overline{\overline{xy}\,y}} - \frac{\overline{\overline{\overline{xy}\,y}\,y}}{5760}, \ldots\Big],$$

$$y \to \mathrm{LS}\Big[-\frac{\overline{y}}{2}, \frac{\overline{xy}}{24}, -\frac{1}{96}\overline{\overline{xy}\,y}, -\frac{\overline{x\,\overline{x\,\overline{xy}}}}{1440} + \frac{7\,\overline{x\,\overline{\overline{xy}\,y}}}{5760} - \frac{\overline{\overline{\overline{xy}\,y}\,y}}{2880}, \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[-\frac{\widehat{y}}{2}, \frac{\widehat{xy}}{48} + \frac{\widehat{yy}}{48}, 0, -\frac{\widehat{xxxy}}{2880} - \frac{\widehat{xxyy}}{2880} - \frac{\widehat{xyxy}}{5760} - \frac{\widehat{xyyy}}{2880} - \frac{\widehat{yyyy}}{5760}, \ldots\Big]\Big]$$

V_0 // ρ_3 // ρ_3

$$\mathrm{Es}\Big[\Big\langle x \to \mathrm{LS}\Big[-\frac{\overline{x}}{2}-\frac{\overline{y}}{2},\ -\frac{\overline{x\,y}}{24},\ \frac{1}{96}\,\overline{x\,\overline{x\,y}}-\frac{1}{96}\,\overline{\overline{x\,y}\,y},\ \frac{\overline{x\,\overline{x\,\overline{x\,y}}}}{2880}+\frac{\overline{x\,\overline{\overline{x\,y}\,y}}}{1920}-\frac{\overline{\overline{\overline{x\,y}\,y}\,y}}{5760},\ \ldots\Big],$$

$$y \to \mathrm{LS}\Big[0,\ \frac{\overline{x\,y}}{24},\ 0,\ -\frac{\overline{x\,\overline{x\,\overline{x\,y}}}}{1440}+\frac{7\,\overline{x\,\overline{\overline{x\,y}\,y}}}{5760}-\frac{7\,\overline{\overline{\overline{x\,y}\,y}\,y}}{5760},\ \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[0,\ \frac{\overparen{xx}}{48}+\frac{\overparen{xy}}{48},\ 0,\ -\frac{\overparen{xxxx}}{5760}-\frac{\overparen{xxxy}}{2880}-\frac{\overparen{xxyy}}{2880}-\frac{\overparen{xyxy}}{5760}-\frac{\overparen{xyyy}}{2880},\ \ldots\Big]\Big]$$

V_3 =
(V_0 // ρ_3) ** Es[⟨x → LS[LW@x / 2], y → LS[-LW@y / 2]⟩, (-2 Cap⟦2⟧ // tσ[x, y]) - 2 Cap⟦2⟧]

$$\mathrm{Es}\Big[\Big\langle x \to \mathrm{LS}\Big[\frac{\overline{x}}{2}-\frac{\overline{y}}{2},\ \frac{5\,\overline{x\,y}}{24},\ -\frac{1}{32}\,\overline{x\,\overline{x\,y}}+\frac{1}{96}\,\overline{\overline{x\,y}\,y},\ \frac{\overline{x\,\overline{x\,\overline{x\,y}}}}{2880}-\frac{17\,\overline{x\,\overline{\overline{x\,y}\,y}}}{5760}-\frac{\overline{\overline{\overline{x\,y}\,y}\,y}}{5760},\ \ldots\Big],$$

$$y \to \mathrm{LS}\Big[-\overline{y},\ \frac{\overline{x\,y}}{24},\ -\frac{1}{48}\,\overline{\overline{x\,y}\,y},\ -\frac{\overline{x\,\overline{x\,\overline{x\,y}}}}{1440}+\frac{7\,\overline{x\,\overline{\overline{x\,y}\,y}}}{5760}+\frac{13\,\overline{\overline{\overline{x\,y}\,y}\,y}}{5760},\ \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[-\frac{\overparen{y}}{2},\ \frac{\overparen{xx}}{48}+\frac{\overparen{xy}}{48}+\frac{\overparen{yy}}{24},\ 0,\ -\frac{\overparen{xxxx}}{5760}-\frac{\overparen{xxxy}}{2880}-\frac{\overparen{xxyy}}{2880}-\frac{\overparen{xyxy}}{5760}-\frac{\overparen{xyyy}}{2880}-\frac{\overparen{yyyy}}{2880},\ \ldots\Big]\Big]$$