

Pensieve header: A $\mathbb{Z}\langle \nu^{-1} \rangle$ -like computation, except with the middle strand punctured, as suggested by Zsuzsi ... version of July 2018.

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO3"];
<< "../WKO4/FreeLie.m";
<< "../WKO4/AwCalculus.m";
$SeriesShowDegree = 4;
```

FreeLie` implements / extends
 {*, +, **, \$SeriesShowDegree, <>, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords, AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization, Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support, t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ℓ, Δ, σ, ħ, ↦, ↠}.

FreeLie` is in the public domain. Dror Bar-Natan is committed to support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA, hm, hS, hΔ, hη, hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Δ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed to support it within reason until July 15, 2022. This is version 150909.

```
In[*]:= Φs[2, 1] = Φs[3, 1] = Φs[3, 2] = 0; Φs[3, 1, 2] = 1/24; Φ₀ = DKS[3, Φs];
SeriesSolve[Φ₀, (Φ₀^σ[3,2,1] ≡ -Φ₀) ∧ (Φ₀ ** Φ₀^σ[1,2,3,4] ** Φ₀^σ[2,3,4] ≡ Φ₀^σ[12,3,4] ** Φ₀^σ[1,2,3,4])];
Φ₀@{6}
```

SeriesSolve: In degree 3 arbitrarily setting {Φs[3, 1, 1, 2] → 0}.

SeriesSolve: In degree 5 arbitrarily setting {Φs[3, 1, 1, 1, 2] → 0}.

$$\text{Out[*]} = \text{DKS}\left[0, \frac{1}{24} t_{13} t_{23}, 0, -\frac{7 t_{13} t_{23} t_{23} t_{23}}{5760} + \frac{7 t_{13} t_{13} t_{23} t_{23}}{5760} - \frac{t_{13} t_{13} t_{13} t_{23}}{1440}, \right. \\
0, \frac{31 t_{13} t_{23} t_{23} t_{23} t_{23}}{967680} - \frac{157 t_{13} t_{13} t_{23} t_{23} t_{13} t_{23}}{1935360} - \frac{31 t_{13} t_{23} t_{13} t_{23} t_{23} t_{23}}{387072} - \\
\frac{31 t_{13} t_{13} t_{23} t_{23} t_{23} t_{23}}{483840} + \frac{11 t_{13} t_{13} t_{13} t_{23} t_{13} t_{23}}{290304} + \frac{31 t_{13} t_{13} t_{23} t_{13} t_{23} t_{23}}{725760} + \\
\left. \frac{83 t_{13} t_{13} t_{13} t_{23} t_{23} t_{23}}{967680} - \frac{13 t_{13} t_{13} t_{13} t_{13} t_{23} t_{23}}{241920} + \frac{t_{13} t_{13} t_{13} t_{13} t_{13} t_{23}}{60480}, \dots \right]$$

```
In[*]:= DK2Es[s___][s_] := El[s // aMap[s], CWS[0]] // r;
DK2Es[1, 2, 3][x_0]
```

$$\begin{aligned}
 \text{Out[*]} = & \text{Es} \left[\left\langle 1 \rightarrow \text{LS} \left[0, \frac{\overline{23}}{24}, 0, -\frac{\overline{1123}}{1440} + \frac{\overline{71223}}{5760} + \frac{\overline{1233}}{5760} - \frac{\overline{72223}}{5760} + \right. \right. \\
 & \left. \frac{\overline{72233}}{5760} + \frac{1}{480} \frac{\overline{1213}}{1213} - \frac{\overline{1323}}{1920} + \frac{1}{640} \frac{\overline{1232}}{1232} - \frac{\overline{1322}}{1152} - \frac{\overline{1332}}{1152} - \frac{\overline{2333}}{1440}, \dots \right\rangle, \\
 & 2 \rightarrow \text{LS} \left[0, -\frac{\overline{13}}{24}, 0, \frac{\overline{1113}}{1440} - \frac{\overline{1123}}{1152} + \frac{\overline{71223}}{1920} - \frac{1}{480} \frac{\overline{1132}}{1132} - \frac{\overline{1133}}{5760} + \frac{\overline{1233}}{1152} + \right. \\
 & \left. \frac{\overline{71213}}{5760} + \frac{19}{5760} \frac{\overline{1323}}{1323} + \frac{7}{1920} \frac{\overline{1232}}{1232} + \frac{7}{5760} \frac{\overline{1322}}{1322} + \frac{7}{5760} \frac{\overline{1332}}{1332} + \frac{\overline{1333}}{1440}, \dots \right\rangle, \\
 & 3 \rightarrow \text{LS} \left[0, \frac{\overline{12}}{24}, 0, -\frac{\overline{1112}}{1440} + \frac{\overline{1123}}{5760} + \frac{\overline{71223}}{5760} + \frac{\overline{71122}}{5760} - \frac{\overline{1132}}{1440} - \frac{\overline{1233}}{1440} + \frac{\overline{1213}}{5760} \right. \\
 & \left. \frac{\overline{1323}}{1440} - \frac{\overline{1232}}{1152} - \frac{7}{5760} \frac{\overline{1222}}{1222} - \frac{7}{5760} \frac{\overline{1322}}{1322} - \frac{\overline{1332}}{1440}, \dots \right\rangle, \text{CWS}[0, 0, 0, 0, \dots] \right]
 \end{aligned}$$

```
In[*]:= vinv = x_0 // DK2Es[1, 2, 3] // dS[2] // dm[3, 2, 2] // dm[2, 1, x]
```

$$\text{Out[*]} = \text{Es} \left[\langle x \rightarrow \text{LS}[0, 0, 0, 0, \dots] \rangle, \text{CWS} \left[0, \frac{\overline{xx}}{24}, 0, -\frac{\overline{xxxx}}{2880}, \dots \right] \right]$$

```
In[*]:= vinv@{6}
```

$$\text{Out[*]} = \text{Es} \left[\langle x \rightarrow \text{LS}[0, 0, 0, 0, 0, 0, \dots] \rangle, \text{CWS} \left[0, \frac{\overline{xx}}{24}, 0, -\frac{\overline{xxxx}}{2880}, 0, \frac{\overline{xxxxxx}}{181440}, \dots \right] \right]$$

```
In[*]:= zv = x_0 // DK2Es[1, 2, 3] // dS[2] // teta[2] // dm[3, 2, 2] // dm[2, 1, x];
zv@{6}
```

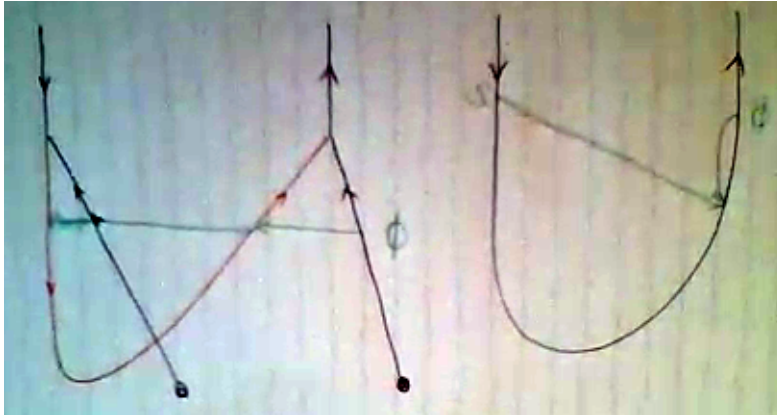
$$\text{Out[*]} = \text{Es} \left[\langle x \rightarrow \text{LS}[0, 0, 0, 0, 0, 0, \dots] \rangle, \text{CWS} \left[0, \frac{\overline{xx}}{24}, 0, -\frac{7}{2880} \frac{\overline{xxxx}}{xxxx}, 0, \frac{31}{181440} \frac{\overline{xxxxxx}}{xxxxxx}, \dots \right] \right]$$

```
In[*]:= (zv ** vinv ** zv ** vinv)@{6}
```

$$\text{Out[*]} = \text{Es} \left[\langle x \rightarrow \text{LS}[0, 0, 0, 0, 0, 0, \dots] \rangle, \text{CWS} \left[0, \frac{\overline{xx}}{6}, 0, -\frac{\overline{xxxx}}{180}, 0, \frac{\overline{xxxxxx}}{2835}, \dots \right] \right]$$

```
In[*]:= (vinv // dDelta[x, x, y] // dm[x, y, x])@{6}
```

$$\text{Out[*]} = \text{Es} \left[\langle x \rightarrow \text{LS}[0, 0, 0, 0, 0, 0, \dots] \rangle, \text{CWS} \left[0, \frac{\overline{xx}}{6}, 0, -\frac{\overline{xxxx}}{180}, 0, \frac{\overline{xxxxxx}}{2835}, \dots \right] \right]$$



$In[*]:=$ $zv1 = \overline{\emptyset} // DK2Es[1, 2, 3] // dS[1] // t\eta[2] // dm[1, 2, 1] // dm[1, 3, x];$
 $zv1@{6}$

$Out[*]:=$ $Es[\langle x \rightarrow LS[0, 0, 0, 0, 0, 0, \dots] \rangle, CWS[0, -\frac{\overline{xx}}{24}, 0, \frac{\overline{xxxx}}{2880}, 0, -\frac{\overline{xxxxxx}}{181440}, \dots]]$

$In[*]:=$ $(zv1 ** vinv)@{6}$

$Out[*]:=$ $Es[\langle x \rightarrow LS[0, 0, 0, 0, 0, 0, \dots] \rangle, CWS[0, 0, 0, 0, 0, 0, \dots]]$

$In[*]:=$ $zv1u = \overline{\emptyset} // DK2Es[1, 2, 3] // dS[1] // dm[1, 2, 1] // dm[1, 3, x];$
 $zv1u@{6}$

$Out[*]:=$ $Es[\langle x \rightarrow LS[0, 0, 0, 0, 0, 0, \dots] \rangle, CWS[0, -\frac{\overline{xx}}{24}, 0, \frac{7 \overline{xxxx}}{2880}, 0, -\frac{31 \overline{xxxxxx}}{181440}, \dots]]$

$In[*]:=$ $zv2 = \overline{\emptyset} // DK2Es[1, 2, 3] // dS[1] // t\eta[2] // dS[3] // dm[1, 2, 1] // dm[1, 3, x];$
 $zv2@{6}$

$Out[*]:=$ $Es[\langle x \rightarrow LS[0, 0, 0, 0, 0, 0, \dots] \rangle, CWS[0, \frac{\overline{xx}}{24}, 0, -\frac{7 \overline{xxxx}}{2880}, 0, \frac{31 \overline{xxxxxx}}{181440}, \dots]]$

$In[*]:=$ $(zv1u ** zv2)@{6}$

$Out[*]:=$ $Es[\langle x \rightarrow LS[0, 0, 0, 0, 0, 0, \dots] \rangle, CWS[0, 0, 0, 0, 0, 0, \dots]]$

$In[*]:=$ $\phi = LS[Sum[\overline{\emptyset}[k], \{k, 6\}][2]]$

$Out[*]:=$ $LS[0, \frac{\overline{12}}{24}, 0, -\frac{\overline{1112}}{1440} + \frac{7 \overline{1122}}{5760} - \frac{7 \overline{1222}}{5760}, \dots]$

$In[*]:=$ $(J_1[\phi] // LieMorphism[LW@1 \rightarrow LW@1, LW@2 \rightarrow LW@1])@{6}$

$Out[*]:=$ $CWS[0, -\frac{\overline{11}}{24}, 0, \frac{13 \overline{1111}}{5760}, 0, -\frac{53 \overline{11111}}{362880}, \dots]$

$In[*]:=$ $(j[\langle LW[1] \rightarrow \phi, LW[2] \rightarrow 0 \rangle] // LieMorphism[LW@1 \rightarrow LW@1, LW@2 \rightarrow -LW@1])@{6}$

$Out[*]:=$ $CWS[0, \frac{\overline{11}}{24}, 0, -\frac{\overline{1111}}{1440}, 0, \frac{\overline{11111}}{60480}, \dots]$

$$\text{In[*]}:= \phi1 = \text{LieMorphism}[\text{LW}[1] \rightarrow -\text{LW}[1]] [\phi]$$

$$\text{Out[*]}:= \text{LS} \left[0, -\frac{\overline{12}}{24}, 0, \frac{\overline{1112}}{1440} + \frac{71\overline{122}}{5760} + \frac{7\overline{1222}}{5760}, \dots \right]$$

$$\text{In[*]}:= (\mathbf{J}_1[\phi1] // \text{LieMorphism}[\text{LW}@1 \rightarrow \text{LW}@1, \text{LW}@2 \rightarrow -\text{LW}@1]) @ \{6\}$$

$$\text{Out[*]}:= \text{CWS} \left[0, -\frac{\overline{11}}{24}, 0, \frac{13\overline{1111}}{5760}, 0, -\frac{53\overline{111111}}{362880}, \dots \right]$$

$$\text{In[*]}:= (\mathbf{j}[\langle \text{LW}[1] \rightarrow \phi1, \text{LW}[2] \rightarrow 0 \rangle] // \text{LieMorphism}[\text{LW}@1 \rightarrow \text{LW}@1, \text{LW}@2 \rightarrow \text{LW}@1]) @ \{6\}$$

$$\text{Out[*]}:= \text{CWS} \left[0, \frac{\overline{11}}{24}, 0, -\frac{\overline{1111}}{1440}, 0, \frac{\overline{111111}}{60480}, \dots \right]$$