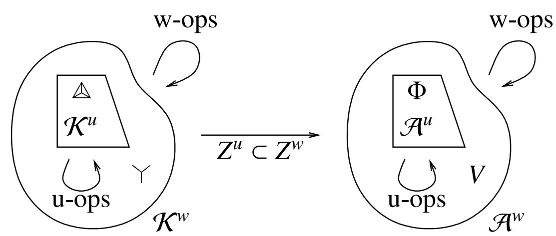
WKO3: Executive Summary

This section is followed by a more traditional introduction.

A "homomorphic expansion" for a certain class of topological objects $\mathcal{K}$ is an invariant $Z\colon \mathcal{K} \to \mathcal{A}$ whose target space $\mathcal{A}$ is canonically associated with $\mathcal{K}$ (its "associated graded"), and which satisfies a certain universality property, and which respects a certain collection of operations which exist on $\mathcal{K}$, and therefore also on $\mathcal{A}$. Homomorphic expansions are often hard to find, and when they are found, they often correspond to some deep mathematics:

- Many classes of knotted objects in 3-dimensional spaces do not have homomorphic expansions — one would have loved ordinary tangles to have homomorphic expansions, for example, but they don't.
- Yet a certain class $\mathcal{K}^u$ of knotted objects in 3-space, the class of "parenthesized tangles", or nearly-equivalently, the class of knotted trivalent graphs (which we adopt in this paper) does have homomorphic expansions. A homomorphic expansion $Z^u\colon \mathcal{K}^u \to \mathcal{A}^u$ is defined by its values on a couple of elements of $\mathcal{K}^u$ which generate $\mathcal{K}^u$ using the operations $\mathcal{K}^u$ is equipped with. The most interesting of these generators is the tetrahedron $\triangle$, and $\Phi = Z^u(\triangle)$ turns out to be equivalent to "a Drinfel'd associator".
- A certain class $\mathcal{K}^w$ of graphs, which is conjectured to be equivalent to a certain class of 2-dimensional knotted objects in 4-space, also has homomorphic expansions. The most interesting generator of $\mathcal{K}^w$ is the "vertex" $\curlyvee$, and if $Z^w\colon \mathcal{K}^w \to \mathcal{A}^w$ is a homomorphic expansion, then it turns out that $V = Z^w(\curlyvee)$ is equivalent to "a solution of the Kashiwara-Vergne problem".



Roughly speaking, $\mathcal{K}^u$ is a part of $\mathcal{K}^w$ and $\mathcal{A}^u$ is a part of $\mathcal{A}^w$, as in the figure on the right (more precisely, there are natural maps $a\colon \mathcal{K}^u \to \mathcal{K}^w$ and $\alpha\colon \mathcal{A}^u \to \mathcal{A}^w$). The main purpose of this paper is to prove the following theorem:

Theorem (precise version in Theorem 1.1). Any homomorphic expansion Z^u for $\mathcal{K}^u$ extends uniquely to a homomorphic expansion Z^w for $\mathcal{K}^w$, and therefore, any Drinfel'd associator Φ yields a solution V of the Kashiwara-Vergne problem.

The proof of this theorem is almost banal. We simply show that the generators of $\mathcal{K}^w$ can be explicitly expressed using the generators of $\mathcal{K}^u$ and the operations of $\mathcal{K}^w$, and that the resulting explicit formulas for $Z^w(\curlyvee)$ (and for Z^w of the other generators) satisfies all the required relations.

The devil is in the details. It is in fact impossible to express the generators of $\mathcal{K}^w$ in terms of the generators of $\mathcal{K}^u$ — to do that, one first has to pass to a larger space $\tilde{\mathcal{K}}^w$ that has more objects and more operations, and in which the desired explicit expressions do exist. But even in $\tilde{\mathcal{K}}^w$ these expressions are complicated, and are best described within a certain "double tree construction" which also provides the framework for the verification of relations. Here's an unexplained summary; the explanations make the bulk of this paper: