FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS III: SOME COMPUTATIONS

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Abstract. In the previous two papers in this series, [BND1, BND2], Z. Dancso and I studied a certain theory of "homomorphic expansions" of "w-knotted objects", a certain class of knotted objects in 4-dimensional space. When all layers of interpretation are stripped off, what remains is a study of a certain number of equations written in a family of spaces $\mathcal{A}^w$, closely related to degree-completed free Lie algebras and to degree-completed spaces of cyclic words.

The purpose of this paper is to introduce mathematical and computational tools that enable explicit computations (up to a certain degree) in these $\mathcal{A}^w$ spaces and to use these tools to solve the said equations and verify some properties of their solutions, and as a consequence, to carry out the computation (up to a certain degree) of certain knot-theoretic invariants discussed in [BND1, BND2] and in ~~the related~~ [BN1].
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1. Introduction

Within the previous two papers in this series [BND1, BND2][1] a number of intricate equations written in various graded spaces related to free Lie algebras and to spaces of cyclic words were examined in detail, for good reasons that were explained there and elsewhere. The purpose of this paper is to introduce mathematical and computational tools that allow for the explicit solution of these equations, at least up to a certain degree.

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[1] Also within my [BN1], and within papers by Alekseev, Enriquez, and Torossian [AT, AET], and within Kashiwara's and Vergne's [KV], and also within many older papers about Drinfel'd associators (e.g. [Dr1, Dr2, BN2].

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Why bother? What do limited explict computations add, given that these intricate equations are known to be soluble, and given that the conceptual framework within which these equations make sense is reasonably well understood [BND1, BND2]? My answers are three:

(1) Personally, my belief in what I can't compute decays quite rapidly as a function of the complexity involved. Even if the overall picture is clear, the details will surely go wrong, and sooner or later, something bigger than a detail will go wrong. Even a limited computation may serve as a wonderful sanity check. In situations such as ours, where many signs and conventions need to be decided and may well go wrong, even a low-degree computation increases my personal confidence level by a great degree. Given computations that work to degree 6 (say), it is hard to imagine that a detail was missed or that conventions were established in an inconsistent manner. In fact, if the computer programs are clear enough and are shown to work, these programs become the authoritative declerations of the details and conventions.
(2) The computational tools introduced here may well be used in other contexts where free Lie algebras and/or cyclic words arise.
(3) The paper [BND1] (and likewise [BN1]) is about equations, but even more so, about the construction of certain knot invariants. With the tools presented here, the invariants of arbitrary knotted objects of the types studied in [BND1, BN1] may be computed.

The equations of [BND1, BND2] always involve group-like, or "exponential" elements, and are written in some spaces of "arrow diagrams" that go under the umbrella name $\mathcal{A}^w$. Hence a crucial first step is to find convenient presentations for the group-like elements $\mathcal{A}^w_{\exp}$ in $\mathcal{A}^w$-spaces. It turns out that there are (at least) two such presentations, each with its own advantages and disadvantages. Hence in Section 2 we recall $\mathcal{A}^w$ briefly (2.1), discuss the "AT" and the "KBH" presentations of $\mathcal{A}^w_{\exp}$ (2.2 and 2.3), and describe how to convert between the two presentations (2.4).

MORE: summaries of the remaining sections.

2. Group-like elements in $\mathcal{A}^w$

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2.1. A brief review of $\mathcal{A}^w$. MORE.

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Comment 2.1. Why are there two presentations to elements of $\mathcal{A}^w_{\exp}$?

Roughly speaking, $\mathcal{A}^w$ is a combinatorial model of (tensor powers of a completion of) the universal enveloping algebra $\mathcal{U}(I\mathfrak{g})$ of the semi-direct product $I\mathfrak{g} = \mathfrak{g}^* \rtimes \mathfrak{g}$, for any finite-dimensional Lie algebra $\mathfrak{g}$, and where $\mathfrak{g}^*$ is taken as an Abelian Lie algebra and $\mathfrak{g}$ acts on $\mathfrak{g}^*$ using the co-adjoint action.

By PBW, $\mathcal{U}(I\mathfrak{g}) \simeq \mathcal{S}(\mathfrak{g}^*) \otimes \mathcal{U}(\mathfrak{g})$, and hence group-like elements in $\mathcal{U}(I\mathfrak{g})$ can either be written in "mixed form", as exponentials of elements of $\mathfrak{g}^* \rtimes \mathfrak{g}$, or in "factored form", as product of an exponential in $\mathcal{S}(\mathfrak{g}^*)$ with an exponential in $\mathcal{U}(\mathfrak{g})$. Very roughly speaking, the "mixed form" corresponds to the "AT presentation" below, and the "factored form" to the "KBH presentation" below.

The reality is a bit more delicate, though. $\mathcal{A}^w$ is only a model of the $\mathfrak{g}$-invariant part of $\mathcal{U}(I\mathfrak{g})$, and the notions of being group-like in $\mathcal{A}^w$ and in $\mathcal{U}(I\mathfrak{g})$ do not match. Yet the flavour remains — in the AT presentation arrow tails ("elements of $\mathfrak{g}^*$") mix with arrow heads ("$\mathfrak{g}$"), while in the KBH presentation heads and tails are kept apart.

2.2. The AT presentation of $\mathcal{A}^w_{\exp}$. MORE.

2.3. The KBH presentation of $\mathcal{A}^w_{\exp}$. MORE.

2.4. The conversion between the AT and the KBH presentations. MORE.

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Everything below is to be blanked out before the completion of this paper. sectionIntroduction This paper being a third in a series [BND1, BND2], as well as a continuation of [AT, AET] and of [BN1], we will forgo a description of the context and the motivations and forgo the precise definitions, and instead jump right into the heart of the matter — the equations we seek to solve, and the spaces in which they are written. Our fundamental quantities are

- $R = \exp(\uparrow\!\!\rightarrow\!\!\uparrow)$, the Z^w-value of a crossing, a member of the space $\mathcal{A}^w(\uparrow_2)$ defined in [BND1] and reviewed in Section 2.4 below.
- V, the Z^w-value of a vertex, a member of $\mathcal{A}^w(\uparrow_3)$.
- $C \in \mathcal{A}^w(\uparrow)$, the Z^w-value of a cap.
- A Drinfel'd associator Φ and a braiding element for u-braids $\Theta = \exp(\frac{1}{2}\uparrow\!\!-\!\!\uparrow)$.

subsectionThe Equations

- Reidemeister 4, R4 $$R_{23}R_{13}V = VR_{12,3} \qquad (1)$$

subsectionThe Spaces

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