Pensieve header: Calculations appearing in the WKO4 paper.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
```

Section 1 - Introduction

Initialization

```
<< FreeLie.m;
<< AwCalculus.m;
$SeriesShowDegree = 4;
```

Initialization

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop,
  CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, EulerE, Exp,
  Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
  Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve,
  Support, t, tb, TopBracketForm, tr, UndeterminedCoefficients, Γ, ι, Λ, σ, ℏ, ⌐, ⌢}.
```

Initialization

```
AwCalculus` implements / extends {*, **, ≡, dA, dc, deg,
  dm, dS, dΔ, dη, dσ, El, Es, hA, hm, hS, hη, hσ, tA, tha, tm, tS, tσ, Γ, Λ}.
```

Section 2.2 - Some Preliminaries on Lie Algebras and Cyclic Words

alphabetagamma

```
x1 = LW[1]; x2 = LW[2];
{α, β, γ} = LS /@ {x1 + b[x1, x2], x2 - b[x1, b[x1, x2]], x1 + x2 - 2 b[x1, x2]}
```

alphabetagamma

{LS[$\overline{1}$, $\overline{12}$, 0, 0, ...], LS[$\overline{2}$, 0, $-\overline{1\overline{12}}$, 0, ...], LS[$\overline{1}+\overline{2}$, $-2\,\overline{12}$, 0, 0, ...]}

BracketExample

```
{b[α, β], b[α, b[β, γ]] + b[β, b[γ, α]] + b[γ, b[α, β]]}
```

BracketExample

{LS[0, $\overline{12}$, $\overline{\overline{12}2}$, $-\overline{1\overline{1\overline{12}}}$, ...], LS[0, 0, 0, 0, ...]}

bch

```
bch = BCH[LW@x, LW@y]
```

bch

LS[$\overline{x}+\overline{y}$, $\frac{\overline{xy}}{2}$, $\frac{1}{12}\overline{x\overline{xy}}+\frac{1}{12}\overline{\overline{xy}y}$, $\frac{1}{24}\overline{x\overline{\overline{xy}y}}$, ...]

bch16

```
Timing@{Length@(bch@16), (bch@16)〚1090 ;; 1092〛 // TopBracketForm}
```

bch16

$$\left\{53.633144,\ \left\{2181,\ \frac{53\ x\,x\,x\,\overline{x\,y}\,\overline{x\,y}\,x\,\overline{x\,y}\,x\,\overline{x\,y}\,\overline{x\,y}\,y}{1\,089\,728\,640} - \frac{17\ x\,x\,x\,\overline{x\,y}\,\overline{x\,y}\,x\,\overline{x\,y}\,x\,x\,\overline{x\,y}\,y\,y}{179\,625\,600} + \frac{389\ x\,x\,x\,\overline{x\,y}\,\overline{x\,y}\,x\,\overline{x\,y}\,\overline{x\,y}\,x\,\overline{x\,y}}{1\,320\,883\,200}\right\}\right\}$$

omegas

```
{ω1, ω2} = CWS /@ {CW[1] - 3 CW[2, 1, 1], CW[2] + CW[2, 2]}
```

omegas

$$\{\mathrm{CWS}[\widehat{1},\ 0,\ -3\,\widehat{112},\ 0,\ \ldots],\ \mathrm{CWS}[\widehat{2},\ \widehat{22},\ 0,\ 0,\ \ldots]\}$$

DegreeScale

```
DegreeScale[h] /@ {ω1, ω2}
```

DegreeScale

$$\{\mathrm{CWS}[h\,\widehat{1},\ 0,\ -3\,h^3\,\widehat{112},\ 0,\ \ldots],\ \mathrm{CWS}[h\,\widehat{2},\ h^2\,\widehat{22},\ 0,\ 0,\ \ldots]\}$$

TangentialDerivative

```
{λ = ⟨1 → α, 2 → β⟩, γ // Dλ}
```

TangentialDerivative

$$\{\langle 1\to \mathrm{LS}[\overline{1},\ \overline{12},\ 0,\ 0,\ \ldots],\ 2\to \mathrm{LS}[\overline{2},\ 0,\ -\overline{1\,\overline{12}},\ 0,\ \ldots]\rangle,\ \mathrm{LS}[0,\ 0,\ \overline{1\,\overline{12}},\ -\overline{1\,\overline{\overline{12}\,2}},\ \ldots]\}$$

tb

```
λ1 = λ; λ2 = ⟨1 → β, 2 → γ⟩; tb[λ1, λ2]
```

tb

$$\langle 1\to \mathrm{LS}[0,\ 0,\ \overline{1\,\overline{12}},\ -\overline{1\,\overline{\overline{12}\,2}},\ \ldots],\ 2\to \mathrm{LS}[0,\ 0,\ \overline{1\,\overline{12}},\ -\overline{1\,\overline{\overline{12}\,2}},\ \ldots]\rangle$$

tb2

```
lhs = D_tb[λ1,λ2][ω1]; rhs = b[Dλ1, Dλ2][ω1];
{lhs@{8}, (lhs ≡ rhs)@{8}}
```

tb2

$$\{\mathrm{CWS}[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 18\,\widehat{11112122} - 18\,\widehat{11112212} - 36\,\widehat{11121122} + 36\,\widehat{11122112},\ \ldots],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\}$$

TestingGammaODE

```
lhs = ∂t Γt[λ]; rhs = λ // e^(-t Dλ) // adSeries[ad/(e^ad - 1), Γt[λ]];
{Γ0[λ], lhs, (lhs ≡ rhs)@{6}}
```

TestingGammaODE

$$\{\langle 1\to \mathrm{LS}[0,\ 0,\ 0,\ 0,\ \ldots],\ 2\to \mathrm{LS}[0,\ 0,\ 0,\ 0,\ \ldots]\rangle,$$
$$\langle 1\to \mathrm{LS}[\overline{1},\ \overline{12},\ -t\,\overline{1\,\overline{12}},\ \frac{1}{4}t^2\,\overline{1\,\overline{1\,\overline{12}}} - t\,\overline{1\,\overline{\overline{12}\,2}},\ \ldots],$$
$$2\to \mathrm{LS}[\overline{2},\ 0,\ -\overline{1\,\overline{12}},\ -t\,\overline{1\,\overline{\overline{12}\,2}},\ \ldots]\rangle,\ \mathrm{BS}[7\ \mathrm{True},\ \ldots]\}$$

TestingGamma

$\{\gamma\ //\ e^{-t\,D_\lambda},\ \gamma\ //\ CC[\Gamma_t[\lambda]]\}$

TestingGamma

$\{LS[\overline{1}+\overline{2},\ -2\,\overline{12},\ -t\,\overline{1\,\overline{12}},\ t\,\overline{1\,\overline{\overline{12}2}},\ \ldots],\ LS[\overline{1}+\overline{2},\ -2\,\overline{12},\ -t\,\overline{1\,\overline{12}},\ t\,\overline{1\,\overline{\overline{12}2}},\ \ldots]\}$

TestingLambdaODE

$lhs = \partial_t\Lambda_t[\lambda];\ rhs = \lambda\ //\ e^{D_{\Lambda_t[\lambda]}}\ //\ adSeries\left[\frac{ad}{e^{ad}-1},\ \Lambda_t[\lambda],\ tb\right];$

$\{\Lambda_0[\lambda],\ lhs,\ (lhs \equiv rhs)@\{6\}\}$

TestingLambdaODE

$\{\langle 1 \to LS[0,\ 0,\ 0,\ 0,\ \ldots],\ 2 \to LS[0,\ 0,\ 0,\ 0,\ \ldots]\rangle,$

$\left\langle 1 \to LS\left[\overline{1},\ \overline{12},\ t\,\overline{1\,\overline{12}},\ \frac{1}{2}t^2\,\overline{1\,\overline{1\,\overline{12}}}+t\,\overline{1\,\overline{\overline{12}2}},\ \ldots\right],\ 2 \to LS\left[\overline{2},\ 0,\ -\overline{1\,\overline{12}},\ t\,\overline{1\,\overline{\overline{12}2}},\ \ldots\right]\right\rangle,$

$BS[7\ True,\ \ldots]\}$

TestingLambda

$\{\gamma\ //\ CC[t\,\lambda],\ \gamma\ //\ e^{-D_{\Lambda_t[\lambda]}}\}$

TestingLambda

$\{LS\left[\overline{1}+\overline{2},\ -2\,\overline{12},\ -t\,\overline{1\,\overline{12}},\ -\frac{1}{2}t^2\,\overline{1\,\overline{1\,\overline{12}}}+t\,\overline{1\,\overline{\overline{12}2}},\ \ldots\right],$

$LS\left[\overline{1}+\overline{2},\ -2\,\overline{12},\ -t\,\overline{1\,\overline{12}},\ -\frac{1}{2}t^2\,\overline{1\,\overline{1\,\overline{12}}}+t\,\overline{1\,\overline{\overline{12}2}},\ \ldots\right]\}$

Unclassified aside: an alternative formulation of Λ (on March 1, 2015, this took 61 Seconds):

$\lambda 2 = \langle 1 \to RandomLieSeries[\{1,\ 2\}],\ 2 \to RandomLieSeries[\{1,\ 2\}]\rangle;$

$\{lhs = \lambda 2\ //\ EulerE\ //\ adSeries\left[\frac{e^{ad}-1}{ad},\ \lambda 2\right]\ //\ RC[-\lambda 2],$

$rhs = \Lambda[\lambda 2]\ //\ EulerE\ //\ adSeries\left[\frac{e^{ad}-1}{ad},\ \Lambda[\lambda 2],\ tb\right];\ (lhs \equiv rhs)@\{8\}\}\ //\ Timing$

$\{60.949591,$

$\{\left\langle 1 \to LS\left[-\overline{1}-\overline{2},\ 4\,\overline{12},\ -\frac{23}{4}\overline{1\,\overline{12}}-\frac{23}{4}\overline{\overline{12}2},\ 11\,\overline{1\,\overline{1\,\overline{12}}}+\frac{37}{3}\overline{1\,\overline{\overline{12}2}}+\frac{39}{4}\overline{\overline{\overline{12}2}2},\ \ldots\right],\right.$

$\left. 2 \to LS\left[\overline{2},\ \overline{12},\ \frac{5}{2}\overline{1\,\overline{12}}+\frac{11}{4}\overline{\overline{12}2},\ \frac{11}{2}\overline{1\,\overline{1\,\overline{12}}}-10\,\overline{1\,\overline{\overline{12}2}}-\frac{35}{4}\overline{\overline{\overline{12}2}2},\ \ldots\right]\right\rangle,\ BS[$

$9\ True,\ \ldots]\}\}$

CCAndRC

$\{\alpha\ //\ CC_1[-\gamma],\ \alpha\ //\ CC_1[-\gamma]\ //\ RC_1[\gamma],\ \alpha\ //\ CC_1[-\gamma]\ //\ CC_1[\gamma]\}$

CCAndRC

$\{LS\left[\overline{1},\ 2\,\overline{12},\ -\frac{5}{2}\overline{1\,\overline{12}}+\frac{3}{2}\overline{\overline{12}2},\ \frac{7}{6}\overline{1\,\overline{1\,\overline{12}}}-\frac{23}{6}\overline{1\,\overline{\overline{12}2}}+\frac{2}{3}\overline{\overline{\overline{12}2}2},\ \ldots\right],$

$LS[\overline{1},\ \overline{12},\ 0,\ 0,\ \ldots],\ LS\left[\overline{1},\ \overline{12},\ -\overline{1\,\overline{12}},\ 2\,\overline{1\,\overline{1\,\overline{12}}}+\overline{1\,\overline{\overline{12}2}},\ \ldots\right]\}$

tru

$With[\{\gamma = b[b[LW@v,\ LW@u],\ LW@u]\},\ tr_u[\gamma]]\ //\ TopBracketForm$

tru

$-\overline{uv}$

divu

```
With[{γ = LW@u + b[b[LW@v, LW@u], LW@u]}, divu[γ]] // TopBracketForm
```

divu

$\overparen{u} - \overparen{uuv}$

Ju

```
J1[γ]
```

Ju

$\text{CWS}\left[\overparen{1}, \frac{5\,\overparen{12}}{2}, -\frac{7\,\overparen{112}}{6} + \frac{7\,\overparen{122}}{6}, \frac{3\,\overparen{1112}}{8} - \frac{11\,\overparen{1122}}{4} - \frac{3\,\overparen{1212}}{4} + \frac{3\,\overparen{1222}}{8}, \ldots\right]$

j

```
{div[λ]@{5}, j[λ]@{5}}
```

j

$\{\text{CWS}[\overparen{1} + \overparen{2}, -\overparen{12}, -\overparen{112}, 0, 0, \ldots],$
$\text{CWS}[\overparen{1} + \overparen{2}, -\overparen{12}, -\overparen{112}, -\overparen{1122} + \overparen{1212}, -\overparen{11122} + \overparen{11212}, \ldots]\}$

cocycle4j

```
lhs = j[BCHtb[λ1, λ2]]; rhs = j[λ1] + e^(Dλ1)[j[λ2]];
{lhs, (lhs ≡ rhs)@{8}}
```

cocycle4j

$\left\{\text{CWS}\left[2\,\overparen{2}, -2\,\overparen{12}, \frac{5\,\overparen{112}}{6} - \frac{8\,\overparen{122}}{3}, -\frac{7\,\overparen{1112}}{12} - \frac{35\,\overparen{1122}}{6} + \frac{23\,\overparen{1212}}{3} - \frac{25\,\overparen{1222}}{12}, \ldots\right], \text{BS}[9\,\text{True}, \ldots]\right\}$

```
lhs = j[BCHb[λ1, λ2]]; rhs = j[λ1] + e^(Dλ1)[j[λ2]];
{lhs, (lhs ≡ rhs)}
```

$\left\{\text{CWS}\left[2\,\overparen{2}, -\frac{3\,\overparen{12}}{2}, -\frac{\overparen{112}}{4} - \frac{17\,\overparen{122}}{6}, -\frac{11\,\overparen{1112}}{24} - \frac{49\,\overparen{1122}}{12} + \frac{71\,\overparen{1212}}{12} - \frac{35\,\overparen{1222}}{12}, \ldots\right],\right.$

$\text{BS}\left[2\,\text{True}, -\frac{3\,\overparen{12}}{2} == -2\,\overparen{12}, -\frac{3\,\overparen{12}}{2} == -2\,\overparen{12}\ \&\&\ -\frac{\overparen{112}}{4} - \frac{17\,\overparen{122}}{6} == \frac{5\,\overparen{112}}{6} - \frac{8\,\overparen{122}}{3},\right.$

$-\frac{3\,\overparen{12}}{2} == -2\,\overparen{12}\ \&\&\ -\frac{\overparen{112}}{4} - \frac{17\,\overparen{122}}{6} == \frac{5\,\overparen{112}}{6} - \frac{8\,\overparen{122}}{3}\ \&\&\ -\frac{11\,\overparen{1112}}{24} - \frac{49\,\overparen{1122}}{12} +$

$\left.\left.\frac{71\,\overparen{1212}}{12} - \frac{35\,\overparen{1222}}{12} == -\frac{7\,\overparen{1112}}{12} - \frac{35\,\overparen{1122}}{6} + \frac{23\,\overparen{1212}}{3} - \frac{25\,\overparen{1222}}{12}, \ldots\right]\right\}$

dj

```
ϵ /: ϵ^2 = 0;
{j[ϵ λ], j[ϵ λ] ≡ ϵ div[λ]}
```

dj

$\{\text{CWS}[\epsilon\,\overparen{1} + \epsilon\,\overparen{2}, -\epsilon\,\overparen{12}, -\epsilon\,\overparen{112}, 0, \ldots], \text{BS}[5\,\text{True}, \ldots]\}$

Section 2.3 - The [AT]-inspired presentation El of $A^w{}_{exp}$

ElSetup

```
x1 = LW[1]; x2 = LW[2];
{ζa =
   El[⟨1 → LS[x1 + b[x1, x2]], 2 → LS[x2 - b[x1, b[x1, x2]]]⟩, CWS[CW[1] - 3 CW[1, 2, 1]]],
  ζb = El[⟨1 → LS[x2 - b[x1, x2]], 2 → LS[x1 + x2 + b[x2, b[x1, x2]]]⟩,
    CWS[CW[2] - 2 CW[1, 2]]],
  ζc = El[⟨1 → LS[x1 - b[b[x1, x2], b[x1, x2]]], 2 → LS[x2 + 3 b[x1, b[x1, x2]]]⟩,
    CWS[CW[1] - 2 CW[1, 2] + CW[1, 2, 1]]]}
```

ElSetup

```
{El[⟨1 → LS[1, 12, 0, 0, ...], 2 → LS[2, 0, -112, 0, ...]⟩, CWS[1, 0, -3 112, 0, ...]],
 El[⟨1 → LS[2, -12, 0, 0, ...], 2 → LS[1 + 2, 0, -122, 0, ...]⟩,
  CWS[2, -2 12, 0, 0, ...]], El[
  ⟨1 → LS[1, 0, 0, 0, ...], 2 → LS[2, 0, 3 112, 0, ...]⟩, CWS[1, -2 12, 112, 0, ...]]}
```

ElAssociativity

```
lhs = ζa ** (ζb ** ζc); rhs = (ζa ** ζb) ** ζc;
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

ElAssociativity

```
{El[⟨1 → LS[2 1 + 2, 0, 1/2 112, ...], 2 → LS[1 + 3 2, 0, 5/2 112 - 122, ...]⟩,
  CWS[2 1 + 2, -4 12, -2 112, ...]], BS[9 True, ...]}
```

detaExample

```
{ζa // dη^1, ζa // dη^2}
```

detaExample

```
{El[⟨2 → LS[2, 0, 0, 0, ...]⟩, CWS[0, 0, 0, 0, ...]],
 El[⟨1 → LS[1, 0, 0, 0, ...]⟩, CWS[1, 0, 0, 0, ...]]}
```

dA1

```
{ζd = El[λ, CWS[0]], ζd // dA}
```

dA1

```
{El[⟨1 → LS[1, 12, 0, 0, ...], 2 → LS[2, 0, -112, 0, ...]⟩, CWS[0, 0, 0, 0, ...]],
 El[⟨1 → LS[-1, -12, 0, 0, ...], 2 → LS[-2, 0, 112, 0, ...]⟩,
  CWS[-1 - 2, 12, 112, 1122 - 1212, ...]]}
```

dA2

```
(ζd ≡ (ζd // dA // dA))@{8}
```

dA2

```
BS[9 True, ...]
```

dA3

```
lhs = (ξa ** ξb) // dA; rhs = (ξb // dA) ** (ξa // dA);
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

dA3

```
{El[⟨1 → LS[-1̄ - 2̄, 0, -1/2 1̄1̄2̄, ...], 2 → LS[-1̄ - 2 2̄, 0, 1/2 1̄1̄2̄ + 1̄2̄2̄, ...]⟩,
  CWS[-2̂, -2 12̂, -2 112̂ - 122̂, ...]], BS[9 True, ...]}
```

dS

```
ξd // dS
```

dS

```
El[⟨1 → LS[1̄, -1̄2̄, 0, 0, ...], 2 → LS[2̄, 0, -1̄1̄2̄, 0, ...]⟩,
 CWS[1̂ + 2̂, 12̂, -112̂, 1122̂ - 1212̂, ...]]
```

dD1

```
{ξa, ξa // dΔ[2, 2, 3]}
```

dD1

```
{El[⟨1 → LS[1̄, 1̄2̄, 0, 0, ...], 2 → LS[2̄, 0, -1̄1̄2̄, 0, ...]⟩, CWS[1̂, 0, -3 112̂, 0, ...]],
 El[⟨1 → LS[1̄, 1̄2̄ + 1̄3̄, 0, 0, ...], 2 → LS[2̄ + 3̄, 0, -1̄1̄2̄ - 1̄1̄3̄, 0, ...],
   3 → LS[2̄ + 3̄, 0, -1̄1̄2̄ - 1̄1̄3̄, 0, ...]⟩, CWS[1̂, 0, -3 112̂ - 3 113̂, 0, ...]]}
```

dD2

```
lhs = (ξa ** ξb) // dΔ[2, 2, 3]; rhs = (ξa // dΔ[2, 2, 3]) ** (ξb // dΔ[2, 2, 3]);
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

dD2

```
{El[⟨1 → LS[1̄ + 2̄ + 3̄, 0, 1/2 1̄1̄2̄ + 1/2 1̄1̄3̄, ...],
   2 → LS[1̄ + 2 2̄ + 2 3̄, 0, -1/2 1̄1̄2̄ - 1/2 1̄1̄3̄ - 1̄2̄3̄ - 1̄2̄2̄ - 2 1̄3̄2̄ - 1̄3̄3̄, ...],
   3 → LS[1̄ + 2 2̄ + 2 3̄, 0, -1/2 1̄1̄2̄ - 1/2 1̄1̄3̄ - 1̄2̄3̄ - 1̄2̄2̄ - 2 1̄3̄2̄ - 1̄3̄3̄, ...]⟩,
  CWS[1̂ + 2̂ + 3̂, -2 12̂ - 2 13̂, -3 112̂ - 3 113̂, ...]], BS[9 True, ...]}
```

Section 2.4 - The factored presentation Ef of $A^w{}_{exp}$ and its stronger precursor Es

EsSetup1

```
u = LW@"u"; v = LW@"v";
ξa = Es[⟨1 → LS[u + b[u, v]], 2 → LS[v - b[u, b[u, v]]], 3 → LS[u - b[b[u, v], b[u, v]]]⟩,
   CWS[CW["u"] - 3 CW["u", "v", "u"]]]
```

EsSetup1

```
Es[⟨1 → LS[ū, ūv̄, 0, 0, ...], 2 → LS[v̄, 0, -ūūv̄, 0, ...], 3 → LS[ū, 0, 0, 0, ...]⟩,
 CWS[û, 0, -3 uuv̂, 0, ...]]
```

EsSetup2

```
SeedRandom[0]; ξb = Es[
  ⟨Table[i → RandomLieSeries[{1, 2, 3, 4}], {i, 4}]⟩, RandomCWSeries[{1, 2, 3, 4}]];
ξb@
 {2}
```

EsSetup2

$$\begin{aligned}&\mathrm{Es}\Big[\Big\langle 1\to \mathrm{LS}\Big[-\overline{1}-2\,\overline{2}+2\,\overline{3}-2\,\overline{4},\ 2\,\overline{1\,2}+\frac{\overline{1\,3}}{2}+\overline{1\,4}-\frac{\overline{2\,3}}{2}-\frac{\overline{2\,4}}{2}+2\,\overline{3\,4},\ \ldots\Big],\\ &\quad 2\to \mathrm{LS}\Big[2\,\overline{1}-\overline{2}-2\,\overline{3}+\overline{4},\ 2\,\overline{1\,2}+\frac{3\,\overline{1\,3}}{2}-2\,\overline{1\,4}-\overline{2\,3}-\overline{2\,4}-\frac{\overline{3\,4}}{2},\ \ldots\Big],\\ &\quad 3\to \mathrm{LS}\Big[-\overline{1}+\overline{2}+2\,\overline{4},\ -2\,\overline{1\,2}+2\,\overline{1\,3}-\overline{1\,4}-\frac{3\,\overline{2\,3}}{2}+2\,\overline{2\,4}-2\,\overline{3\,4},\ \ldots\Big],\\ &\quad 4\to \mathrm{LS}\Big[-2\,\overline{1}+2\,\overline{2}+2\,\overline{3}+\overline{4},\ -\frac{\overline{1\,2}}{2}+\frac{3\,\overline{1\,3}}{2}-2\,\overline{2\,4}+\overline{3\,4},\ \ldots\Big]\Big\rangle,\\ &\quad \mathrm{CWS}\Big[\widehat{3}-\widehat{4},\ \frac{3\,\widehat{11}}{2}+\frac{3\,\widehat{12}}{2}-2\,\widehat{13}+\widehat{14}+\widehat{22}+2\,\widehat{23}-\frac{\widehat{24}}{2}-2\,\widehat{33}-\widehat{34}+\widehat{44},\ \ldots\Big]\Big]\end{aligned}$$

haction

```
lhs = ξa // hm[1, 2, 4] // tha[u, 4];
rhs = ξa // tha[u, 1] // tha[u, 2] // hm[1, 2, 4];
{lhs, (lhs ≡ rhs)@{8}}
```

haction

$$\begin{aligned}&\Big\{\mathrm{Es}\Big[\Big\langle 3\to \mathrm{LS}\Big[\overline{u},\ -\overline{u\,v},\ -\overline{u\,\overline{u\,v}}+\frac{1}{2}\,\overline{\overline{u\,v}\,v},\ \frac{3}{2}\,\overline{u\,\overline{u\,\overline{u\,v}}}+\overline{u\,\overline{\overline{u\,v}\,v}}-\frac{1}{6}\,\overline{\overline{\overline{u\,v}\,v}\,v},\ \ldots\Big],\\ &\quad 4\to \mathrm{LS}\Big[\overline{u}+\overline{v},\ \frac{\overline{u\,v}}{2},\ -\frac{23}{12}\,\overline{u\,\overline{u\,v}}-\frac{5}{12}\,\overline{\overline{u\,v}\,v},\ \overline{u\,\overline{u\,\overline{u\,v}}}+\frac{13}{24}\,\overline{u\,\overline{\overline{u\,v}\,v}}+\frac{1}{12}\,\overline{\overline{\overline{u\,v}\,v}\,v},\ \ldots\Big]\Big\rangle,\\ &\quad \mathrm{CWS}\Big[2\,\widehat{u},\ -\widehat{uv},\ -\frac{3\,\widehat{uuv}}{2},\ -\frac{\widehat{uuuv}}{6}+\widehat{uuvv}-\widehat{uvuv},\ \ldots\Big]\Big],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\Big\}\end{aligned}$$

metaassoc

```
lhs = ξb // dm[1, 2, 1] // dm[1, 3, 1]; rhs = ξb // dm[2, 3, 2] // dm[1, 2, 1];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

metaassoc

$$\begin{aligned}&\Big\{\mathrm{Es}\Big[\Big\langle 1\to \mathrm{LS}\Big[-2\,\overline{1}+\overline{4},\ -\frac{3\,\overline{1\,4}}{2},\ 20\,\overline{1\,\overline{1\,4}}-\frac{19}{3}\,\overline{\overline{1\,4}\,4},\ \ldots\Big],\\ &\quad 4\to \mathrm{LS}\Big[2\,\overline{1}+\overline{4},\ \overline{1\,4},\ -\frac{31}{2}\,\overline{1\,\overline{1\,4}}-\frac{13}{6}\,\overline{\overline{1\,4}\,4},\ \ldots\Big]\Big\rangle,\\ &\quad \mathrm{CWS}\Big[3\,\widehat{1}-\widehat{4},\ -3\,\widehat{11}+\frac{\widehat{14}}{2}+\widehat{44},\ \frac{71\,\widehat{111}}{4}+\frac{19\,\widehat{114}}{4}-\frac{7\,\widehat{144}}{6}-\frac{2\,\widehat{444}}{3},\ \ldots\Big]\Big],\ \mathrm{BS}[6\ \mathrm{True},\ \ldots]\Big\}\end{aligned}$$

Section 3.1 - Tangle Invariants

Section 3.1.1 - The General Framework

RDefs

```
Rl[a_, b_] := El[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRl[a_, b_] := El[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
```

R3

```
lhs = Rl[1, 2] ** Rl[1, 3] ** Rl[2, 3]; rhs = Rl[2, 3] ** Rl[1, 3] ** Rl[1, 2];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

R3

$$\{\mathrm{El}[\langle 1 \to \mathrm{LS}[0, 0, 0, \ldots], 2 \to \mathrm{LS}[\overline{1}, 0, 0, \ldots], 3 \to \mathrm{LS}[\overline{1} + \overline{2}, 0, 0, \ldots]\rangle, \mathrm{CWS}[0, 0, 0, \ldots]], \mathrm{BS}[6\ \mathrm{True}, \ldots]\}$$

Section 3.1.2 - The Knot 8_{17} and the Borromean Tangle

817

```
t1 = iRs[12, 1] iRs[2, 7] iRs[8, 3] iRs[4, 11] Rs[16, 5] Rs[6, 13] Rs[14, 9] Rs[10, 15];
Do[t1 = t1 // dm[1, k, 1], {k, 2, 16}];
t1@{6}
```

817

$$\mathrm{Es}\left[\langle 1 \to \mathrm{LS}[0, 0, 0, 0, 0, 0, \ldots]\rangle, \mathrm{CWS}\left[0, -\widehat{11}, 0, -\frac{31\,\widehat{1111}}{12}, 0, -\frac{1351\,\widehat{111111}}{360}, \ldots\right]\right]$$

Borromean

```
t2 = iRs[r, 6] Rs[2, 4] iRs[g, 9] Rs[5, 7] iRs[b, 3] Rs[8, 1];
(Do[t2 = t2 // dm[r, k, r], {k, 1, 3}]; Do[t2 = t2 // dm[g, k, g], {k, 4, 6}];
 Do[t2 = t2 // dm[b, k, b], {k, 7, 9}]; t2)
```

Borromean

$$\mathrm{Es}\Big[\Big\langle b \to \mathrm{LS}\Big[0, \overline{g r}, \frac{1}{2}\overline{g\overline{g r}} + \overline{\overline{b r} g} + \frac{1}{2}\overline{\overline{g r} r},$$
$$-\frac{1}{2}\overline{b\overline{\overline{b r} g}} + \frac{1}{6}\overline{g\overline{g\overline{g r}}} + \frac{1}{4}\overline{g\overline{\overline{g r} r}} - \frac{1}{2}\overline{\overline{b g}\,\overline{b r}} - \frac{1}{2}\overline{\overline{\overline{b r} g} g} - \frac{1}{2}\overline{\overline{\overline{b r} r} g} + \frac{1}{6}\overline{\overline{\overline{g r} r} r}, \ldots\Big], g \to$$
$$\mathrm{LS}\Big[0, -\overline{b r}, \frac{1}{2}\overline{b\overline{b r}} - \overline{b\overline{g r}} - \overline{\overline{b r} g} + \frac{1}{2}\overline{\overline{b r} r}, -\frac{1}{6}\overline{b\overline{b\overline{b r}}} - \frac{1}{2}\overline{b\overline{b\overline{g r}}} - \frac{1}{2}\overline{b\overline{g\overline{g r}}} - \frac{1}{2}\overline{b\overline{\overline{b r} g}} -$$
$$\frac{1}{4}\overline{b\overline{\overline{b r} r}} + \frac{1}{2}\overline{b\overline{\overline{g r} r}} + \frac{1}{2}\overline{\overline{b g}\,\overline{b r}} + \overline{\overline{b r}\,\overline{g r}} - \overline{\overline{b\overline{g r}} g} - \frac{1}{2}\overline{\overline{\overline{b r} g} g} + \frac{1}{2}\overline{\overline{\overline{b r} r} g} - \frac{1}{6}\overline{\overline{\overline{b r} r} r}, \ldots\Big],$$
$$r \to \mathrm{LS}\Big[0, \overline{b g}, \frac{1}{2}\overline{b\overline{b g}} + \overline{b\overline{g r}} + \frac{1}{2}\overline{\overline{b g} g}, \frac{1}{6}\overline{b\overline{b\overline{b g}}} + \frac{1}{2}\overline{b\overline{b\overline{g r}}} +$$
$$\frac{1}{2}\overline{b\overline{g\overline{g r}}} + \frac{1}{4}\overline{b\overline{\overline{b g} g}} + \frac{1}{2}\overline{b\overline{\overline{g r} r}} + \frac{1}{6}\overline{\overline{\overline{b g} g} g}, \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\big[0, 0, 2\,\widehat{bgr}, \widehat{bbgr} - \widehat{bgbr} + \widehat{bggr} - \widehat{bgrg} + \widehat{bgrr} - \widehat{brgr}, \ldots\big]\Big]$$

Section 3.2 - Solutions of the Kashiwara-Vergne Equations

Continues pensieve://2013-10/SolvingWKO.nb.

VSetup

```
α = LS[{x, y}, αs]; β = LS[{x, y}, βs]; γ = CWS[{x, y}, γs];
V = Es[⟨x → α, y → β⟩, γ];
```

CapSetup

```
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
```

VCapEqns

```
R4Eqn = V ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V;
UnitarityEqn = (V ** (V // dA[x] // dA[y]) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]]);
CapEqn = ((V ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap (Cap // dσ[x, y]) // dc[x] // dc[y]));
```

VCapSolution

```
αs[x] = 0; αs[y] = -1/2;
SeriesSolve[{α, β, γ, κ}, (ℏ^-1 R4Eqn) && UnitarityEqn && CapEqn];
{V, κ}
```

VCapSolution

Arbitrarily setting {κs[x] → 0}.

VCapSolution

Arbitrarily setting {αs[x, y, y] → 0}.

VCapSolution

$$\Big\{\mathrm{Es}\Big[\Big\langle x \to \mathrm{LS}\Big[-\frac{\overline{y}}{2}, \frac{\overline{xy}}{12}, 0, -\frac{1}{720}\overline{x\overline{x\overline{xy}}} + \frac{1}{720}\overline{x\overline{\overline{xy}y}} - \frac{\overline{\overline{\overline{xy}y}y}}{5760}, \ldots\Big],$$

$$y \to \mathrm{LS}\Big[0, \frac{\overline{xy}}{24}, 0, -\frac{\overline{x\overline{x\overline{xy}}}}{1440} + \frac{7\,\overline{x\overline{\overline{xy}y}}}{5760} - \frac{7\,\overline{\overline{\overline{xy}y}y}}{5760}, \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[0, -\frac{\widehat{xy}}{48}, 0, \frac{\widehat{xxxy}}{2880} + \frac{\widehat{xxyy}}{2880} + \frac{\widehat{xyxy}}{5760} + \frac{\widehat{xyyy}}{2880}, \ldots\Big]\Big], \mathrm{CWS}\Big[0, -\frac{\widehat{xx}}{96}, 0, \frac{\widehat{xxxx}}{11\,520}, \ldots\Big]\Big\}$$

LambdaV

```
Λ[V]
```

LambdaV

$$\mathrm{El}\Big[\Big\langle x \to \mathrm{LS}\Big[-\frac{\overline{y}}{2}, \frac{\overline{xy}}{12}, -\frac{1}{96}\overline{\overline{xy}y}, -\frac{1}{720}\overline{x\overline{x\overline{xy}}} + \frac{1}{320}\overline{x\overline{\overline{xy}y}} - \frac{1}{960}\overline{\overline{\overline{xy}y}y}, \ldots\Big],$$

$$y \to \mathrm{LS}\Big[0, \frac{\overline{xy}}{24}, -\frac{1}{96}\overline{\overline{xy}y}, -\frac{\overline{x\overline{x\overline{xy}}}}{1440} + \frac{1}{480}\overline{x\overline{\overline{xy}y}} - \frac{\overline{\overline{\overline{xy}y}y}}{2880}, \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[0, -\frac{\widehat{xy}}{48}, 0, \frac{\widehat{xxxy}}{2880} + \frac{\widehat{xxyy}}{2880} + \frac{\widehat{xyxy}}{5760} + \frac{\widehat{xyyy}}{2880}, \ldots\Big]\Big]$$

From the AT
paper:

THEOREM 5.8. *An element $F \in \mathrm{TAut}_2$ is a solution of the generalized* KV *problem if and only if $u = \kappa(F) = (A(x,y), B(x,y))$ satisfies the following two properties:*

$$x + y - \mathrm{ch}(y,x) = (1 - \exp(-\mathrm{ad}_x))A(x,y) + (\exp(\mathrm{ad}_y) - 1)B(x,y) \tag{21}$$

and

$$\mathrm{div}(u) \in \mathrm{im}(\tilde{\delta}). \tag{22}$$

logF

```
logF = Λ[V]⟦1⟧ // dσ[{x, y} → {y, x}]
```

logF

$$\left\langle x \to \mathrm{LS}\left[0, -\frac{\overline{xy}}{24}, -\frac{1}{96}\overline{x\overline{xy}}, \frac{\overline{x\overline{x\overline{xy}}}}{2880} - \frac{1}{480}\overline{x\overline{\overline{xy}y}} + \frac{\overline{\overline{\overline{xy}y}y}}{1440}, \ldots\right], \right.$$
$$\left. y \to \mathrm{LS}\left[-\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, -\frac{1}{96}\overline{x\overline{xy}}, \frac{1}{960}\overline{x\overline{x\overline{xy}}} - \frac{1}{320}\overline{x\overline{\overline{xy}y}} + \frac{1}{720}\overline{\overline{\overline{xy}y}y}, \ldots\right]\right\rangle$$

atkv

```
atkv = logF // EulerE // adSeries[(e^ad - 1)/ad, logF, tb];
{f = atkv_x, g = atkv_y}
```

atkv

$$\left\{\mathrm{LS}\left[0, -\frac{\overline{xy}}{12}, -\frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} - \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \ldots\right], \right.$$
$$\left. \mathrm{LS}\left[-\frac{\overline{x}}{2}, -\frac{\overline{xy}}{6}, -\frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} - \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \ldots\right]\right\}$$

```
logF1 = V⟦1⟧ // dσ[{x, y} → {y, x}];
atkv1 = logF1 // EulerE // adSeries[(e^ad - 1)/ad, logF1] // RC[-logF1]
```

$$\left\langle x \to \mathrm{LS}\left[0, -\frac{\overline{xy}}{12}, -\frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} - \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \ldots\right], \right.$$
$$\left. y \to \mathrm{LS}\left[-\frac{\overline{x}}{2}, -\frac{\overline{xy}}{6}, -\frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} - \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \ldots\right]\right\rangle$$

On March 1, 2015, the following took 74 Seconds:

```
(atkv ≡ atkv1)@{8} // Timing
```

```
Arbitrarily setting {αs[x, x, x, y, y] → 0}.
Arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.
Arbitrarily setting {αs[x, x, x, x, y, x, y, y] → 0}.
{89.809776, BS[9 True, ...]}
```

```
LS[LW@x + LW@y] - BCH[LW@y, LW@x] ≡ A - B - Ad[-LW@x][A] + Ad[LW@y][B]
```

```
BS[5 True, ...]
```

```
ϕ = CWS[{x}, ϕs];
SeriesSolve[ϕ,
  (divx[f] + divy[g] ≡
    ϕ + LieMorphism[LW@x → LW@y][ϕ] - LieMorphism[LW@x → BCH[LW@x, LW@y]][ϕ])
 ];
ϕ@{8}
```

Arbitrarily setting {ϕs[x] → 0}.

$\text{CWS}\left[0, \frac{\widehat{xx}}{24}, 0, -\frac{\widehat{xxxx}}{1440}, 0, \frac{\widehat{xxxxxx}}{60\,480}, 0, -\frac{\widehat{xxxxxxxx}}{2\,419\,200}, \ldots\right]$

```
{A1 = LS[{x, y}, as], B1 = LS[{x, y}, bs], ϕ1 = CWS[{x}, ϕ1s]};
SeriesSolve[{A1, B1, ϕ1},
  ℏ^-1 (LS[LW@x + LW@y] - BCH[LW@y, LW@x] ≡ A1 - B1 - Ad[-LW@x][A1] + Ad[LW@y][B1])
   && (divx[A1] + divy[B1] ≡
     ϕ1 + LieMorphism[LW@x → LW@y][ϕ1] - LieMorphism[LW@x → BCH[LW@x, LW@y]][ϕ1])
 ];
{A1, B1, ϕ1}
```

Arbitrarily setting {as[y] → 0, ϕ1s[x] → 0}.

Arbitrarily setting {as[x, y, y] → 0}.

$\left\{\text{LS}\left[0, -\frac{\overline{xy}}{12}, -\frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} - \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \ldots\right],\right.$

$\text{LS}\left[-\frac{\overline{x}}{2}, -\frac{\overline{xy}}{6}, -\frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} - \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \ldots\right],$

$\left.\text{CWS}\left[0, \frac{\widehat{xx}}{24}, 0, -\frac{\widehat{xxxx}}{1440}, \ldots\right]\right\}$

On March 1, 2015, the following took 64 Seconds:

```
ϕ1@{11} // Timing
```

Arbitrarily setting {as[x, x, x, y, y] → 0}.

Arbitrarily setting {as[x, x, x, x, x, y, y] → 0}.

Arbitrarily setting {as[x, x, x, x, y, x, y, y] → 0}.

Arbitrarily setting {as[x, x, x, x, x, x, x, y, y] → 0}.

Arbitrarily setting {as[x, x, x, x, x, x, y, x, y, y] → 0}.

Arbitrarily setting
 {as[x, x, x, x, x, x, x, x, x, y, y] → 0, as[x, x, x, x, x, x, y, x, y, y, y] → 0}.

$\left\{63.617208, \text{CWS}\left[0, \frac{\widehat{xx}}{24}, 0, -\frac{\widehat{xxxx}}{1440}, -\frac{\widehat{xxxxx}}{1200}, \frac{\widehat{xxxxxx}}{60\,480},\right.\right.$

$\left.\left.\frac{\widehat{xxxxxxx}}{14\,112}, -\frac{\widehat{xxxxxxxx}}{2\,419\,200}, -\frac{\widehat{xxxxxxxxx}}{172\,800}, \frac{\widehat{xxxxxxxxxx}}{95\,800\,320}, \frac{17\,\widehat{xxxxxxxxxxx}}{35\,126\,784}, \ldots\right]\right\}$

Conjecture. *For any Lie algebra $\mathfrak{g}$ of finite dimension, we can find F and G such that they satisfy*

a) $x+y-\log e^y e^x = (1-e^{-\operatorname{ad} x})F + (e^{\operatorname{ad} y}-1)G.$

b) *F and G give $\mathfrak{g}$-valued convergent power series on $(x,y)\in\mathfrak{g}\times\mathfrak{g}$.*

c) $\operatorname{tr}((\operatorname{ad} x)(\partial_x F);\mathfrak{g}) + \operatorname{tr}((\operatorname{ad} y)(\partial_y G);\mathfrak{g})$

$$= \tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1} + \frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1} - \frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1} - 1;\mathfrak{g}\right).$$

Here $z=\log e^x e^y$ and $\partial_x F$ (resp. $\partial_y G$) is the $\mathbf{End}(\mathfrak{g})$-valued real analytic function defined by

$$\mathfrak{g}\ni a\mapsto \frac{d}{dt}F(x+ta,y)|_{t=0} \quad \left(\text{resp. } \mathfrak{g}\ni a\mapsto \frac{d}{dt}G(x,y+ta)|_{t=0}\right),$$

and tr *denotes the trace of an endomorphism of $\mathfrak{g}$.*

On March 1, 2015, the following took 379 seconds:

KVTest

$$\left(\hbar^{-1}\,(\text{LS}[\text{LW@x}+\text{LW@y}]-\text{BCH}[\text{LW@y},\text{LW@x}] \equiv f-g-\text{Ad}[-\text{LW@x}][f]+\text{Ad}[\text{LW@y}][g])\ \&\&\ \text{div}_x[f]+\text{div}_y[g] \equiv \frac{1}{2}\,\text{tr}_u\left[\text{adSeries}\left[\frac{\text{ad}}{e^{\text{ad}}-1},\ \text{LW@x}\right][\text{LW@u}]+\text{adSeries}\left[\frac{\text{ad}}{e^{\text{ad}}-1},\ \text{LW@y}\right][\text{LW@u}]-\text{adSeries}\left[\frac{\text{ad}}{e^{\text{ad}}-1},\ \text{BCH}[\text{LW@x},\text{LW@y}]\right][\text{LW@u}]\right]\right)\text{@}\{8\}\ \text{//}\ \text{Timing}$$

KVTest

Arbitrarily setting {αs[x, x, x, y, y] → 0}.

KVTest

Arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.

KVTest

Arbitrarily setting {αs[x, x, x, x, y, x, y, y] → 0}.

KVTest

Arbitrarily setting {αs[x, x, x, x, x, x, x, y, y] → 0}.

KVTest

{480.951083, BS[9 True, ...]}

KVDirect

$$\{F=\text{LS}[\{x,y\},\ \text{Fs}],\ G=\text{LS}[\{x,y\},\ \text{Gs}]\};$$
$$\text{SeriesSolve}\Big[\{F,G\},\ \hbar^{-1}\,(\text{LS}[\text{LW@x}+\text{LW@y}]-\text{BCH}[\text{LW@y},\text{LW@x}] \equiv F-G-\text{Ad}[-\text{LW@x}][F]+\text{Ad}[\text{LW@y}][G])\ \&\&\ \text{div}_x[F]+\text{div}_y[G] \equiv \frac{1}{2}\,\text{tr}_u\left[\text{adSeries}\left[\frac{\text{ad}}{e^{\text{ad}}-1},\ \text{LW@x}\right][\text{LW@u}]+\text{adSeries}\left[\frac{\text{ad}}{e^{\text{ad}}-1},\ \text{LW@y}\right][\text{LW@u}]-\text{adSeries}\left[\frac{\text{ad}}{e^{\text{ad}}-1},\ \text{BCH}[\text{LW@x},\text{LW@y}]\right][\text{LW@u}]\right]\Big]$$

KVDirect

Arbitrarily setting {Fs[y] → 0}.

KVDirect

$$\left\{\text{LS}\left[0,\ -\frac{\overline{xy}}{12},\ -\frac{1}{24}\overline{x\overline{xy}},\ -\frac{1}{180}\overline{x\overline{x\overline{xy}}}-\frac{1}{120}\overline{x\overline{\overline{xy}y}}+\frac{1}{360}\overline{\overline{\overline{xy}y}y},\ \ldots\right],\right.$$
$$\left.\text{LS}\left[-\frac{\overline{x}}{2},\ -\frac{\overline{xy}}{6},\ -\frac{1}{24}\overline{x\overline{xy}},\ -\frac{1}{360}\overline{x\overline{x\overline{xy}}}-\frac{1}{80}\overline{x\overline{\overline{xy}y}}+\frac{1}{180}\overline{\overline{\overline{xy}y}y},\ \ldots\right]\right\}$$

```
TrueQ[atkv ≡ ⟨x → F, y → G⟩]@{8}
```

```
Arbitrarily setting {fs[x, x, x, x, y, x, y, y] → 0}.
```

```
BS[8 True, False, ...]
```

On March 1, 2015, a complete run of this notebook took 712 seconds.

```
TimeUsed[]
```

```
711.926
```