All u/caps are downcaps.

We need an index of notations?

FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS III: THE DOUBLE TREE CONSTRUCTION

DROR BAR-NATAN AND ZSUZSANNA DANCSO

ABSTRACT. This is the third in a series of papers studying the finite type invariants of various w-knotted objects and their relationship to the Kashiwara-Vergne problem and Drinfel'd associators. In this paper we present a topological solution to the Kashiwara-Vergne problem. In particular we recover via a topological argument the Alkeseev-Enriquez-Torossian [AET] formula for explicit solutions of the Kashiwara-Vergne equations in terms of associators.

We study a class of w-knotted objects: knottings of *2-dimensional foams* and various associated features in four-dimensioanl space. We use a topological construction which we name the double tree construction to show that every *expansion* (also known as *universal finite type invariant*) of parenthesized braids extends first to an expansion of knotted trivalent graphs (a well known result), and then extends uniquely to an expansion of the w-knotted objects mentioned above.

In algebraic language, an expansion for parenthesized braids is the same as a *Drinfel'd associator* Φ, and an expansion for w-knotted objects is the same as a solution V of the Kashiwara-Vergne problem [KV] as reformulated by Alekseev and Torossian [AT]. Hence our result provides a topological framework for the result of [AET] that "there is a formula for V in terms of Φ", along with an independent topological proof that the said formula works — namely that the equations satisfied by V follow from the equations satisfied by Φ.

CONTENTS

Date: first edition in future, this edition Dec. 27, 2021. The arXiv:????.???? edition may be older.

2020 *Mathematics Subject Classification.* 57M25.

Key words and phrases. virtual knots, w-braids, w-knots, w-tangles, knotted graphs, finite type invariants, Kashiwara-Vergne, associators, double tree, free Lie algebras.

add RGPIN-2018-04350

The first author's work was partially supported by NSERC grant RGPIN 264374, and wishes to thank the Sydney Mathematics Research Institute for their hospitality and support. The second author was partially supported by NSF grant no. 0932078 000 while in residence at the Mathematical Sciences Research Institute, and by the Australian Research Council DECRA DE170101128. Electronic version and related files at [WKO3], http://www.math.toronto.edu/~drorbn/papers/WKO3/.

Add Chu Family Foundation (NYC)

Now we show that V determines C uniquely. Assume there are different values C and C' in $\mathcal{A}^{sw}(\uparrow)$ so that (V, C) and (V, C') are both vertex-cap value pairs of Z^u-compatible homomorphic expansions. Let c denote the lowest degree term of $C - C'$, then c is a scalar multiple of a single wheel. The Cap Equation of Fact 2.5 implies $c_{(12)} = c_1 + c_2$ in $\mathcal{A}^{sw}(\uparrow_2)$.

There is a well-defined linear map $\omega : \mathcal{A}^{sw}(\uparrow_2) \to \mathbb{Q}[x, y]$ sending an arrow diagram – which has arrow tails only on each strand – to "x to the power of the number of tails on strand 1, times y to the power of the number of tails on strand 2". Assume $c = \alpha w_r$, where w_r denotes the r-wheel, and $\alpha \in \mathbb{Q}$. Then $0 = \omega(c_{(12)} - c_1 - c_2) = \alpha((x+y)^r - x^r - y^r)$, so either $r = 1$ or $\alpha = 0$. But $w_1 = 0$ in $\mathcal{A}^{sw}$ by the RI relation, hence $\alpha = 0$ and thus $c = 0$, a contradiction. $\square$

3.2. **Proof of Part (2).** In this section we compute V, the value of the vertex, from Φ, the Drinfel'd associator determining Z^b, using the construction of Part (1). We then show that the result translates to the [AET] formula for Kashiwara-Vergne solutions in terms of Drinfel'd associators.

3.2.1. *From Φ to V.* To compute V, consider once again the w-tangled foam K on the right of Figure 11. On one hand, $Z^w(K)$ can be computed directly from the generators: $Z^w(K) = C_1C_2V_{12} \in \mathcal{A}^{sw}(\uparrow_2)$, since the values of the left-punctured vertices are trivial. On the other hand, one can compute $Z^w(K)$, using the compatibility with Z^u, from $\beta^u = Z^u(B^u)$, where B^u is the closure of the parenthesised braid B^b shown in Figure 12. In particular, $Z^w(K) = C_{12}\varphi(p_1p_3\alpha(\beta^u))$, where φ is the isomorphism of Lemma 2.4. In summary, $V = C_1^{-1}C_2^{-1} \cdot \varphi(p_1p_3\alpha(\beta^u))C_{12}$. To obtain the formula (2) of Theorem 1.1, one needs to compute $p_1p_3\alpha(\beta^u)$ in terms of Φ.

Too big a leap! needs explanation.

By the compatibility of Z^u and Z^b, it is enough to compute $\beta^b := Z^b(B^b)$. The result can be read from the picture in Figure 12:

$$\beta^b = \Phi^{-1}_{(13)24}\Phi_{132}R_{32}\Phi^{-1}_{123}\Phi_{(12)34}.$$

To interpret this formula, recall that the associator Φ is an element of $\mathcal{A}^{hor}(\uparrow_3)$, and the subscripts show which strands diagrams are placed on. For example, the notation $\Phi^{-1}_{(13)24}$ means doubling the first strand of Φ^{-1} and placing the resulting chord endings on strands 1 and 3, as well as placing the chord endings from the other two strands of Φ^{-1} on strands 2 and 4. Also recall that $R = e^{c/2}$, where c is a single horizontal chord between two strands (and in this case R_{32} means that this chord runs between strands 3 and 2).

As β^u is the tree closure of β^b, it is given by the same formula interpreted as an element of $\mathcal{A}^u(\uparrow_4)$. One then applies α to obtain $\beta^w = \alpha(\beta^u)$, followed by puncturing strands 1 and 3 and caping strands 2 and 4.

To begin understanding the effect of these operations, we note that $p_3\alpha(R_{32}) = e^{a_{23}/2}$, where a_{ij} is a single arrow pointing from strand i to strand j.

Recall that Φ_{123} is a horizontal chord associator which can be expressed as a power series in non-commuting variables c_{12} and c_{23} (i.e., chords between strands 1—2 and 2—3, respectively). The image of Φ in the quotient where c_{12} and c_{23} commute is 1. Hence, $p_1p_3(\alpha\Phi^{-1}_{123}) = 1$, as the punctured strands only support arrow heads, and tails on the middle strand commute by TC. Similarly, $p_1p_3(\alpha\Phi_{132}) = 1$ because the punctures kill the entire "left side" of the associator (that is, $p_1p_3\alpha(c_{13}) = 0$).

From here to next arrow, re-style as "formulas interlaced with explanations" as in my attached handwritten page.

Dror must reread after re-styling is done.

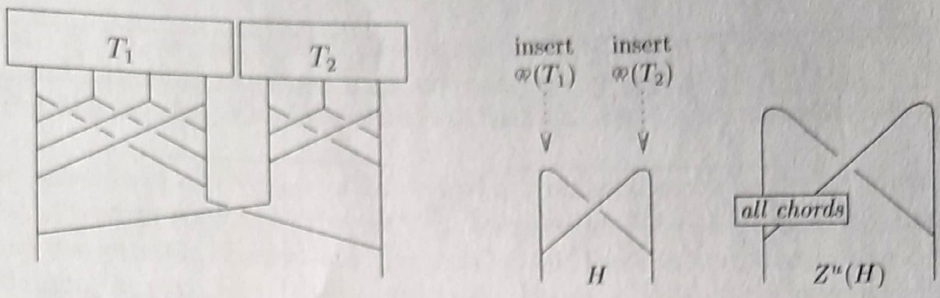

Figure 26. The double tree map applied to a disjoint union of $u\mathcal{T}T$-s is the same as inserting the double tree of each individual $u\mathcal{T}T$ into the $sKTG$ H. In $Z^u(H)$ all chords can be pushed into the rectangle shown, using VI relations when necessary.

the same algorithm as before, but we can save ourselves the work, as follows. All chords in $Z^u(H)$ can be assumed to be located in the rectangle shown in Figure 26. After applying α, both supporting strands are punctured, meaning that after punctures $p\alpha(Z^u(H)) = 1$ in $\mathcal{A}^{sw}$. This implies that $\xi(T_1 \sqcup T_2) = \xi(T_1) \sqcup \xi(T_2)$, and it follows immediately that $Z^w(T_1 \sqcup T_2) = Z^w(T_1) \sqcup Z^w(T_2)$.

Contractions. Proving that Z^w commutes with contractions is more involved. By Lemma 3.2, we can assume ~~without loss of generality~~ that the ends contracted are the last (rightmost) two ends of T. Hence we will drop the subscript from c_i and denote this operation simply by c. of the n ends of T

Consider re-arranging this proof in the same spirit of my suggestion for section 3.2.1, one advantage is that Fig 27 breaks into many smaller squares, each of which can be made larger & clearer.

We need to show that $Z^w(cT) = cZ^w(T)$, for any $T \in u\mathcal{T}T$. The homomorphic expansion Z^w is constructed as the composition of many maps, so this can be restated as the commutativity of the perimeter of a large diagram, which can be broken down to the commutativity of its smaller parts: the diagram is shown in Figure 27. To complete the proof, we analyze the commutativity of each numbered square (and one pentagon) of the diagram:

(1) This square plays out at *skeleton level* in $u\mathcal{T}T$ and $sKTG$. It clearly commutes, for example by inspection of the first square of the bottom diagram in Figure 27. In the diagram, the three strands to delete are indicated by dotted lines. Deletion in the consecutive square of the diagram always refers to these same strands.

(2) To show that this square commutes, one needs to understand the properties of Z^u with respect to deleting edges in $sKTG$. When an edge – which ends in a vertex at both ends – is deleted, those two vertices cease to be vertices. The associated graded operation on chord diagrams deletes the skeleton edge, and chord diagrams with any chord endings on the deleted edge are set equal to 0. See [WKO2, Section 4.6.1] for more detail.

The expansion Z^u is not quite homomorphic with respect to such edge deletions. Recall that in [WKO2, Section 4.6.1] Z^u is constructed from an invariant Z^{old} by adding vertex normalisations, which make it commute with unzips. In fact, Z^{old} does commute with edge deletions[18] [Da, Proposition 6.7], so the deletion errors of Z^u depend only on the vertex normalisations implemented in [WKO2].

[18] The reader might wonder, why normalize so that the expansion respects unzips, rather than deletions? The answer is that for finite generation of knotted trivalent graphs, unzips are crucial but deletions are not.

$$
\begin{tikzcd}
u\mathcal{T}T \arrow[r, "\varphi"] \arrow[d, "c"] \arrow[dr, phantom, "(1)"] & sKTG \arrow[r, "Z^u"] \arrow[d, "d^3"] \arrow[dr, phantom, "(2)"] & \mathcal{A}^u \arrow[r, "\alpha"] \arrow[d, "d^3\circ(\cdot e^{-\frac{c}{4}}\nu^{\frac{1}{2}})"] \arrow[dr, phantom, "(3)"] & \mathcal{A}^{sw} \arrow[r, "\cdot C"] \arrow[d, "d^3\circ(\cdot e^{-\frac{a}{2}}\alpha(\nu^{\frac{1}{2}}))"] \arrow[dr, phantom, "(4)"] & \mathcal{A}^{sw} \arrow[r, "u^{n-1}"] \arrow[d, "d^3\circ(\cdot e^{-\frac{a}{2}}\alpha(\nu^{\frac{1}{2}}))"] \arrow[dr, phantom, "(5)"] & \mathcal{A}^{sw} \arrow[r, "p^{n-2}"] \arrow[d, "d^3\circ(\cdot e^{-\frac{a}{2}}\alpha(\nu^{\frac{1}{2}}))"] \arrow[dr, phantom, "(6)"] & {} \\
u\mathcal{T}T \arrow[r, "\varphi"] & sKTG \arrow[r, "Z^u"] & \mathcal{A}^u \arrow[r, "\alpha"] & \mathcal{A}^{sw} \arrow[r, "\cdot C"] & \mathcal{A}^{sw} \arrow[r, "u^{n-1}\ \ n-3"] & \mathcal{A}^{sw} \arrow[r, "p^{n-2}"] & {}
\end{tikzcd}
$$

add: diagram continues....

$$
\begin{tikzcd}
\mathcal{A}^{sw} \arrow[r, "p^2"] \arrow[dd, "d^3\circ(\cdot e^{-\frac{a}{2}}\alpha(\nu^{\frac{1}{2}}))"] \arrow[ddrr, phantom, "(7)"] & \mathcal{A}^{sw} \arrow[r, "\varphi"] & \mathcal{A}^{sw} \arrow[r, "\cdot(\alpha(\nu^{-\frac{1}{4}}))^n"] \arrow[dd, "c\circ(\cdot\alpha(\nu^{-\frac{1}{2}}))"] \arrow[ddr, phantom, "(8)"] & \mathcal{A}^{sw} \arrow[dd, "c"] \\
& & & \\
\mathcal{A}^{sw} \arrow[rr, "\varphi"] & & \mathcal{A}^{sw} \arrow[r, "\cdot(\alpha(\nu^{-\frac{1}{4}}))^{n-2}"] & \mathcal{A}^{sw}
\end{tikzcd}
$$

Flush to right

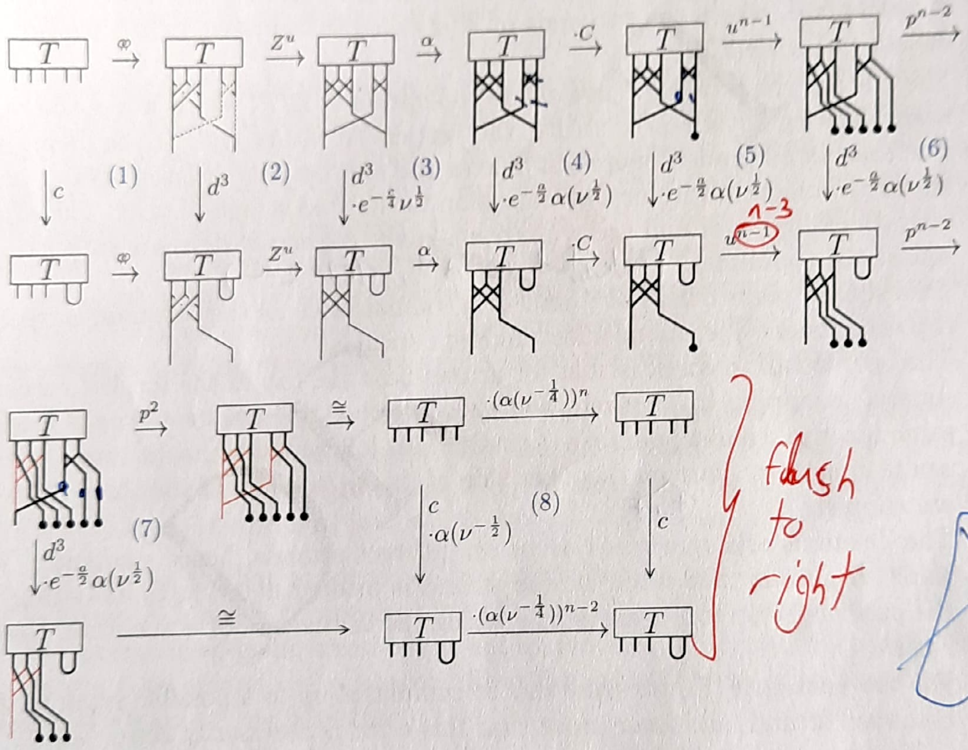

Figure 27. The top diagram summarises the proof that Z^w commutes with contractions. The bottom diagram follows the skeleton changes of $T \in u\mathcal{T}T$ throughout the same diagram. In the first square, the dotted lines indicate which three strands are to be deleted.

fig:Contra

Each vertex of an *sKTG* has one incident edge marked *distinguished*, in our drawings this is the vertical edge. If an edge is not distinguished at either of its ending vertices, then deleting it commutes with Z^u. When deleting a distinguished edge, such as the right vertical edge of square (2), then deletion and Z^u commutes only up to a correction term of $e^{-c/4}\nu^{1/2}$ inserted at the place of the vertex, where c stands for a

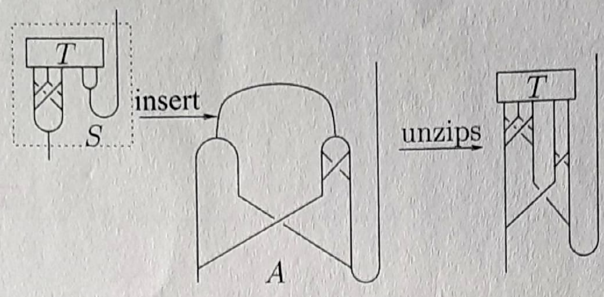

Figure 28. Computing the top left corner of Square 7. Step 1: $\varphi(T)$ can be expressed as the *sKTG* denoted S inserted into the *sKTG* denoted A, followed by some unzips, as shown. Z^u respects insertions, hence computing $Z^u(A)$ determines the value of $Z^u(\varphi(T))$ outside of S.

single chord. In square (2), this correction term appears at the bottom right corner of the square, where the two ends of T are contracted (see in the diagram showing skeleta in Figure 27).

(3) There is a corresponding "edge delete" operation of $\widetilde{wTF}$, which works the same way: when deleting a tube or a string, the vertices at either end[19] cease being vertices. The associated graded operation deletes the appropriate skeleton strand and sends any arrow diagram with arrow endings on the deleted strand to zero. The edge delete operations for chord and arrow diagrams make a commutative square with α, hence square (3) commutes. (note also that in A^{sw}, $\alpha(C)=2a$)

(4) This square plays out in $\mathcal{A}^{sw}$ and it is commutative as the deletion/correction and cap attachment operations affect different strands.

(5) The only detail to check is that for arrows near the cap in the top left corner of this square, unzipping $n-1$ times and then deleting the last two strands leads to the same diagram as unzipping $n-3$ times. This follows from the definition of unzip for arrow diagrams. Note that on the right side of this square two of the three deleted edges are capped.

(6) The deletions and punctures occur on different strands, hence commute. The only thing to note is that when a tube strand is deleted at a "tube-and-string" vertex, the other tube strand deflates to a string (as in the case of a puncture, as shown in Figure 4).

(7) For the pentagon (7), we show that it commutes up to a possible small error on the u-shaped strand, and later prove that this error is necessarily zero.

To show commutativity up to an error, a better understanding of the arrow diagram in the top left corner is necessary. This arrow diagram is the result of a sequence of operations $(p^{n-2} \circ u^{n-1} \circ \cdot C \circ \alpha \circ Z^u \circ \varphi)$. All of these operations with the exception of Z^u are "easy" in the sense that we have a complete understanding of their effect. Z^u is "hard", but the proof of [WKO2, Proposition 4.13] presents a technique for computing the relevant part of its value, see Figures 28 and 29 and their captions.

The value at the top left corner of the pentagon (7) is the chord diagram D of Figure 29, with $Z^u(A)$ inserted, then α, cap attachment, unzips and punctures performed.

[19]It is also possible to delete a capped edge.