Aw-Calculus Programs for the WKO4 Paper

Pensieve header: Aw-calculus programs for the WKO4 paper.

"d" is "ht": along tube strands, heads appear before tails.

Utilities

```
⟨{λ___}⟩ := ⟨λ⟩;
Support[⟨λ___⟩] := First /@ {λ};
λ_ \ key_ := DeleteCases[λ, key → _];
λ_ \ keys_List := Fold[#1 \ #2 &, λ, keys];
⟨λ___⟩s_ := s /. {λ};
(λ_AngleBracket)[d_] := ⟨Table[u → λu[d], {u, Support[λ]}]⟩;
Crop[λ_AngleBracket, d_] := MapAt[Crop[#, d] &, λ, {All, 2}];
AngleBracket /: Plus[λs__AngleBracket] := ⟨Table[
    u → Total[#u & /@ {λs}],
    {u, Union @@ (Support[#] & /@ {λs})}
   ]⟩;
AngleBracket /: c_ * λ_AngleBracket := ⟨Table[
    u → Expand[c λu],
    {u, Support[λ]}
   ]⟩;
AngleBracket /: λ_AngleBracket // mor_LieMorphism := MapAt[mor, λ, {All, 2}];
λ_AngleBracket // DegreeScale[h_] := MapAt[DegreeScale[h], λ, {All, 2}];
deg /: (h_)^deg := DegreeScale[h];
```

Tangential Derivations, the Tree Bracket tb, div and j

```
TangentialDerivation[⟨λ___⟩] :=
  LieDerivation[{λ} /. (s_ → λs_) :↦ (LW[s] → b[LW[s], λs])];
tb[λ1_AngleBracket, λ2_AngleBracket] /; Support[λ1] == Support[λ2] := ⟨Table[
    s →
     b[λ1s, λ2s] + TangentialDerivation[λ1][λ2s] - TangentialDerivation[λ2][λ1s],
    {s, Support[λ1]}
   ]⟩;
div[λ_AngleBracket] := Sum[divs[λs], {s, Support[λ]}];
j[λ_AngleBracket] :=
  DerivationSeries[(e^der - 1)/der, TangentialDerivation[λ]][div[λ]];
```

Evaluating Lie Series in ⟨…⟩

```
LieMorphism[rules__Rule, keys_AngleBracket, br_] :=
  LieMorphism[{rules}, keys, br];
LieMorphism[rules_List, keys_AngleBracket, br_] := New[LieMorphism[mor],
   mor[Support] = First /@ rules;
   (mor[w_LW] /; Deg[w] == 1) := (mor[w] = w /. rules);
   mor[w_LW] := (mor[w] = br @@ (mor /@ LyndonFactorization[w]));
   mor[expr_][d_] := Expand[expr /. w_LW :> mor[w][d]];
   mor[ls_LieSeries] := mor[ls] = Module[{ser},
      ⟨Table[
        ReleaseHold[Hold[
           ser[] = Hold[AngleBracketFromLieMorphism[mor, ls, ss]];
           ser[d_Integer] := ser[d] = (Sum[mor[ls[k]][d], {k, 1, d}])_ss;
           ss → LieSeries[ser]
         ]] /. {ss → s, ser → Unique[LieSeriesInAngleBracket]}],
        {s, List @@ keys}
       ]⟩]
  ];
BCH[x_, y_, keys_AngleBracket, br_] :=
  LieMorphism[{LW["x"] → x, LW["y"] → y}, keys, br][BCHBase];
```

The AT Presentation E_l of A^w

```
El /: El[λ1_, ω1_] ⋃ El[λ2_, ω2_] /; Support[λ1] ⋂ Support[λ2] == {} :=
  El[λ1 ⋃ λ2, ω1 + ω2];
El /: El[λ1_, ω1_] ** El[λ2_, ω2_] /; Support[λ1] == Support[λ2] :=
  El[BCH[λ1, λ2, ⟨1, 2⟩, tb], ω1 + DerivationExp[TangentialDerivation[λ1]][ω2]];
El[λ_, ω_] // dA := El[-λ, DerivationExp[TangentialDerivation[λ]][ω] - j[λ]];
El[λ_, ω_] // dS := El[
   -λ // (-1)^deg,
   (DerivationExp[TangentialDerivation[λ]][ω] - j[λ]) // (-1)^deg
  ];
```

The KBH Presentation E_s of A^w

```
Es /: Es[λ1_, ω1_] ⋃ Es[λ2_, ω2_] /; Support[λ1] ⋂ Support[λ2] == {} :=
  Es[λ1 ⋃ λ2, ω1 + ω2];
```

```
tσ[us_List → vs_List][ser_LieSeries | ser_CWSeries | ser_AngleBracket] :=
  ser // LieMorphism[Thread[(LW /@ us) → (LW /@ vs)]];
tσ[u_, v_] := tσ[{u} → {v}];
tσ[us_List → vs_List][ζ_Es] := tσ[us → vs] /@ ζ;
hσ[xs_List → ys_List][λ_AngleBracket] :=
  Union[λ \ xs, ⟨Thread[ys → Table[λx, {x, xs}]]⟩];
hσ[x_, y_] := hσ[{x} → {y}];
hσ[xs_List → ys_List][Es[λ_, ω_]] := Es[λ // hσ[xs → ys], ω];
dσ[as_List → bs_List][ζ_] := ζ // tσ[as → bs] // hσ[as → bs];
dσ[a_, b_][ζ_] := ζ // tσ[a, b] // hσ[a, b];

tm[u_, v_, w_][λ_AngleBracket] := λ // LieMorphism[LW@u → LW@w, LW@v → LW@w];
tm[u_, v_, w_][Es[λ_, ω_]] := LieMorphism[LW@u → LW@w, LW@v → LW@w] /@ Es[λ, ω];
hm[x_, y_, z_][λ_AngleBracket] := Union[λ \ {x, y}, ⟨z → BCH[λx, λy]⟩];
hm[x_, y_, z_][Es[λ_, ω_]] := Es[λ // hm[x, y, z], ω];
tha[u_LW, x_][λ_AngleBracket] := λ // RCu[λx];
tha[u_LW, x_][Es[λ_, ω_]] := Es[λ // tha[u, x], (ω + Ju[λx]) // RCu[λx]];
dm[a_, b_, c_][ζ_] := ζ // tha[LW@a, b] // tm[LW@a, LW@b, LW@c] // hm[a, b, c];
dm[a_, b_, rest__, c_][ζ_] := ζ // dm[b, rest, b] // dm[a, b, c];

Es /: Es[λ1_, ω1_] ** Es[λ2_, ω2_] /; Support[λ1] == Support[λ2] := Module[
   {supp, temps, ζ},
   supp = Support[λ1];
   temps = Complement[Characters["0123456789abcdefghijklmnopqrstuvwxyz"],
      ToString /@ supp][[1 ;; Length@supp]];
   ζ = Es[λ1, ω1] ⋃ (Es[λ2, ω2] // dσ[supp → temps]);
   MapThread[(ζ = ζ // dm[#1, #2, #1]) &, {supp, temps}] // Last
  ];

tA[u_][expr_] := expr;
hA[x_][Es[λ_, ω_]] := Es[Union[λ \ x, ⟨x → -λx⟩], ω];
dA[a_][μ_] := μ // hA[a] // tha[LW@a, a];
dA[a_, rest__][μ_] := μ // dA[a] // dA[rest];
Es[λ_, ω_] // dA := Es[λ, ω] // (dA @@ Support[λ])

tS[u_][λ_AngleBracket] :=
  ⟨Table[x → LieMorphism[LW@u → -LW@u][λx], {x, Support[λ]}]⟩;
tS[u_][Es[λ_, ω_]] := Es[λ // tS[u], ω // LieMorphism[LW@u → -LW@u]];
hS[x_][Es[λ_, ω_]] := Es[Union[λ \ x, ⟨x → -λx⟩], ω];
dS[a_][μ_] := μ // tS[a] // hS[a] // tha[LW@a, a];
dS[a_, rest__][μ_] := μ // dS[a] // dS[rest];
Es[λ_, ω_] // dS := Es[λ, ω] // (dS @@ Support[λ])
```

The $E_l \leftrightarrow E_s$ Conversions

```
Γ[-1, λ_AngleBracket, _] := ⟨Table[s → MakeLieSeries[0], {s, Support[λ]}]⟩;
Γ[n_, λ_AngleBracket, t_] := Γ[n, λ, t] = Module[{τ, Γ0},
    Γ0 = Γ[n - 1, λ, τ];
    ⟨Table[
      s → ∫_0^t (λ_s // DerivationExp[-τ TangentialDerivation[λ]] //
           adSeries[ad/(e^ad - 1), Γ0_s]) dτ,
      {s, Support[λ]}
     ]⟩
   ];
Γ[λ_AngleBracket, t_] := Γ[λ, t] = ⟨Table[
     s → New[LieSeries[ser],
       With[{s = s}, ser[d_Integer] := ser[d] = (Γ[d - 1, λ, t]_s)[d]];
      ],
     {s, Support[λ]}
    ]⟩;
Γ[λ_AngleBracket] := Γ[λ, 1];
Γi[λ_AngleBracket] := Γi[λ] = ⟨Table[With[{s = s},
      s → New[LieSeries[ser],
        ser[d_Integer] := ser[d] = λ_s[d] - Γ[Crop[Γi[λ], d - 1]]_s[d]
       ]],
     {s, Support[λ]}
    ]⟩;
Γ /: Γ^-1 = Γi;
Es[El[λ_, ω_]] := Es[Γ[λ], ω];
El[Es[λ_, ω_]] := El[Γ^-1[λ], ω];
```