Pensieve header: Calculations appearing in the WKO4 paper.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
```

Section 1 - Introduction

Initialization

```
<< FreeLie.m;
<< AwCalculus.m;
$SeriesShowDegree = 4;
```

Initialization

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords,
  AllLyndonWords, AllWords, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS,
  CC, Crop, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, EulerE,
  Exp, InvertLieMorphism, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW,
  LyndonFactorization, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve,
  Support, tb, TopBracketForm, tr, UndeterminedCoefficients, Γ, ι, Λ, ℏ, ⌐, ⌢}.
```

Initialization

```
AwCalculus` implements / extends {*, **, E, ≡, dA, dc, deg,
  dm, dS, dΔ, dη, dσ, E₁, Eₛ, hA, hm, hS, hη, hσ, tA, tha, tm, tS, tσ, Γ, Λ}.
```

Section 2.2 - Some Preliminaries on Lie Algebras and Cyclic Words

alphabetagamma

```
x₁ = LW[1]; x₂ = LW[2];
{α, β, γ} = LS /@ {x₁ + b[x₁, x₂], x₂ - b[x₁, b[x₁, x₂]], x₁ + x₂ - 2 b[x₁, x₂]}
```

alphabetagamma

$$\{\mathrm{LS}[\overline{1},\ \overline{12},\ 0,\ 0,\ \ldots],\ \mathrm{LS}[\overline{2},\ 0,\ -\overline{1\,\overline{12}},\ 0,\ \ldots],\ \mathrm{LS}[\overline{1}+\overline{2},\ -2\,\overline{12},\ 0,\ 0,\ \ldots]\}$$

BracketExample

```
{b[α, β], b[α, b[β, γ]] + b[β, b[γ, α]] + b[γ, b[α, β]]}
```

BracketExample

$$\{\mathrm{LS}[0,\ \overline{12},\ \overline{\overline{12}\,2},\ -\overline{1\,\overline{1\,\overline{12}}},\ \ldots],\ \mathrm{LS}[0,\ 0,\ 0,\ 0,\ \ldots]\}$$

bch

```
bch = BCH[LW@x, LW@y]
```

bch

$$\mathrm{LS}\left[\overline{x}+\overline{y},\ \frac{\overline{xy}}{2},\ \frac{1}{12}\overline{x\,\overline{xy}}+\frac{1}{12}\overline{\overline{xy}\,y},\ \frac{1}{24}\overline{x\,\overline{\overline{xy}\,y}},\ \ldots\right]$$

bch16

```
Timing@{Length@(bch@16), (bch@16)[[1090 ;; 1092]] // TopBracketForm}
```

bch16

```
{45.474291, {2181, (53 x x x x y x y x x y x x y x y y)/1089728640 -
   (17 x x x x y x y x x y x x x y y y)/179625600 + (389 x x x x y x y x x y x x y x x y)/1320883200}}
```

omegas

```
{ω1, ω2} = CWS /@ {CW["1"] - 3 CW["211"], CW["2"] + CW["22"]}
```

omegas

```
{CWS[1, 0, -3 112, 0, ...], CWS[2, 22, 0, 0, ...]}
```

DegreeScale

```
DegreeScale[h] /@ {ω1, ω2}
```

DegreeScale

```
{CWS[h 1, 0, -3 h^3 112, 0, ...], CWS[h 2, h^2 22, 0, 0, ...]}
```

TangentialDerivative

```
{λ = ⟨1 → α, 2 → β⟩, γ // Dλ}
```

TangentialDerivative

```
{⟨1 → LS[1, 12, 0, 0, ...], 2 → LS[2, 0, -1 1 2, 0, ...]⟩, LS[0, 0, 1 1 2, -1 1 2 2, ...]}
```

tb

```
λ1 = λ; λ2 = ⟨1 → β, 2 → γ⟩; tb[λ1, λ2]
```

tb

```
⟨1 → LS[0, 0, 1 1 2, -1 1 2 2, ...], 2 → LS[0, 0, 1 1 2, -1 1 2 2, ...]⟩
```

tb2

```
lhs = D_tb[λ1,λ2][ω1]; rhs = b[Dλ1, Dλ2][ω1];
{lhs@{8}, (lhs ≡ rhs)@{8}}
```

tb2

```
{CWS[0, 0, 0, 0, 0, 0, 0, 18 11112122 - 18 11112212 - 36 11121122 + 36 11122112, ...],
 BS[9 True, ...]}
```

TestingGammaODE

```
lhs = ∂t Γt[λ]; rhs = λ // e^(-t Dλ) // adSeries[ad/(e^ad - 1), Γt[λ]];
{Γ0[λ], lhs, (lhs ≡ rhs)@{6}}
```

TestingGammaODE

```
{⟨1 → LS[0, 0, 0, 0, ...], 2 → LS[0, 0, 0, 0, ...]⟩,
 ⟨1 → LS[1, 12, -t 1 1 2, 1/4 t^2 1 1 1 2 - t 1 1 2 2, ...],
  2 → LS[2, 0, -1 1 2, -t 1 1 2 2, ...]⟩, BS[7 True, ...]}
```

TestingGamma

$\{\gamma\ //\ e^{-t\,D_\lambda},\ \gamma\ //\ CC[\Gamma_t[\lambda]]\}$

TestingGamma

$\{LS[\overline{1}+\overline{2},\ -2\,\overline{12},\ -t\,\overline{1\overline{12}},\ t\,\overline{1\overline{\overline{12}2}},\ \ldots],\ LS[\overline{1}+\overline{2},\ -2\,\overline{12},\ -t\,\overline{1\overline{12}},\ t\,\overline{1\overline{\overline{12}2}},\ \ldots]\}$

TestingLambdaODE

$\text{lhs} = \partial_t\Lambda_t[\lambda];\ \text{rhs} = \lambda\ //\ e^{D_{\Lambda_t[\lambda]}}\ //\ \text{adSeries}\left[\frac{ad}{e^{ad}-1},\ \Lambda_t[\lambda],\ tb\right];$

$\{\Lambda_0[\lambda],\ \text{lhs},\ (\text{lhs} \equiv \text{rhs})@\{6\}\}$

TestingLambdaODE

$\{\langle 1 \to LS[0,\ 0,\ 0,\ 0,\ \ldots],\ 2 \to LS[0,\ 0,\ 0,\ 0,\ \ldots]\rangle,$
$\left\langle 1 \to LS\left[\overline{1},\ \overline{12},\ t\,\overline{1\overline{12}},\ \frac{1}{2}t^2\,\overline{1\overline{1\overline{12}}} + t\,\overline{1\overline{\overline{12}2}},\ \ldots\right],\ 2 \to LS[\overline{2},\ 0,\ -\overline{1\overline{12}},\ t\,\overline{1\overline{\overline{12}2}},\ \ldots]\right\rangle,$
$BS[7\ \text{True},\ \ldots]\}$

TestingLambda

$\{\gamma\ //\ CC[t\,\lambda],\ \gamma\ //\ e^{-D_{\Lambda_t[\lambda]}}\}$

TestingLambda

$\{LS[\overline{1}+\overline{2},\ -2\,\overline{12},\ -t\,\overline{1\overline{12}},\ -\frac{1}{2}t^2\,\overline{1\overline{1\overline{12}}} + t\,\overline{1\overline{\overline{12}2}},\ \ldots],$
$LS[\overline{1}+\overline{2},\ -2\,\overline{12},\ -t\,\overline{1\overline{12}},\ -\frac{1}{2}t^2\,\overline{1\overline{1\overline{12}}} + t\,\overline{1\overline{\overline{12}2}},\ \ldots]\}$

CCAndRC

$\{\alpha\ //\ CC_1[-\gamma],\ \alpha\ //\ CC_1[-\gamma]\ //\ RC_1[\gamma],\ \alpha\ //\ CC_1[-\gamma]\ //\ CC_1[\gamma]\}$

CCAndRC

$\{LS[\overline{1},\ 2\,\overline{12},\ -\frac{5}{2}\,\overline{1\overline{12}} + \frac{3}{2}\,\overline{\overline{12}2},\ \frac{7}{6}\,\overline{1\overline{1\overline{12}}} - \frac{23}{6}\,\overline{1\overline{\overline{12}2}} + \frac{2}{3}\,\overline{\overline{\overline{12}2}2},\ \ldots],$
$LS[\overline{1},\ \overline{12},\ 0,\ 0,\ \ldots],\ LS[\overline{1},\ \overline{12},\ -\overline{1\overline{12}},\ 2\,\overline{1\overline{1\overline{12}}} + \overline{1\overline{\overline{12}2}},\ \ldots]\}$

divu

$\text{With}[\{\gamma = LW@u + b[b[LW@v,\ LW@u],\ LW@u]\},\ \text{div}_u[\gamma]]\ //\ \text{TopBracketForm}$

divu

$\widehat{u} - \widehat{uuv}$

Ju

$J_1[\gamma]$

Ju

$CWS\left[\widehat{1},\ \frac{5\,\widehat{12}}{2},\ -\frac{7\,\widehat{112}}{6} + \frac{7\,\widehat{122}}{6},\ \frac{3\,\widehat{1112}}{8} - \frac{11\,\widehat{1122}}{4} - \frac{3\,\widehat{1212}}{4} + \frac{3\,\widehat{1222}}{8},\ \ldots\right]$

j

$\{\text{div}[\lambda]@\{5\},\ j[\lambda]@\{5\}\}$

j

$\{CWS[\widehat{1}+\widehat{2},\ -\widehat{12},\ -\widehat{112},\ 0,\ 0,\ \ldots],$
$CWS[\widehat{1}+\widehat{2},\ -\widehat{12},\ -\widehat{112},\ -\widehat{1122}+\widehat{1212},\ -\widehat{11122}+\widehat{11212},\ \ldots]\}$

cocycle4j

$\text{lhs} = j[BCH_{tb}[\lambda 1,\ \lambda 2]];\ \text{rhs} = j[\lambda 1] + e^{D_{\lambda 1}}[j[\lambda 2]];$

$\{\text{lhs},\ (\text{lhs} \equiv \text{rhs})@\{8\}\}$

cocycle4j

$\{CWS[\widehat{1}+2\,\widehat{2},\ -3\,\widehat{12},\ 0,\ -9\,\widehat{1122}+9\,\widehat{1212},\ \ldots],\ BS[9\ \text{True},\ \ldots]\}$

```
lhs = j[BCHb[λ1, λ2]]; rhs = j[λ1] + e^(Dλ1)[j[λ2]];
{lhs, (lhs ≡ rhs)}
```

$\{\mathrm{CWS}[\widehat{1} + 2\,\widehat{2}, -4\,\widehat{12}, -\frac{5\,\widehat{122}}{12}, \widehat{1112} - \frac{101\,\widehat{1122}}{6} + \frac{53\,\widehat{1212}}{3} - \frac{\widehat{1222}}{24}, \ldots],$
$\mathrm{BS}[2\,\mathrm{True}, -4\,\mathrm{CW}[12] == -3\,\mathrm{CW}[12], -4\,\mathrm{CW}[12] == -3\,\mathrm{CW}[12]\ \&\&\ -\frac{5\,\mathrm{CW}[122]}{12} == 0,$
$-4\,\mathrm{CW}[12] == -3\,\mathrm{CW}[12]\ \&\&\ -\frac{5\,\mathrm{CW}[122]}{12} == 0\ \&\&$
$\mathrm{CW}[1112] - \frac{101\,\mathrm{CW}[1122]}{6} + \frac{53\,\mathrm{CW}[1212]}{3} - \frac{\mathrm{CW}[1222]}{24} == -9\,\mathrm{CW}[1122] + 9\,\mathrm{CW}[1212], \ldots]\}$

dj

```
ϵ /: ϵ^2 = 0;
{j[ϵ λ], j[ϵ λ] ≡ ϵ div[λ]}
```

dj

$\{\mathrm{CWS}[\epsilon\,\widehat{1} + \epsilon\,\widehat{2}, -\epsilon\,\widehat{12}, -\epsilon\,\widehat{112}, 0, \ldots], \mathrm{BS}[5\,\mathrm{True}, \ldots]\}$

Section 2.3 - The [AT]-inspired presentation E_l of $A^w{}_{exp}$

ElSetup

```
x1 = LW[1]; x2 = LW[2];
{ζa =
  El[⟨1 → LS[x1 + b[x1, x2]], 2 → LS[x2 - b[x1, b[x1, x2]]]⟩, CWS[CW["1"] - 3 CW["121"]]],
 ζb = El[⟨1 → LS[x2 - b[x1, x2]], 2 → LS[x1 + x2 + b[x2, b[x1, x2]]]⟩,
   CWS[CW["2"] - 2 CW["12"]]],
 ζc = El[⟨1 → LS[x1 - b[b[x1, x2], b[x1, x2]]], 2 → LS[x2 + 3 b[x1, b[x1, x2]]]⟩,
   CWS[CW["1"] - 2 CW["12"] + CW["121"]]]}
```

ElSetup

$\{\mathrm{E_l}[\langle 1 \to \mathrm{LS}[\overline{1}, \overline{12}, 0, 0, \ldots], 2 \to \mathrm{LS}[\overline{2}, 0, -\overline{1\overline{12}}, 0, \ldots]\rangle, \mathrm{CWS}[\widehat{1}, 0, -3\,\widehat{112}, 0, \ldots]],$
$\mathrm{E_l}[\langle 1 \to \mathrm{LS}[\overline{2}, -\overline{12}, 0, 0, \ldots], 2 \to \mathrm{LS}[\overline{1} + \overline{2}, 0, -\overline{\overline{12}2}, 0, \ldots]\rangle,$
$\mathrm{CWS}[\widehat{2}, -2\,\widehat{12}, 0, 0, \ldots]],$
$\mathrm{E_l}[\langle 1 \to \mathrm{LS}[\overline{1}, 0, 0, 0, \ldots], 2 \to \mathrm{LS}[\overline{2}, 0, 3\,\overline{1\overline{12}}, 0, \ldots]\rangle, \mathrm{CWS}[\widehat{1}, -2\,\widehat{12}, \widehat{112}, 0, \ldots]]\}$

ElAssociativity

```
lhs = ζa ** (ζb ** ζc); rhs = (ζa ** ζb) ** ζc;
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

ElAssociativity

$\{\mathrm{E_l}[\langle 1 \to \mathrm{LS}[2\,\overline{1} + \overline{2}, 0, \frac{1}{2}\,\overline{1\overline{12}}, \ldots], 2 \to \mathrm{LS}[\overline{1} + 3\,\overline{2}, 0, \frac{5}{2}\,\overline{1\overline{12}} - \overline{\overline{12}2}, \ldots]\rangle,$
$\mathrm{CWS}[2\,\widehat{1} + \widehat{2}, -4\,\widehat{12}, -2\,\widehat{112}, \ldots]], \mathrm{BS}[9\,\mathrm{True}, \ldots]\}$

detaExample

$$\{\zeta_a\ //\ d\eta^1,\ \zeta_a\ //\ d\eta^2\}$$

detaExample

$$\{E_l[\langle 2 \to LS[\overline{2}, 0, 0, 0, \ldots]\rangle, CWS[0, 0, 0, 0, \ldots]], E_l[\langle 1 \to LS[\overline{1}, 0, 0, 0, \ldots]\rangle, CWS[\widehat{1}, 0, 0, 0, \ldots]]\}$$

dA1

$$\{\zeta_d = E_l[\lambda, CWS[0]],\ \zeta_d\ //\ dA\}$$

dA1

$$\{E_l[\langle 1 \to LS[\overline{1}, \overline{1\,2}, 0, 0, \ldots], 2 \to LS[\overline{2}, 0, -\overline{1\,\overline{1\,2}}, 0, \ldots]\rangle, CWS[0, 0, 0, 0, \ldots]], E_l[\langle 1 \to LS[-\overline{1}, -\overline{1\,2}, 0, 0, \ldots], 2 \to LS[-\overline{2}, 0, \overline{1\,\overline{1\,2}}, 0, \ldots]\rangle, CWS[-\widehat{1} - \widehat{2}, \widehat{12}, \widehat{112}, \widehat{1122} - \widehat{1212}, \ldots]]\}$$

dA2

$$(\zeta_d \equiv (\zeta_d\ //\ dA\ //\ dA))@\{8\}$$

dA2

$$BS[9\ True, \ldots]$$

dA3

$$lhs = (\zeta_a ** \zeta_b)\ //\ dA;\ rhs = (\zeta_b\ //\ dA) ** (\zeta_a\ //\ dA);$$
$$\{lhs@\{3\}, (lhs \equiv rhs)@\{8\}\}$$

dA3

$$\{E_l[\langle 1 \to LS[-\overline{1} - \overline{2}, 0, -\frac{1}{2}\overline{1\,\overline{1\,2}}, \ldots], 2 \to LS[-\overline{1} - 2\,\overline{2}, 0, \frac{1}{2}\overline{1\,\overline{1\,2}} + \overline{\overline{1\,2}\,2}, \ldots]\rangle, CWS[-\widehat{2}, -2\,\widehat{12}, -2\,\widehat{112} - \widehat{122}, \ldots]], BS[9\ True, \ldots]\}$$

dS

$$\zeta_d\ //\ dS$$

dS

$$E_l[\langle 1 \to LS[\overline{1}, -\overline{1\,2}, 0, 0, \ldots], 2 \to LS[\overline{2}, 0, -\overline{1\,\overline{1\,2}}, 0, \ldots]\rangle, CWS[\widehat{1} + \widehat{2}, \widehat{12}, -\widehat{112}, \widehat{1122} - \widehat{1212}, \ldots]]$$

dD1

$$\{\zeta_a,\ \zeta_a\ //\ d\Delta[2, 2, 3]\}$$

dD1

$$\{E_l[\langle 1 \to LS[\overline{1}, \overline{1\,2}, 0, 0, \ldots], 2 \to LS[\overline{2}, 0, -\overline{1\,\overline{1\,2}}, 0, \ldots]\rangle, CWS[\widehat{1}, 0, -3\,\widehat{112}, 0, \ldots]], E_l[\langle 1 \to LS[\overline{1}, \overline{1\,2} + \overline{1\,3}, 0, 0, \ldots], 2 \to LS[\overline{2} + \overline{3}, 0, -\overline{1\,\overline{1\,2}} - \overline{1\,\overline{1\,3}}, 0, \ldots], 3 \to LS[\overline{2} + \overline{3}, 0, -\overline{1\,\overline{1\,2}} - \overline{1\,\overline{1\,3}}, 0, \ldots]\rangle, CWS[\widehat{1}, 0, -3\,\widehat{112} - 3\,\widehat{113}, 0, \ldots]]\}$$

dD2

```
lhs = (ξa ** ξb) // dΔ[2, 2, 3]; rhs = (ξa // dΔ[2, 2, 3]) ** (ξb // dΔ[2, 2, 3]);
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

dD2

$$\Big\{E_1\Big[\Big\langle 1 \to \mathrm{LS}\Big[\overline{1} + \overline{2} + \overline{3},\ 0,\ \frac{1}{2}\,\overline{1\,\overline{1\,2}} + \frac{1}{2}\,\overline{1\,\overline{1\,3}},\ \ldots\Big],$$
$$2 \to \mathrm{LS}\Big[\overline{1} + 2\,\overline{2} + 2\,\overline{3},\ 0,\ -\frac{1}{2}\,\overline{1\,\overline{1\,2}} - \frac{1}{2}\,\overline{1\,\overline{1\,3}} - \overline{1\,\overline{2\,3}} - \overline{\overline{1\,2}\,2} - 2\,\overline{\overline{1\,3}\,2} - \overline{\overline{1\,3}\,3},\ \ldots\Big],$$
$$3 \to \mathrm{LS}\Big[\overline{1} + 2\,\overline{2} + 2\,\overline{3},\ 0,\ -\frac{1}{2}\,\overline{1\,\overline{1\,2}} - \frac{1}{2}\,\overline{1\,\overline{1\,3}} - \overline{1\,\overline{2\,3}} - \overline{\overline{1\,2}\,2} - 2\,\overline{\overline{1\,3}\,2} - \overline{\overline{1\,3}\,3},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[\widehat{1} + \widehat{2} + \widehat{3},\ -2\,\widehat{12} - 2\,\widehat{13},\ -3\,\widehat{112} - 3\,\widehat{113},\ \ldots\Big]\Big],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\Big\}$$

Section 2.4 - The factored presentation E_f of $A^w{}_{\exp}$ and its stronger precursor E_s

EsSetup1

```
u = LW@u; v = LW@v;
ξa = Es[⟨1 → LS[u + b[u, v]], 2 → LS[v - b[u, b[u, v]]], 3 → LS[u - b[b[u, v], b[u, v]]]⟩,
  CWS[CW["u"] - 3 CW["uvu"]]]
```

EsSetup1

$$E_s\Big[\Big\langle 1 \to \mathrm{LS}[\overline{u},\ \overline{u\,v},\ 0,\ 0,\ \ldots],\ 2 \to \mathrm{LS}\big[\overline{v},\ 0,\ -\overline{u\,\overline{u\,v}},\ 0,\ \ldots\big],\ 3 \to \mathrm{LS}[\overline{u},\ 0,\ 0,\ 0,\ \ldots]\Big\rangle,$$
$$\mathrm{CWS}[\widehat{u},\ 0,\ -3\,\widehat{uuv},\ 0,\ \ldots]\Big]$$

EsSetup2

```
SeedRandom[0]; ξb =
  Es[⟨Table[i → RandomLieSeries[{1, 2, 3, 4}], {i, 4}]⟩, RandomCWSeries[{1, 2, 3, 4}]];
ξb@
  {2}
```

EsSetup2

$$E_s\Big[\Big\langle 1 \to \mathrm{LS}\Big[-\overline{1} - 2\,\overline{2} + 2\,\overline{3} - 2\,\overline{4},\ 2\,\overline{1\,2} + \frac{\overline{1\,3}}{2} + \overline{1\,4} - \frac{\overline{2\,3}}{2} - \frac{\overline{2\,4}}{2} + 2\,\overline{3\,4},\ \ldots\Big],$$
$$2 \to \mathrm{LS}\Big[2\,\overline{1} - \overline{2} - 2\,\overline{3} + \overline{4},\ 2\,\overline{1\,2} + \frac{3\,\overline{1\,3}}{2} - 2\,\overline{1\,4} - \overline{2\,3} - \overline{2\,4} - \frac{\overline{3\,4}}{2},\ \ldots\Big],$$
$$3 \to \mathrm{LS}\Big[-\overline{1} + \overline{2} + 2\,\overline{4},\ -2\,\overline{1\,2} + 2\,\overline{1\,3} - \overline{1\,4} - \frac{3\,\overline{2\,3}}{2} + 2\,\overline{2\,4} - 2\,\overline{3\,4},\ \ldots\Big],$$
$$4 \to \mathrm{LS}\Big[-2\,\overline{1} + 2\,\overline{2} + 2\,\overline{3} + \overline{4},\ -\frac{\overline{1\,2}}{2} + \frac{3\,\overline{1\,3}}{2} - 2\,\overline{2\,4} + \overline{3\,4},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[\widehat{3} - \widehat{4},\ \frac{3\,\widehat{11}}{2} + \frac{3\,\widehat{12}}{2} - 2\,\widehat{13} + \widehat{14} + \widehat{22} + 2\,\widehat{23} - \frac{\widehat{24}}{2} - 2\,\widehat{33} - \widehat{34} + \widehat{44},\ \ldots\Big]\Big]$$

haction

```
lhs = ξa // hm[1, 2, 4] // tha[u, 4];
rhs = ξa // tha[u, 1] // tha[u, 2] // hm[1, 2, 4];
{lhs, (lhs ≡ rhs)@{8}}
```

haction

$\{E_s[\langle 3 \to LS[\overline{u}, -\overline{uv}, -u\overline{u\overline{v}} + \frac{1}{2}\overline{\overline{uv}v}, \frac{3}{2}u\overline{u\overline{uv}} + u\overline{\overline{uv}v} - \frac{1}{6}\overline{\overline{\overline{uv}v}v}, \ldots],$
$4 \to LS[\overline{u} + \overline{v}, \frac{\overline{uv}}{2}, -\frac{23}{12}u\overline{u\overline{v}} - \frac{5}{12}\overline{\overline{uv}v}, u\overline{u\overline{uv}} + \frac{13}{24}u\overline{\overline{uv}v} + \frac{1}{12}\overline{\overline{\overline{uv}v}v}, \ldots]\rangle,$
$CWS[2\widehat{u}, -\widehat{uv}, -\frac{3\widehat{uuv}}{2}, -\frac{\widehat{uuuv}}{6} + \widehat{uuvv} - \widehat{uvuv}, \ldots]], BS[9\,True, \ldots]\}$

metaassoc

```
lhs = ξb // dm[1, 2, 1] // dm[1, 3, 1]; rhs = ξb // dm[2, 3, 2] // dm[1, 2, 1];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

metaassoc

$\{E_s[\langle 1 \to LS[-2\overline{1} + \overline{4}, -\frac{3\overline{1\,4}}{2}, 20\overline{1\overline{1\,4}} - \frac{19}{3}\overline{\overline{1\,4}4}, \ldots],$
$4 \to LS[2\overline{1} + \overline{4}, \overline{1\,4}, -\frac{31}{2}\overline{1\overline{1\,4}} - \frac{13}{6}\overline{\overline{1\,4}4}, \ldots]\rangle,$
$CWS[3\widehat{1} - \widehat{4}, -3\widehat{11} + \frac{\widehat{14}}{2} + \widehat{44}, \frac{71\widehat{111}}{4} + \frac{19\widehat{114}}{4} - \frac{7\widehat{144}}{6} - \frac{2\widehat{444}}{3}, \ldots]], BS[6\,True, \ldots]\}$

Section 3.1 - Tangle Invariants

Section 3.1.1 - The General Framework

RDefs

```
R⁺t:(1|s)[a_, b_] := Et[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
R⁻t:(1|s)[a_, b_] := Et[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
```

R3

```
lhs = R⁺1[1, 2] ** R⁺1[1, 3] ** R⁺1[2, 3]; rhs = R⁺1[2, 3] ** R⁺1[1, 3] ** R⁺1[1, 2];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

R3

$\{E_1[\langle 1 \to LS[0, 0, 0, \ldots], 2 \to LS[\overline{1}, 0, 0, \ldots], 3 \to LS[\overline{1} + \overline{2}, 0, 0, \ldots]\rangle,$
$CWS[0, 0, 0, \ldots]], BS[6\,True, \ldots]\}$

Section 3.1.2 - The Knot 8_{17} and the Borromean Tangle

817

```
t1 = R⁻s[12, 1] R⁻s[2, 7] R⁻s[8, 3] R⁻s[4, 11] R⁺s[16, 5] R⁺s[6, 13] R⁺s[14, 9] R⁺s[10, 15];
Do[t1 = t1 // dm[1, k, 1], {k, 2, 16}];
t1@{6}
```

817

$$\mathrm{E_s}\left[\langle 1\to \mathrm{LS}[0,0,0,0,0,0,\ldots]\rangle,\ \mathrm{CWS}\left[0,-\widehat{11},0,-\frac{31\,\widehat{1111}}{12},0,-\frac{1351\,\widehat{111111}}{360},\ldots\right]\right]$$

Borromean

```
t2 = R⁻s[r, 6] R⁺s[2, 4] R⁻s[g, 9] R⁺s[5, 7] R⁻s[b, 3] R⁺s[8, 1];
(Do[t2 = t2 // dm[r, k, r], {k, 1, 3}]; Do[t2 = t2 // dm[g, k, g], {k, 4, 6}];
 Do[t2 = t2 // dm[b, k, b], {k, 7, 9}]; t2)
```

Borromean

$$\mathrm{E_s}\Big[\Big\langle b\to \mathrm{LS}\Big[0,\ \overline{gr},\ \frac12\overline{g\,\overline{gr}}+\overline{\overline{br}\,g}+\frac12\overline{\overline{gr}\,r},$$
$$-\frac12\overline{b\,\overline{\overline{br}\,g}}+\frac16\overline{g\,\overline{g\,\overline{gr}}}+\frac14\overline{g\,\overline{\overline{gr}\,r}}-\frac12\overline{\overline{bg}\,\overline{br}}-\frac12\overline{\overline{\overline{br}\,g}\,g}-\frac12\overline{\overline{\overline{br}\,r}\,g}+\frac16\overline{\overline{\overline{gr}\,r}\,r},\ \ldots\Big],\ g\to$$
$$\mathrm{LS}\Big[0,\ -\overline{br},\ \frac12\overline{b\,\overline{br}}-\overline{b\,\overline{gr}}-\overline{\overline{br}\,g}+\frac12\overline{\overline{br}\,r},\ -\frac16\overline{b\,\overline{b\,\overline{br}}}-\frac12\overline{b\,\overline{b\,\overline{gr}}}-\frac12\overline{b\,\overline{g\,\overline{gr}}}-\frac12\overline{b\,\overline{\overline{br}\,g}}-$$
$$\frac14\overline{b\,\overline{\overline{br}\,r}}+\frac12\overline{b\,\overline{\overline{gr}\,r}}+\frac12\overline{\overline{bg}\,\overline{br}}+\overline{\overline{br}\,\overline{gr}}-\overline{\overline{b\,\overline{gr}}\,g}-\frac12\overline{\overline{\overline{br}\,g}\,g}+\frac12\overline{\overline{\overline{br}\,r}\,g}-\frac16\overline{\overline{\overline{br}\,r}\,r},\ \ldots\Big],$$
$$r\to \mathrm{LS}\Big[0,\ \overline{bg},\ \frac12\overline{b\,\overline{bg}}+\overline{b\,\overline{gr}}+\frac12\overline{\overline{bg}\,g},\ \frac16\overline{b\,\overline{b\,\overline{bg}}}+\frac12\overline{b\,\overline{b\,\overline{gr}}}+$$
$$\frac12\overline{b\,\overline{g\,\overline{gr}}}+\frac14\overline{b\,\overline{\overline{bg}\,g}}+\frac12\overline{b\,\overline{\overline{gr}\,r}}+\frac16\overline{\overline{\overline{bg}\,g}\,g},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[0,\ 0,\ 2\,\widehat{bgr},\ \widehat{bbgr}-\widehat{bgbr}+\widehat{bggr}-\widehat{bgrg}+\widehat{bgrr}-\widehat{brgr},\ \ldots\Big]\Big]$$

Section 3.2 - Solutions of the Kashiwara-Vergne Equations

Continues pensieve://2013-10/SolvingWKO.nb.

VSetup

```
α = LS[{"1", "2"}, αs]; β = LS[{"1", "2"}, βs]; γ = CWS[{"1", "2"}, γs];
V = Es[⟨1 → α, 2 → β⟩, γ];
```

CapSetup

```
κ = CWS[{"1"}, κs]; Cap = Es[⟨1 → LS[0]⟩, κ];
```

VCapEqns

```
R4Eqn = (R⁺s[2, 3] ** R⁺s[1, 3] ** V ≡ V ** (R⁺s[1, 3] // dΔ[1, 1, 2]));
UnitarityEqn = (V ** (V // dA[1] // dA[2]) ≡ Es[⟨1 → LS[0], 2 → LS[0]⟩, CWS[0]]);
CapEqn = ((V ** (Cap // dΔ[1, 1, 2]) // dc[1] // dc[2]) ≡
    (Cap (Cap // dσ[1, 2]) // dc[1] // dc[2]));
```

VCapSolution

```
αs["1"] = 0; αs["2"] = -1/2;
SeriesSolve[{α, β, γ, κ}, ℏ^-1 R4Eqn && UnitarityEqn && CapEqn];
{V, κ}
```

VCapSolution

```
Arbitrator called on {κs[1]}...
```

VCapSolution

```
Arbitrator called on {αs[122]}...
```

VCapSolution

$$\left\{E_s\left[\left\langle 1\to LS\left[-\frac{\overline{2}}{2},\ \frac{\overline{12}}{12},\ 0,\ -\frac{1}{720}\overline{1\,\overline{1\,\overline{12}}}+\frac{1}{720}\overline{1\,\overline{\overline{12}\,2}}-\frac{\overline{\overline{\overline{12}\,2}\,2}}{5760},\ \ldots\right],\right.\right.\right.$$
$$\left.2\to LS\left[0,\ \frac{\overline{12}}{24},\ 0,\ -\frac{\overline{1\,\overline{1\,\overline{12}}}}{1440}+\frac{7\,\overline{1\,\overline{\overline{12}\,2}}}{5760}-\frac{7\,\overline{\overline{\overline{12}\,2}\,2}}{5760},\ \ldots\right]\right\rangle,$$
$$\left.CWS\left[0,\ -\frac{\widehat{12}}{48},\ 0,\ \frac{\widehat{1112}}{2880}+\frac{\widehat{1122}}{2880}+\frac{\widehat{1212}}{5760}+\frac{\widehat{1222}}{2880},\ \ldots\right]\right],\ CWS\left[0,\ -\frac{\widehat{11}}{96},\ 0,\ \frac{\widehat{1111}}{11\,520},\ \ldots\right]\right\}$$

Conjecture. *For any Lie algebra* $\mathfrak{g}$ *of finite dimension, we can find F and G such that they satisfy*

a) $x+y-\log e^y e^x=(1-e^{-\operatorname{ad}x})F+(e^{\operatorname{ad}y}-1)G.$

b) *F and G give* $\mathfrak{g}$*-valued convergent power series on* $(x,y)\in\mathfrak{g}\times\mathfrak{g}$.

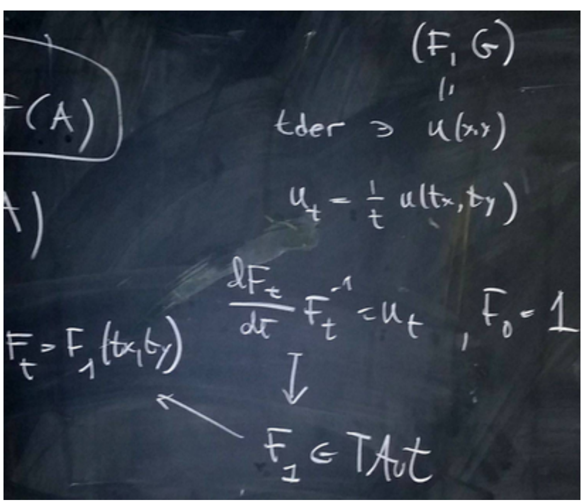
c) $\operatorname{tr}((\operatorname{ad}x)(\partial_x F);\mathfrak{g})+\operatorname{tr}((\operatorname{ad}y)(\partial_y G);\mathfrak{g})$

$$=\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}-1;\mathfrak{g}\right).$$

Here $z=\log e^x e^y$ *and* $\partial_x F$ (resp. $\partial_y G$) *is the* **End**$(\mathfrak{g})$*-valued real analytic function defined by*

$$\mathfrak{g}\ni a\mapsto\frac{d}{dt}F(x+ta,y)|_{t=0}\quad\left(\text{resp. }\mathfrak{g}\ni a\mapsto\frac{d}{dt}G(x,y+ta)|_{t=0}\right),$$

and tr *denotes the trace of an endomorphism of* $\mathfrak{g}$.



```
{F = LS[{x, y}, fs], G = LS[{x, y}, gs]};
SeriesSolve[{F, G},
  ħ^-1 (LS[LW@x + LW@y] - BCH[LW@y, LW@x] ≡ F - G - Ad[-LW@x][F] + Ad[LW@y][G])
   && div_x[F] + div_y[G] ≡ 1/2 tr_LW@u[adSeries[ad/(ⅇ^ad - 1), LW@x][LW@u] +
       adSeries[ad/(ⅇ^ad - 1), LW@y][LW@u] - adSeries[ad/(ⅇ^ad - 1), BCH[LW@x, LW@y]][LW@u]]
]
```

```
kv = ⟨1 → F, 2 → G⟩
```

```
Arbitrator called on {fs[y]}...
```

$$\left\langle 1 \to \mathrm{LS}\left[0, -\frac{\overline{xy}}{12}, -\frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} - \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \ldots\right],\right.$$
$$\left. 2 \to \mathrm{LS}\left[-\frac{\overline{x}}{2}, -\frac{\overline{xy}}{6}, -\frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} - \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \ldots\right]\right\rangle$$

```
at = V⟦1⟧ // dσ[{1, 2} → {2, 1}];
atkv = at // EulerE // adSeries[(ⅇ^ad - 1)/ad, at] // RC[-at] //
  LieMorphism[LW@1 → LW@x, LW@2 → LW@y]
```

$$\left\langle 1 \to \mathrm{LS}\left[0, -\frac{\overline{xy}}{12}, -\frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} - \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \ldots\right],\right.$$
$$\left. 2 \to \mathrm{LS}\left[-\frac{\overline{x}}{2}, -\frac{\overline{xy}}{6}, -\frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} - \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \ldots\right]\right\rangle$$

```
at1 = at // Λ;
atkv1 = at1 // EulerE // adSeries[(ⅇ^ad - 1)/ad, at1, tb] //
  LieMorphism[LW@1 → LW@x, LW@2 → LW@y]
```

$$\left\langle 1 \to \mathrm{LS}\left[0, -\frac{\overline{xy}}{12}, -\frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} - \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \ldots\right],\right.$$
$$\left. 2 \to \mathrm{LS}\left[-\frac{\overline{x}}{2}, -\frac{\overline{xy}}{6}, -\frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} - \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \ldots\right]\right\rangle$$

```
λ2 = ⟨1 → RandomLieSeries[{1, 2}], 2 → RandomLieSeries[{1, 2}]⟩
```

$$\left\langle 1 \to \mathrm{LS}\left[2\,\overline{1} + 2\,\overline{2}, -\overline{12}, -\frac{3}{2}\overline{1\overline{12}} - \overline{\overline{12}2}, \frac{3}{4}\overline{1\overline{1\overline{12}}} - \frac{7}{24}\overline{1\overline{\overline{12}2}} + \frac{4}{3}\overline{\overline{\overline{12}2}2}, \ldots\right],\right.$$
$$\left. 2 \to \mathrm{LS}\left[\overline{1} - 2\,\overline{2}, \frac{3\,\overline{12}}{2}, -\frac{5}{6}\overline{1\overline{12}} + \frac{4}{3}\overline{\overline{12}2}, \frac{3}{2}\overline{1\overline{1\overline{12}}} - \frac{5}{3}\overline{1\overline{\overline{12}2}} + \frac{3}{8}\overline{\overline{\overline{12}2}2}, \ldots\right]\right\rangle$$

{lhs = λ2 // EulerE // adSeries[$\frac{e^{ad}-1}{ad}$, λ2] // RC[-λ2],
rhs = Λ[λ2] // EulerE // adSeries[$\frac{e^{ad}-1}{ad}$, Λ[λ2], tb]; (lhs ≡ rhs)@{8}}

$$\Big\{\Big\langle 1\to \text{LS}\Big[2\,\overline{1}+2\,\overline{2},\ 0,\ -\frac{9}{2}\,\overline{1\,\overline{1\,2}}-11\,\overline{\overline{1\,2}\,2},\ \frac{7}{6}\,\overline{1\,\overline{1\,\overline{1\,2}}}-\frac{10}{3}\,\overline{1\,\overline{\overline{1\,2}\,2}}-\frac{59}{3}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\Big],$$

$$2\to \text{LS}\Big[\overline{1}-2\,\overline{2},\ 7\,\overline{1\,2},\ -\frac{27}{4}\,\overline{1\,\overline{1\,2}}+\frac{45}{2}\,\overline{\overline{1\,2}\,2},\ \frac{47}{6}\,\overline{1\,\overline{1\,\overline{1\,2}}}-54\,\overline{1\,\overline{\overline{1\,2}\,2}}+\frac{89}{2}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\Big]\Big\rangle,\ \text{BS}[9\ \text{True},\ \ldots]\Big\}$$

(atkv ≡ atkv1)@{9}

```
Arbitrator called on {αs[11122]}...
Arbitrator called on {αs[1111122]}...
Arbitrator called on {αs[11112122]}...
Arbitrator called on {αs[111111122]}...
BS[10 True, ...]
```

(atkv ≡ kv)@{8}

```
Arbitrator called on {fs[xxxxyxyy]}...
```

$$\text{BS}\Big[8\ \text{True},\ -\frac{\text{LW[xxxxxxxy]}}{151200}+\frac{\text{LW[xxxxxxyy]}}{37800}-\frac{\text{LW[xxxxxyxy]}}{151200}-\frac{\text{LW[xxxxxyyy]}}{37800}-\frac{\text{LW[xxxxyxxy]}}{50400}-\frac{61\,\text{LW[xxxxyxyy]}}{1360800}-\frac{61\,\text{LW[xxxxyyxy]}}{680400}-\frac{\text{LW[xxxxyyyy]}}{30240}-\frac{829\,\text{LW[xxxyxxyy]}}{10886400}+\frac{7\,\text{LW[xxxyxyxy]}}{259200}-\frac{653\,\text{LW[xxxyxyyy]}}{7257600}+\frac{251\,\text{LW[xxxyyxxy]}}{5443200}+\frac{67\,\text{LW[xxxyyxyy]}}{2419200}+\frac{467\,\text{LW[xxxyyyxy]}}{2419200}+\frac{\text{LW[xxxyyyyy]}}{18900}+\frac{19\,\text{LW[xxyxxyxy]}}{2721600}+\frac{653\,\text{LW[xxyxxyyy]}}{7257600}-\frac{307\,\text{LW[xxyxyxyy]}}{2419200}+\frac{173\,\text{LW[xxyxyyxy]}}{2419200}+\frac{1313\,\text{LW[xxyxyyyy]}}{7257600}-\frac{67\,\text{LW[xxyyxyxy]}}{1209600}+\frac{667\,\text{LW[xxyyxyyy]}}{7257600}-\frac{25\,\text{LW[xxyyyxyy]}}{290304}-\frac{773\,\text{LW[xxyyyyxy]}}{7257600}-\frac{\text{LW[xxyyyyyy]}}{43200}+\frac{59\,\text{LW[xyxyxyyy]}}{259200}-\frac{173\,\text{LW[xyxyyxyy]}}{2419200}-\frac{59\,\text{LW[xyxyyyyy]}}{777600}-\frac{197\,\text{LW[xyyxyyyy]}}{2177280}+\frac{\text{LW[xyyyyyyy]}}{302400}==$$

$$-\frac{\text{LW[xxxxxxxy]}}{151200}+\frac{\text{LW[xxxxxxyy]}}{37800}-\frac{\text{LW[xxxxxyxy]}}{151200}-\frac{\text{LW[xxxxxyyy]}}{37800}-\frac{\text{LW[xxxxyxxy]}}{50400}-\frac{\text{LW[xxxxyyyy]}}{30240}-\frac{13\,\text{LW[xxxyxxyy]}}{241920}-\frac{13\,\text{LW[xxxyxyxy]}}{120960}-\frac{299\,\text{LW[xxxyxyyy]}}{2419200}+\frac{11\,\text{LW[xxxyyxxy]}}{120960}-\frac{59\,\text{LW[xxxyyxyy]}}{806400}+\frac{223\,\text{LW[xxxyyyxy]}}{2419200}+\frac{\text{LW[xxxyyyyy]}}{18900}-\frac{\text{LW[xxyxxyxy]}}{12096}+\frac{299\,\text{LW[xxyxxyyy]}}{2419200}-\frac{\text{LW[xxyxyxyy]}}{38400}+\frac{139\,\text{LW[xxyxyyxy]}}{806400}+\frac{173\,\text{LW[xxyxyyyy]}}{806400}+\frac{59\,\text{LW[xxyyxyxy]}}{403200}+\frac{47\,\text{LW[xxyyxyyy]}}{806400}-\frac{41\,\text{LW[xxyyyxyy]}}{161280}-\frac{113\,\text{LW[xxyyyyxy]}}{806400}-\frac{\text{LW[xxyyyyyy]}}{43200}+\frac{73\,\text{LW[xyxyxyyy]}}{201600}-$$

$$\frac{139\,\text{LW[xyxyyxyy]}}{806400}-\frac{73\,\text{LW[xyxyyyyy]}}{604800}-\frac{7\,\text{LW[xyyxyyyy]}}{34560}+\frac{\text{LW[xyyyyyyy]}}{302400}\;\&\&$$

$$-\frac{\text{LW[xxxxxxxy]}}{302400}+\frac{\text{LW[xxxxxxyy]}}{67200}+\frac{89\,\text{LW[xxxxxyxy]}}{5443200}-\frac{\text{LW[xxxxxyyy]}}{302400}-$$
$$\frac{181\,\text{LW[xxxxyxxy]}}{10886400}-\frac{73\,\text{LW[xxxxyxyy]}}{1451520}-\frac{461\,\text{LW[xxxxyyxy]}}{2419200}-$$
$$\frac{\text{LW[xxxxyyyy]}}{16128}-\frac{643\,\text{LW[xxxyxxyy]}}{7257600}+\frac{\text{LW[xxxyxyxy]}}{12600}-\frac{907\,\text{LW[xxxyxyyy]}}{7257600}+$$
$$\frac{\text{LW[xxxyyxxy]}}{25200}+\frac{31\,\text{LW[xxxyyxyy]}}{518400}+\frac{1987\,\text{LW[xxxyyyxy]}}{7257600}+\frac{23\,\text{LW[xxxyyyyy]}}{302400}+$$
$$\frac{211\,\text{LW[xxyxxyxy]}}{7257600}+\frac{\text{LW[xxyxxyyy]}}{10080}-\frac{667\,\text{LW[xxyxyxyy]}}{3628800}+\frac{59\,\text{LW[xxyxyyxy]}}{1036800}+$$
$$\frac{1207\,\text{LW[xxyxyyyy]}}{5443200}-\frac{1133\,\text{LW[xxyyxyxy]}}{7257600}+\frac{61\,\text{LW[xxyyxyyy]}}{518400}-$$
$$\frac{\text{LW[xxyyyxyy]}}{15120}-\frac{\text{LW[xxyyyyxy]}}{7560}-\frac{\text{LW[xxyyyyyy]}}{28800}+\frac{1493\,\text{LW[xyxyxyyy]}}{5443200}-$$
$$\frac{667\,\text{LW[xyxyyxyy]}}{10886400}-\frac{\text{LW[xyxyyyyy]}}{10080}-\frac{\text{LW[xyyxyyyy]}}{10080}+\frac{\text{LW[xyyyyyyy]}}{151200}==$$
$$-\frac{\text{LW[xxxxxxxy]}}{302400}+\frac{\text{LW[xxxxxxyy]}}{67200}+\frac{37\,\text{LW[xxxxxyxy]}}{604800}-\frac{\text{LW[xxxxxyyy]}}{302400}+\frac{\text{LW[xxxxyxxy]}}{172800}-$$
$$\frac{29\,\text{LW[xxxxyxyy]}}{345600}-\frac{47\,\text{LW[xxxxyyxy]}}{161280}-\frac{\text{LW[xxxxyyyy]}}{16128}-\frac{19\,\text{LW[xxxyxxyy]}}{345600}+$$
$$\frac{\text{LW[xxxyxyxy]}}{12600}-\frac{221\,\text{LW[xxxyxyyy]}}{2419200}+\frac{\text{LW[xxxyyxxy]}}{25200}-\frac{\text{LW[xxxyyxyy]}}{134400}+\frac{83\,\text{LW[xxxyyyxy]}}{345600}+$$
$$\frac{23\,\text{LW[xxxyyyyy]}}{302400}-\frac{11\,\text{LW[xxyxxyxy]}}{2419200}+\frac{\text{LW[xxyxxyyy]}}{10080}-\frac{47\,\text{LW[xxyxyxyy]}}{403200}+$$
$$\frac{73\,\text{LW[xxyxyyxy]}}{806400}+\frac{107\,\text{LW[xxyxyyyy]}}{604800}-\frac{17\,\text{LW[xxyyxyxy]}}{89600}+\frac{61\,\text{LW[xxyyxyyy]}}{1209600}-$$
$$\frac{\text{LW[xxyyyxyy]}}{15120}-\frac{\text{LW[xxyyyyxy]}}{7560}-\frac{\text{LW[xxyyyyyy]}}{28800}+\frac{193\,\text{LW[xyxyxyyy]}}{604800}-$$
$$\frac{47\,\text{LW[xyxyyxyy]}}{1209600}-\frac{\text{LW[xyxyyyyy]}}{10080}-\frac{\text{LW[xyyxyyyy]}}{10080}+\frac{\text{LW[xyyyyyyy]}}{151200},\;\ldots\Big]$$

{A, B} = {atkv$_1$, atkv$_2$}

$$\Big\{\text{LS}\Big[0,\,-\frac{\overline{xy}}{12},\,-\frac{1}{24}\,\overline{x\overline{xy}},\,-\frac{1}{180}\,\overline{x\overline{x\overline{xy}}}-\frac{1}{120}\,\overline{x\overline{\overline{xy}y}}+\frac{1}{360}\,\overline{\overline{\overline{xy}y}y},\,\ldots\Big],$$
$$\text{LS}\Big[-\frac{\overline{x}}{2},\,-\frac{\overline{xy}}{6},\,-\frac{1}{24}\,\overline{x\overline{xy}},\,-\frac{1}{360}\,\overline{x\overline{x\overline{xy}}}-\frac{1}{80}\,\overline{x\overline{\overline{xy}y}}+\frac{1}{180}\,\overline{\overline{\overline{xy}y}y},\,\ldots\Big]\Big\}$$

$\Big(\hbar^{-1}$ (LS[LW@x + LW@y] - BCH[LW@y, LW@x] ≡ A - B - Ad[-LW@x][A] + Ad[LW@y][B]) &&

div_x[A] + div_y[B] ≡ $\frac{1}{2}$ $\text{tr}_{\text{LW@u}}\Big[$adSeries$\Big[\frac{\text{ad}}{e^{\text{ad}}-1}$, LW@x$\Big]$[LW@u] + adSeries$\Big[\frac{\text{ad}}{e^{\text{ad}}-1}$, LW@y$\Big]$[
LW@u] - adSeries$\Big[\frac{\text{ad}}{e^{\text{ad}}-1}$, BCH[LW@x, LW@y]$\Big]$[LW@u]$\Big]\Big)$@{9}

Arbitrator called on {αs[1111112122]}...

BS[10 True, ...]

f = (V // Λ)⟦1⟧

$$\left\langle 1\to \mathrm{LS}\left[0,\ -\frac{\overline{1\,2}}{24},\ \frac{1}{96}\,\overline{1\,\overline{1\,2}},\ \frac{\overline{1\,\overline{1\,\overline{1\,2}}}}{2880}-\frac{1}{480}\,\overline{1\,\overline{\overline{1\,2}\,2}}+\frac{\overline{\overline{\overline{1\,2}\,2}\,2}}{1440},\ \ldots\right],\right.$$
$$\left.2\to \mathrm{LS}\left[\frac{\overline{1}}{2},\ -\frac{\overline{1\,2}}{12},\ \frac{1}{96}\,\overline{1\,\overline{1\,2}},\ \frac{1}{960}\,\overline{1\,\overline{1\,\overline{1\,2}}}-\frac{1}{320}\,\overline{1\,\overline{\overline{1\,2}\,2}}+\frac{1}{720}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\right]\right\rangle$$

f // RC[-f]

$$\left\langle 1\to \mathrm{LS}\left[0,\ -\frac{\overline{1\,2}}{24},\ \frac{1}{32}\,\overline{1\,\overline{1\,2}},\ -\frac{29\,\overline{1\,\overline{1\,\overline{1\,2}}}}{2880}-\frac{11\,\overline{1\,\overline{\overline{1\,2}\,2}}}{2880}+\frac{\overline{\overline{\overline{1\,2}\,2}\,2}}{1440},\ \ldots\right],\right.$$
$$\left.2\to \mathrm{LS}\left[\frac{\overline{1}}{2},\ -\frac{\overline{1\,2}}{12},\ \frac{1}{32}\,\overline{1\,\overline{1\,2}},\ \frac{1}{960}\,\overline{1\,\overline{1\,\overline{1\,2}}}-\frac{19\,\overline{1\,\overline{\overline{1\,2}\,2}}}{2880}+\frac{1}{720}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\right]\right\rangle$$

V⟦1⟧ // RC[-V⟦1⟧]

$$\left\langle 1\to \mathrm{LS}\left[0,\ -\frac{\overline{1\,2}}{24},\ \frac{1}{48}\,\overline{1\,\overline{1\,2}},\ -\frac{23\,\overline{1\,\overline{1\,\overline{1\,2}}}}{5760}-\frac{17\,\overline{1\,\overline{\overline{1\,2}\,2}}}{5760}+\frac{\overline{\overline{\overline{1\,2}\,2}\,2}}{1440},\ \ldots\right],\right.$$
$$\left.2\to \mathrm{LS}\left[\frac{\overline{1}}{2},\ -\frac{\overline{1\,2}}{12},\ \frac{1}{48}\,\overline{1\,\overline{1\,2}},\ \frac{\overline{1\,\overline{1\,\overline{1\,2}}}}{5760}-\frac{7\,\overline{1\,\overline{\overline{1\,2}\,2}}}{1440}+\frac{1}{720}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\right]\right\rangle$$

f - (f // EulerE)

$$\left\langle 1\to \mathrm{LS}\left[0,\ \frac{\overline{1\,2}}{24},\ -\frac{1}{48}\,\overline{1\,\overline{1\,2}},\ -\frac{1}{960}\,\overline{1\,\overline{1\,\overline{1\,2}}}+\frac{1}{160}\,\overline{1\,\overline{\overline{1\,2}\,2}}-\frac{1}{480}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\right],\right.$$
$$\left.2\to \mathrm{LS}\left[0,\ \frac{\overline{1\,2}}{12},\ -\frac{1}{48}\,\overline{1\,\overline{1\,2}},\ -\frac{1}{320}\,\overline{1\,\overline{1\,\overline{1\,2}}}+\frac{3}{320}\,\overline{1\,\overline{\overline{1\,2}\,2}}-\frac{1}{240}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\right]\right\rangle$$

f // EulerE // e^{-D_f}

$$\left\langle 1\to \mathrm{LS}\left[0,\ -\frac{\overline{1\,2}}{12},\ -\frac{1}{96}\,\overline{1\,\overline{1\,2}},\ \frac{19\,\overline{1\,\overline{1\,\overline{1\,2}}}}{2880}-\frac{7\,\overline{1\,\overline{\overline{1\,2}\,2}}}{1440}+\frac{1}{360}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\right],\right.$$
$$\left.2\to \mathrm{LS}\left[\frac{\overline{1}}{2},\ -\frac{\overline{1\,2}}{6},\ -\frac{1}{32}\,\overline{1\,\overline{1\,2}},\ -\frac{1}{960}\,\overline{1\,\overline{1\,\overline{1\,2}}}-\frac{1}{180}\,\overline{1\,\overline{\overline{1\,2}\,2}}+\frac{1}{180}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\right]\right\rangle$$

{A = LS[{x, y}, as], B = LS[{x, y}, bs], ϕ = CWS[{x}, ϕs]}

$$\left\{\mathrm{LS}\left[\mathrm{as}[x]\,\overline{x}+\mathrm{as}[y]\,\overline{y},\ \mathrm{as}[xy]\,\overline{x\,y},\ \mathrm{as}[xxy]\,\overline{x\,\overline{x\,y}}+\mathrm{as}[xyy]\,\overline{\overline{x\,y}\,y},\right.\right.$$
$$\left.\mathrm{as}[xxxy]\,\overline{x\,\overline{x\,\overline{x\,y}}}+\mathrm{as}[xxyy]\,\overline{x\,\overline{\overline{x\,y}\,y}}+\mathrm{as}[xyyy]\,\overline{\overline{\overline{x\,y}\,y}\,y},\ \ldots\right],$$
$$\mathrm{LS}\left[\mathrm{bs}[x]\,\overline{x}+\mathrm{bs}[y]\,\overline{y},\ \mathrm{bs}[xy]\,\overline{x\,y},\ \mathrm{bs}[xxy]\,\overline{x\,\overline{x\,y}}+\mathrm{bs}[xyy]\,\overline{\overline{x\,y}\,y},\right.$$
$$\left.\mathrm{bs}[xxxy]\,\overline{x\,\overline{x\,\overline{x\,y}}}+\mathrm{bs}[xxyy]\,\overline{x\,\overline{\overline{x\,y}\,y}}+\mathrm{bs}[xyyy]\,\overline{\overline{\overline{x\,y}\,y}\,y},\ \ldots\right],$$
$$\left.\mathrm{CWS}\left[\widehat{x}\,\phi s[x],\ \widehat{xx}\,\phi s[xx],\ \widehat{xxx}\,\phi s[xxx],\ \widehat{xxxx}\,\phi s[xxxx],\ \ldots\right]\right\}$$

```
SeriesSolve[{A, B, ϕ},
 ℏ^-1 (LS[LW@x + LW@y] - BCH[LW@y, LW@x] ≡ A - B - Ad[-LW@x][A] + Ad[LW@y][B])
  && (div_x[A] + div_y[B] ≡
     ϕ + LieMorphism[LW@x → LW@y][ϕ] - LieMorphism[LW@x → BCH[LW@x, LW@y]][ϕ])
]
```

A[1]

```
Arbitrator called on {as[y], ϕs[x]}...
```

0

B[1]

$$-\frac{\mathrm{LW}[x]}{2}$$

ϕ[1]

0

{A, B, ϕ}

```
Arbitrator called on {as[xyy]}...
```

$$\Big\{\mathrm{LS}\Big[0, -\frac{\overline{xy}}{12}, -\frac{1}{24}\overline{x\,\overline{xy}}, -\frac{1}{180}\overline{x\,\overline{x\,\overline{xy}}} - \frac{1}{120}\overline{x\,\overline{\overline{xy}\,y}} + \frac{1}{360}\overline{\overline{\overline{xy}\,y}\,y}, \ldots\Big],$$
$$\mathrm{LS}\Big[-\frac{\overline{x}}{2}, -\frac{\overline{xy}}{6}, -\frac{1}{24}\overline{x\,\overline{xy}}, -\frac{1}{360}\overline{x\,\overline{x\,\overline{xy}}} - \frac{1}{80}\overline{x\,\overline{\overline{xy}\,y}} + \frac{1}{180}\overline{\overline{\overline{xy}\,y}\,y}, \ldots\Big],$$
$$\mathrm{CWS}\Big[0, \frac{\widehat{xx}}{24}, 0, -\frac{\widehat{xxxx}}{1440}, \ldots\Big]\Big\}$$

ϕ@{11}

```
Arbitrator called on {as[xxxxxxxxxyy], as[xxxxxxyxyyy]}...
```

$$\mathrm{CWS}\Big[0, \frac{\widehat{xx}}{24}, 0, -\frac{\widehat{xxxx}}{1440}, -\frac{\widehat{xxxxx}}{1200}, \frac{\widehat{xxxxxx}}{60\,480}, \frac{\widehat{xxxxxxx}}{14\,112},$$
$$-\frac{\widehat{xxxxxxxx}}{2\,419\,200}, -\frac{\widehat{xxxxxxxxx}}{172\,800}, \frac{\widehat{xxxxxxxxxx}}{95\,800\,320}, \frac{17\,\widehat{xxxxxxxxxxx}}{35\,126\,784}, \ldots\Big]$$