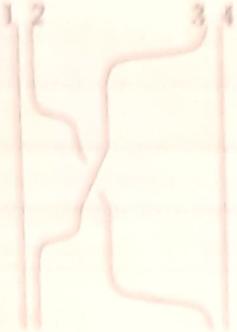


$$\begin{array}{ll}
\Phi_{(12)34} & \longrightarrow \Phi(a_{23}, a_{43}) \\
\Phi^{-1}_{123} & \longrightarrow 1 \\
R_{22} & \longrightarrow e^{a_{22}/2} \\
\Phi_{122} & \longrightarrow 1 \\
\Phi^{-1}_{(13)24} & \longrightarrow \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)})
\end{array}$$

Figure 12. Computing β^u. Strands are numbered at the top and multiplication is read from bottom to top, the rightmost column lists the images of the factors under $p_1p_3\alpha$.

Thus $p_1p_3(\beta^u)$ can be expressed as $p_1p_3(\alpha(\Phi^{-1}_{(13)24}))e^{a_{23}/2}p_1p_3(\alpha(\Phi_{(12)34}))$. Since strands 1 and 3 are both punctured, no arrows can be supported between these strands, hence $p_1p_3\alpha(\Phi_{(12)34}) = p_3\alpha(\Phi_{234})$.

Expressing Φ as a power series in two variables (abusively also denoted by Φ), $\Phi_{234} = \Phi(c_{23}, c_{34})$, and $p_1p_3\alpha(\Phi(c_{23}, c_{34})) = \Phi(a_{23}, a_{43})$. Similarly, $\Phi^{-1}_{(13)24} = \Phi^{-1}(c_{(13)2}, c_{24})$, where $c_{(13)2} = c_{12} + c_{32}$.

A well-known property of associators is $\Phi(c_{ij}, c_{jk}) = \Phi(c_{ij}, -c_{ij} - c_{ik})$. Hence, $\Phi^{-1}(c_{(13)2}, c_{24}) = \Phi^{-1}(c_{(13)2}, -c_{(13)2} - c_{(13)4})$, so $p_1p_3\alpha\Phi^{-1}_{(13)24} = \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)})$. To summarise,

$$p_1p_3\beta^u = \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43}). \tag{3}$$

Combining this with the computation of V at the beginning of this section, we obtain the formula of Theorem 1.1 part (2). Later in Lemma 3.5 we'll also compute the even part of the value of the cap explicitly, and find that it is $\alpha(\nu^{1/4})$, where ν is the Kontsevich integral of the unknot. we will

A 3.2.2. *From V to Kashiwara–Vergne.* This technical section is mainly interesting for readers familiar with the Alekseev–Enriquez–Torossian work on Kashiwara–Vergne solutions. Results of this section are not used afterwards.

move to after the [AET] recap?

The value $\varphi(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43}))$ can be computed more explicitly, which is necessary in order to compare it with the [AET] formulas. The first strand of $\mathcal{A}^w(\uparrow_2)$ joins strands 1 and 2 in a vertex, and the second strand of $\mathcal{A}^w(\uparrow_2)$ joins strands 3 and 4. Strands 1 and 3 are punctured and strands 2 and 4 are capped. Let us call the two strands of $\mathcal{A}^w(\uparrow_2)$ strand I and strand II to avoid confusion. Recall from the construction of φ that one first slides arrow tails from the capped strands "up" through the vertices, then slides all the heads up from the punctured strands 1 and 3. Thus one obtains an element of $\mathcal{A}^w(\uparrow_2)$ in which all arrow heads are below all tails on both strands. The result is shown in Figure 13, and explained in the caption.

This is terrible. Figure 13 clarifies a bit, but it is only mentioned at the end of the paragraph

For a quick re-cap of [AET] notions, let $\mathfrak{lie}_2$ denote the free Lie algebra on two generators x and y. Let $\mathfrak{tder}_2$ denote *tangential derivations* of this Lie algebra, that is, derivations d with the property that $d(x) = [x, a_1]$ and $d(y) = [y, a_2]$, where $a_1, a_2 \in \mathfrak{lie}_2$. Let $\mathrm{TAut}_2 := \exp(\mathfrak{tder}_2)$ denote the group of *tangential automorphisms* of $\mathfrak{lie}_2$. There is a map $\theta : \mathfrak{lie}_2^2 \to \mathfrak{tder}_2$, sending a pair (a_1, a_2) to the derivation d given by $d(x) = [x, a_1], d(y) = [y, a_2]$. The

exp in what sense?

isolated sentence.

A: Mathematicians like to see the top of the mountain before starting to march towards it. Namely, formula (4) should come at the beginning of sec 3.2.2, not at the end.

Perhaps move section 3.2.2 to an appendix?

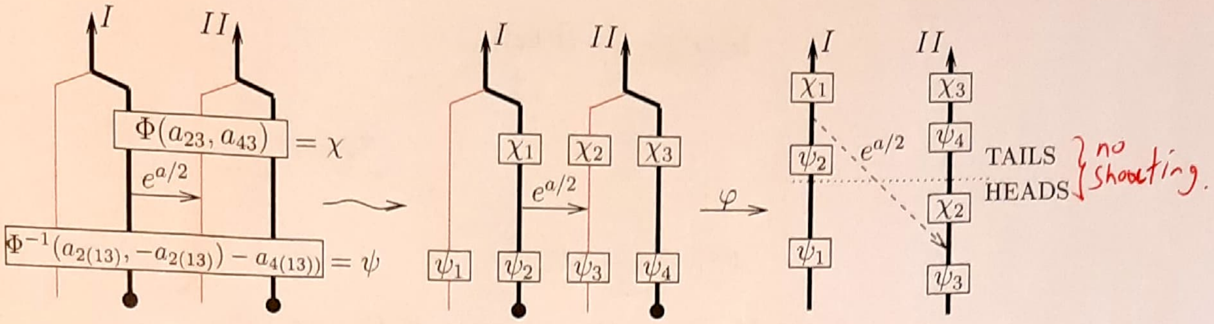


Figure 13. To compute $\varphi(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43}))$ we switch to a *placement notation* in which we mark on each skeleton strand the elements that have arrows ending on it. For this purpose we denote $\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) =: \psi$ and $\Phi(a_{23}, a_{43}) =: \chi$.

kernel of this map consists only of pairs of the form $(\alpha x, \beta y)$ for α, β constants. In other words, $\mathfrak{tder}_2$ is "almost" $\mathfrak{lie}_2^2$, and there is a one-sided inverse $\eta : \mathfrak{tder}_2 \to \mathfrak{lie}_2^2$ which sends a tangential derivation to a pair whose first component has no x term and second component has no y term.

A Lie word in x and y can be represented by a binary tree oriented towards a single "head", with leaves labeled by the letters x and y; for details see "primitive elements of $\mathcal{B}_2$" as in [WKO2, Theorem 3.16] and the discussion following it. There is a *tree attaching* map $l : \mathfrak{tder}_2 \to \mathcal{P}^{sw}(\uparrow_2)$, where $\mathcal{P}^{sw}$ denotes the primitive elements of $\mathcal{A}^{sw}$, as follows. Represent the components of $\eta(D)$ by binary trees, and label the heads with x for Lie words coming from a_1, and y for a_2. Then, attach all x-labeled leaves to strand 1, y-labeled leaves to strand 2, and the head below all tails. The order of tails is irrelevant ~~(TC)~~. Conversely, elements by TC. how? of $\mathcal{P}^{sw}(\uparrow_2)$ act as tangential derivations on $\mathfrak{lie}_2$. Wheels act trivially, and thus one obtains a homomorphism $\delta : \mathcal{P}^{tree}(\uparrow_2) \to \mathfrak{tder}_2$, whose ~~only~~ kernel consists only of short arrows on either strand. The map l is a one-sided inverse to δ, that is, $\delta \circ l = \mathrm{Id}_{\mathfrak{tder}_2}$. For more details see [WKO2, Section 3.2].

Extending δ to exponentials gives a group homomorphism $\delta : \mathcal{A}^w(\uparrow_2)_{exp} \to \mathrm{TAut}_2$, where rewrite, around a formula like $\delta(D)(v) = D^{-1}vD$ elements in $\mathcal{A}^w(\uparrow_2)_{exp}$ denotes the group-like ~~part of~~ $\mathcal{A}^{sw}$. For $D \in \mathcal{A}^w(\uparrow_2)_{exp}$, the map δ can be described diagrammatically in the following way. Add an extra (third) strand, and represent a Lie word $v \in \mathfrak{lie}_2$ by a binary tree whose x-labeled tails are attached to the first strand, y-labeled tails to the second strand, and whose head lies on the extra strand. Its conjugate $D^{-1}vD$ is once again a linear combination of such trees (with heads on the third strand), this is the output of the action. See also [WKO2, Proposition 3.19, "Conceptual argument"].

$V = Z^w(\lambda)$ Let $\cancel{V}$ be the Z^w-value of the vertex for a homomorphic expansion Z^w, then $F = \delta(\pi V)$ is a solution to the Kashiwara-Vergne problem in the sense of [AET]. For details see Section 4 of [WKO2]. In particular the [AET] formulas only concern the *tree-level* part $\pi(V)$.

The formula for F presented in [AET, Theorem 4] is[12],

$$F = (\Phi^{-1}(x, -x-y), e^{-(x+y)/2}\Phi(-x-y, y)e^{y/2}), \tag{4}$$

[12]There are some notational differences between [AT] and [AET], hence we don't switch strands here as we did in [WKO2]. There are sign differences between the formula (4) and [AET] due to notational misalignment, for example our Φ is [AET]'s Φ^{-1}. Our notation is consistent with all other papers in this series and the formulas are computationally verified in [WKO4].

[WKO2, specific place]

$$\begin{tikzcd}
\hat{\mathcal{U}}(\mathfrak{lie}_2^2) \arrow[d, "L"'] & \hat{\mathcal{U}}(\mathfrak{lie}_2^2)_{exp} \arrow[l, hook'] \arrow[d, "L"] \arrow[dr, "\Theta"] & \\
 & & \mathrm{TAut}_2 \\
\mathcal{A}^{sw}(\uparrow_2) & \mathcal{A}^{sw}(\uparrow_2)_{exp} \arrow[l, hook'] \arrow[ur, "\delta"'] &
\end{tikzcd}$$

Figure 14. The connection between $\mathcal{A}^{sw}(\uparrow_2)$ and TAut_2.

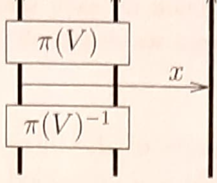


Figure 15. The action of $\pi(V)$ on the generator x of $\mathfrak{lie}_2$.

meaning that the automorphism F conjugates x by $\Phi^{-1}(x,-x-y)$, and conjugates y by $e^{-(x+y)/2}\Phi(-x-y,y)e^{y/2}$. This implicitly assumes an "interpretation map" $\Theta : \mathcal{U}(\mathfrak{lie}_2^2)_{exp} \to \mathrm{TAut}_2$. That is, an element $(e^{\lambda_1}, e^{\lambda_2}) \in \mathcal{U}(\mathfrak{lie}_2^2)_{exp}$ is mapped to the automorphism of $\mathfrak{lie}_2$ which sends the generator x of $\mathfrak{lie}_2$ to $e^{-\lambda_1}xe^{\lambda_1}$, and the generator y to $e^{-\lambda_2}ye^{\lambda_2}$. Note that this is not a group homomorphism: composition in TAut_2 is not given by piecewise multiplication of the conjugators.

We relate $\Theta(\Phi^{-1}(x,-x-y), e^{-(x+y)/2}\Phi(-x-y,y)e^{y/2})$ to $\delta(\pi V)$, by constructing a map L which completes a commutative triangle as in Figure 14. At the level of primitives, the map $l \circ \theta$ has the property that $\delta \circ (l \circ \theta) = \theta$. Extend this to the (completed) enveloping algebra $\hat{\mathcal{U}}(\mathfrak{lie}_2^2)$ as follows. An element of $\hat{\mathcal{U}}(\mathfrak{lie}_2^2)$ is an (infinite) linear combination of products of Lie words. As with l, represent each Lie word as a labeled tree, but then attach the products of these labeled trees to the two strands by attaching *all heads* below *all tails*. The order of tails doesn't matter, the order of heads is in the order in which the words were multiplied. Call this map L, and note that L is *not* an algebra homomorphism: it does not respect multiplication in $\hat{\mathcal{U}}(\mathfrak{lie}_2^2)$. However, the restriction of L to the group-like part $\hat{\mathcal{U}}(\mathfrak{lie}_2^2)_{exp}$, also denoted L (and which does *not* equal e^l) fits into a commutative triangle $\Theta = \Delta \circ L$.

Now we are ready to compute how $\pi(V) \in \mathcal{A}^w(\uparrow_2)$ acts on the generator x of $\mathfrak{lie}_2$ and match this to the formula (4.) Recall the value of $\pi(V)$ shown in Figure 13. The generator x is represented by an arrow from the first strand to the added third strand, and the result of the action is $\pi(V)^{-1}x\pi(V)$, as shown in Figure 15. To compute this, one commutes the tail of x to the top of the strand across $\pi(V)$ using $\overrightarrow{STU}$ relations, thereby $\pi(V)$ and $\pi(V)^{-1}$ cancel, and the result of the action remains. Observe that due to the TC relation, only arrows with heads on strand I act nontrivially on x, in other words only ψ_1 matters, which came from $\Phi^{-1}(a_{2(13),-a_{2(13)}-a_{4(13)}})$. The arrows a_{23} and a_{43} act trivially on x, so, more simply stated, the action on x is by $\varphi(\Phi^{-1}(a_{21},-a_{21}-a_{41}))$. Note that $L(\Phi^{-1}(x,-x-y),0) = \varphi(\Phi^{-1}(a_{21},-a_{21}-a_{41}))$, so Theorem 1.1 agrees with Formula (4) in the first component.

too wordy.

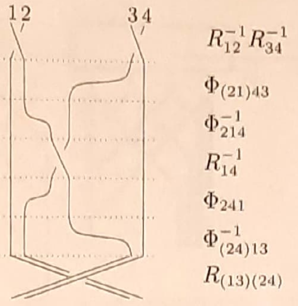


Figure 16. A different expression of β^b.

fig:Buckle

One can proceed similarly for the second component: the action on y is by

$$\varphi(\Phi^{-1}(a_{23}, -a_{23} - a_{43})e^{a_{23}}\Phi(a_{23}, a_{43})) = L(0, \Phi^{-1}(x, -x - y)e^{x/2}\Phi(x, y)).$$

While this does not match the second component of Formula (4), it only differs from it by a hexagon relation. Alternatively, note that one can obtain the second component of the Formula (4) "on the nose" by starting from an equivalent (isotopic) expression[13] of β^b, as shown in Figure 16.

I need to re-read 3.3.2 after it's been appendixed.

3.3. **Proof of part (3): the double tree construction.** It remains to prove that the values of V and C, which we proved in Section 3.1.2 are determined by Z^u, indeed give rise to a homomorphic expansion of $\widetilde{wTF}$. In other words, one needs to show that they satisfy the three equations of Fact 2.5. Unfortunately, doing this directly seems difficult.

Note that $R4$, which is in some sense the "main equation", is an equality between different planar algebra compositions of generators. Hence, the proof would be much easier if Z^u were to be a planar algebra map. This unfortunately makes no sense, as *sKTG* is not a planar algebra but a different, more complicated algebraic structure. The reader might ask, why work with a space as inconvenient as *sKTG* instead of, say, the planar algebra of trivalent tangles? The answer is that the existence of a homomorphic expansion is a highly non-trivial property, and in particular ordinary trivalent tangles do not have one. Even without trivalent vertices, ordinary tangles, or u-tangles, do not have have a homomorphic expansion[14]. ~~Only~~ *parenthesized tangles* (a.k.a. *q-tangles*) [LM, BN2] do have homomorphic expansions, yet ~~and~~ in fact these are almost equivalent to *sKTG* [T, BND1, Da]. Foot 14

Nonetheless, the unsuccessful thought above leads to the following, feasible, strategy: we map ordinary trivalent tangles into *sKTG* via a *double tree* construction, and use this to define Z^w for the a-images of all usual trivalent tangles. Then we use the planar algebra structure to prove that this Z^w is a homomorphic expansion, and finally show that the Z^w constructed this way is in fact the same as the one arising from part (1).

3.3.1. *Defining Z^w.* We start by defining (classical, or *usual*) trivalent tangles, denoted *uTT*: switch to <gens|rels|ops> notation?

$$uTT := \mathrm{PA}\left\langle \nwarrow\!\!\nearrow, \nearrow\!\!\nwarrow, \curlywedge, \curlyvee \;\middle|\; \mathrm{R1}^s, \mathrm{R2}, \mathrm{R3}, \mathrm{R4} \right\rangle$$

Here PA stands for *planar algebra*, an algebraic structure similar to a circuit algebra, except with *planar* wiring diagrams. (This is a slightly more simple-minded notion than the ~~standard~~ original use of the term in [J], in particular we do not use checkerboard shadings.) The elements

[13] We thank Karene Chu for this idea.

[14] We only need to know that the planar algebra of u-tangles does not have a homomorphic expansion so as to explain why we are not using one. This said, the non-existence of $Z^{tangles}$ is easy: By an explicit calculation in degree 2 one may show that there is no linear combination of chord diagrams to serve as $Z^{tangles}(\times)$ which satisfies R3.

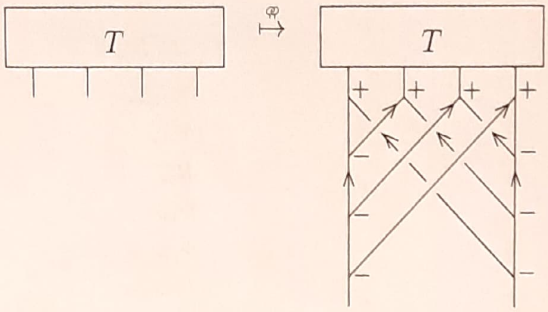


Figure 17. The *double tree map*: connect the ends of T by two binary trees (hence "double tree"), as shown. Note that the tree on the left always crosses over the tree on the right, and all edges of both trees are oriented towards T.

a little picture on the side?

of uTT can be seen as usual (i.e., classical) trivalent tangles with signed vertices and no virtual crossings, modulo all of the relations which make sense in this context. The planar algebra uTT is equipped with orientation switch and edge unzip operations, where an edge unzips is defined for an edge if it ends in one positive and one negative vertex and the edge is meaning? "distinguished" at both vertices. Note that uTT does not have a homomorphic expansion.

We define a *double tree* map $\wp : \mathit{uTT} \to \mathit{sKTG}$, as in Figure 17. The map $\wp$ depends on two choices of binary trees: in Figure 17 we chose a particular example. It is important that, regardless of this choice, the "left side" tree always crosses over the "right side" tree. We will demonstrate that in fact the choice of trees becomes irrelevant after some post-compositions, ~~as in Equation (5).~~ (see lemma 3.1)

Working towards a construction of Z^w, we post-compose $\wp$ with the following sequence of maps, which is explained in the paragraph below:

$$T \in \mathit{uTT} \xrightarrow{\wp} \mathit{sKTG} \xrightarrow{Z^u} \mathcal{A}^u(\wp(T)) \xrightarrow{a} \mathcal{A}^{sw}(\wp(T)) \xrightarrow{\kappa,u,p} \mathcal{A}^{sw}(\check{T}) \overset{\varphi}{\cong} \mathcal{A}^{sw}(T). \tag{5}$$

Here T stands for an arbitrary tangle in uTT. The double tree construction maps T into sKTG, and by applying Z^u one obtains a value in $\mathcal{A}^u$, namely a chord diagram on the skeleton of $\wp(T)$. We denote the space of chord diagrams on this skeleton by $\mathcal{A}^u(\wp(T))$. Now α maps this to arrow diagrams on the skeleton of $\wp(T)$, that is, to $\mathcal{A}^{sw}(\wp(T))$. In order to revert the skeleton back to that of T, we apply some operations in $\mathcal{A}^{sw}$: a cap attachment κ, unzips and punctures (as shown in Figure 18 and explained in the next paragraph), resulting in a slightly modified version of the desired skeleton, denoted $\check{T}$. Finally, we use that $\mathcal{A}^{sw}(\check{T}) \cong \mathcal{A}^{sw}(T)$ via the isomorphism φ of Lemma 2.4, and hence we obtain a value in $\mathcal{A}^{sw}(T)$, as needed, which, we will later see, is *almost* $Z^w(a(T))$. Note that the proof of Lemma 2.4 applies even though the punctured strands all connect in a binary tree: VI relations can be used as part of the isomorphism.

The cap attachment, unzip and puncture operations are done in the ~~following order~~ order shown in Figure 18. First attach a cap – a capped strand with no arrows on it – to the end of the right vertical strand: this is a circuit algebra operation in $\mathcal{A}^{sw}$. If T has n ends, perform $(n-1)$ consecutive disc unzips on the right vertical strand, as shown in Figure 18. Then puncture the strand marked with "1" on the left of Figure 18. Then puncture strands 2, 3, and 4 in that order, and recall that these punctures also spread to the connecting diagonal strands, as in Figure 4. Punctures could be ~~done~~ applied in any other "legal" order without changing the result. Note that

Name lemma 2.4 "the sifting lemma", name φ "the sifting map", use the names often. In particular, replace every occurrence of the reference "Lemma 2.4" with "the sifting lemma (2.4)". Also, a 1-sentence explanation of the name should be given where introduced.

Thinner black lines, Thicker caps.

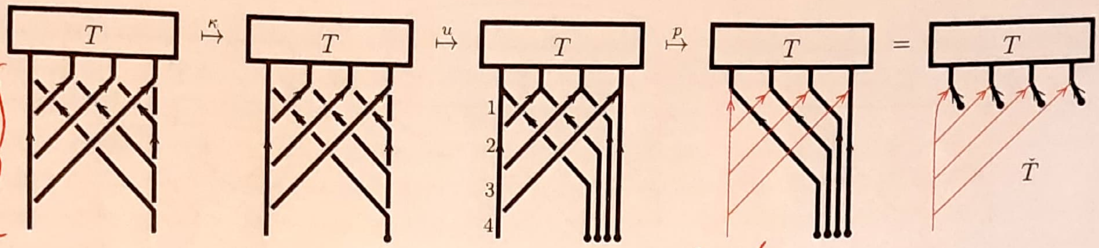
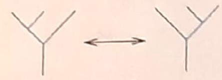


Figure 18. The cap attachment, unzips and punctures. ~~Unzips~~ Punctures are performed in the order of the edge numberings. While these operations are applied in $\mathcal{A}^{sw}$, the picture only shows the effect on the skeleton.

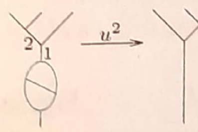
since the punctured tree had originally crossed over the capped tree, these crossings become virtual after puncturing, hence the last equality in Figure 18.

Let us denote the composition of the maps and operations shown in Equation (5) by ξ, that is, $\xi = \varphi \circ p \circ u \circ \kappa \circ \alpha \circ Z^u \circ \wp$. To summarise, $\xi(T) \in \mathcal{A}^{sw}(T)$. We need to show that $\xi(T)$ is well-defined, that is, it doesn't depend on the choice of binary trees.

Lemma 3.1. *The choice of binary trees in the double tree construction does not affect $\xi(T)$.*



Proof. Any binary tree can be changed into any other binary tree via a sequence of "I to H" moves, as shown on the right. Hence, it is enough to analyze how an I to H move on one of the trees affects the value of $Z^u(\wp(T))$, and prove that the difference does not "survive" the cap and puncture operations.



Suppose τ_1 and τ_2 are two binary trees which differ by a single I to H move, and let $\wp_{\tau_1}$ and $\wp_{\tau_2}$ denote the two resulting double-tree maps, assuming the "other side tree" is unchanged. The I to H move can be realised by inserting[14] an associator, followed by unzipping the edge marked '1' on the right, then the edge marked '2'. By the homomorphicity of Z^u, the values $Z^u(\wp_{\tau_2}(T))$ and $Z^u(\wp_{\tau_1}(T))$, only differ in an inserted horizontal chord associator Φ on the three strands involved. In sloppy notation,. $Z^u(\wp_{\tau_2}(T)) = Z^u(\wp_{\tau_1}(T)) * \Phi$. If the I to H move was done on the left side tree, then all the strands involved are later punctured, killing any arrow diagram that lived on them by the TF relation. As a result, the only surviving part of Φ is its constant term, 1, and the resulting values of ξ are equal.

If the I to H move is done on the right side tree, then the all participating strands are capped and disk unzipped. If $\alpha(\Phi)$ is immediately adjacent to the caps, then it cancels by the CP relation. However, it is a priori possible that there are other arrow ending separating it from the cap. Note that in $\mathcal{A}^u$, any chord endings can be can be commuted from below the associator to above, using VI relations and the *invariance property* of chord diagrams. Thus, one can assume that $\alpha(\Phi)$ is adjacent to the caps and hence cancels. This concludes the proof. □

There is an action of $\mathbb{Z}/n\mathbb{Z}$ on elements of $u\mathcal{T}\mathcal{T}$ with n ends, by cyclic permutations of the ends. The following lemma will be useful later in proving that Z^w is a planar algebra map; we present it now because its proof is similar to that of Lemma 3.1.

Lemma 3.2. *The map ξ is invariant under cyclic permutation of the ends of T.*

[14]See [WKO2, Section 4.6] for a detailed description of tangle insertion in *sKTG*.

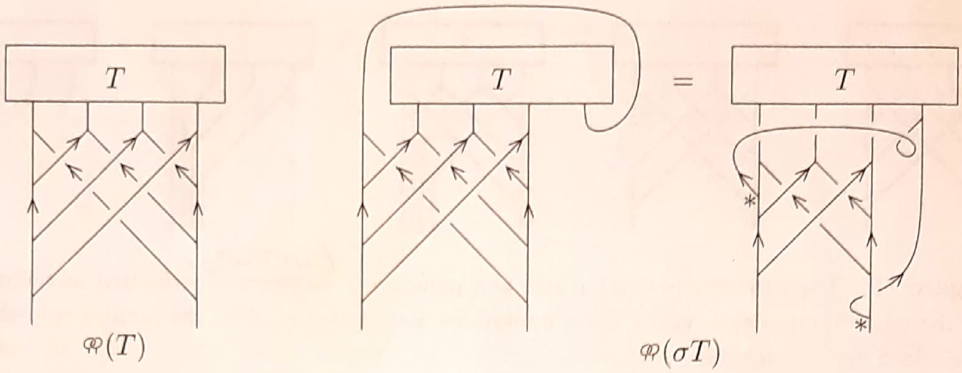


Figure 19. Double tree construction for cyclically permuted ends of T.

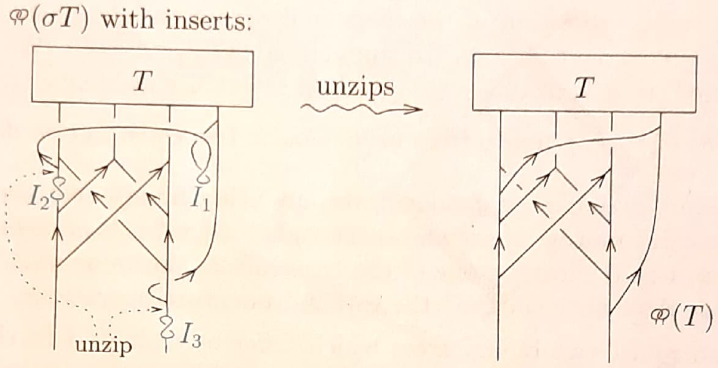


Figure 20. The difference between $\wp(T)$ and $\wp(\sigma T)$, understood via insertions.

Proof. To show that $\xi(T)$ is invariat under cyclic permutations of ends of T, it is enough to show that $\xi(T)$ does not change when the rightmost end of T is moved to the far left (denote this by σT), as shown in the left versus middle pictures of Figure 19.

The rightmost picture of Figure 19 is equivalent as *sKTG*s to $\wp(\sigma T)$. It differs from $\wp(T)$ in three ways:

- the binary trees connecting the ends of T are different;
- two tree branches are connected to the trunk "on the wrong side", that is, these trivalent vertices have opposite cyclic orientation (marked by $*$ in Figure 19);
- one tree branch has a kink in it.

As before, we need to analyse how $Z^u(\wp(\sigma T))$ differs from $Z^u(\wp(T))$, and show that the difference doesn't survive the puncture, cap and unzip operations.

To achieve this, we transform $\wp(\sigma T)$ into $\wp(T)$ using tangle insertions. First, cancel the kink by inserting an opposite kink I_1 on the same strand, as shown in Figure 20 in blue[15]. As Z^u is compatible with insertion, the Z^u values will differ by the value of a kink: a chord diagram on the one strand involved. Later in the process this strand is punctured, and no arrow diagram can live on a single punctured strand (TF), so the value of the kink cancels.

[15] Or grey in black and white print.

Similarly, switching the side that the tree branches are attached on amounts to inserting twists I_2 and I_3, and unzipping the connecting edges, also shown in Figure 20. Each of these operations changes the value of Z^u by inserting the value of a twist: $e^{c/2}$, where c denotes a single chord between the appropriate strands. Applying α maps this to $e^{(a_L+a_R)/2}$, where a_L and a_R denote horizontal left and right arrows, respectively. On the left side tree, this cancels after punctures, as before. On the right side tree, the strand directly underneath the twist is capped and uzipped, and hence the value of the twist cancels by the CP relation.

Now observe that the right side picture of Figure 20 only differs from $\varphi(T)$ in the choices of binary trees, which do not change the value of ξ by Lemma 3.1. □

The purpose of the following lemma is two-fold: first, it clarifies the relationship between the map ξ and the homomorphic expansion Z^w that we're aiming to construct. Second, making this relationship explicit allows us to compute the even part of Z^w-value C of the cap in Corollary 3.5.

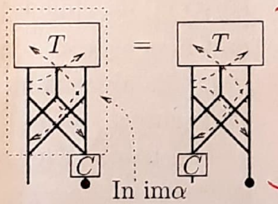
Lemma 3.3. *If there exists a homomorphic expansion Z^w for $\widetilde{wTF}$ compatible with Z^u, then for a tangle $T \in a(u\mathcal{T}T)$, the value $Z^w(T)$ is equal to $\xi(T)$ multiplied by C^{-1} at each end of T, where C is the Z^w-value of the cap.*

these two paragraphs seem to say the same thing twice.

Proof. Assume there exists a homomorphic expansion Z^w compatible with Z^u. We use, as in Figure 21, the homomorphicity of Z^w and its compatibility with Z^u to show that $\xi(T) = Z^w(\check{T})$, where $\check{T}$ is as ~~defined in Equation 5 and the paragraph following it.~~ illustrated in Fig 18.

We show that the three squares A and one heptagon B of the diagram in Figure 21 commute. This means that, for any $T \in u\mathcal{T}T$ and any Z^u-compatible Z^w, and with $\check{T}$ denoting the w-foam derived from T as in Figure 18, we have $Z^w(\check{T}) = \xi(T)$. The result is then obtained by comparing $Z^w(\check{T})$ with $Z^w(T)$ using homomorphicity.

The leftmost square of the diagram commutes by the compatibility of Z^u and Z^w. In the middle square, recall the top horizontal κ map denotes the circuit algebra operation of attaching a cap at the bottom right end of the w-foam, and the same on the bottom horizontal arrow. The $\hat{C}$ on the bottom horizontal arrow denotes the circuit algebra operation which attaches a cap with a value C at the end of the strand. The commutativity of this square is implied by the homomorphicity of Z^w with respect to circuit algebra composition (as a binary operation). The rightmost square is commutative once again due to the homomorphicity of Z^w with respect to punctures and disc unzips.



a bit bigger

The commutativity of the heptagon would be true by definition, if not for the map $\hat{C}$ (the insertion of the cap value). We show that, in fact, the value C cancels after punctures, by a property of arrow diagrams in the image of α, called *tail-invariance*. This property is the arrow diagram ~~implication~~ version of the topological fact that as long as a w-foam is in the image of a, one can slide a strand under it. In the current situation this means that the value C, which has only tails on the skeleton, can be moved from one tangle end to the other, as shown on the right. Consequently, C cancels when the left strand is punctured. For more details on tail-invariance, see [WKO2], Remark 3.14 and early in Section 3.3.

Finally, note that since Z^w is a circuit algebra homomorphism, $Z^w(\check{T})$ can be obtained from $Z^w(T)$ by attaching the Z^w-value of a left-punctured right-capped vertex at each tangle end. By Lemma 2.7 the value of the left-punctured vertex is 1, so the only additions are C

A: (labeled (2),(3), and (4))
B: (labeled (1))

$$
\begin{tikzcd}
T \arrow[r, mapsto] & \varpi(T) \arrow[r, mapsto] & \varpi(T) \arrow[r, mapsto] & \kappa\varpi(T) \arrow[r, mapsto] & \hat{T} \\
\Cap & \Cap & \Cap & \Cap & \Cap \\
uTT \arrow[r, "\varpi"] \arrow[dr, phantom, "(1)"] \arrow[ddrrrr, bend right=20, "\xi"'] & sKTG \arrow[r, "a"] \arrow[d, "Z^u"] \arrow[dr, phantom, "(2)"] & \widetilde{wTF} \arrow[r, "\kappa"] \arrow[d, "Z^w"] \arrow[dr, phantom, "(3)"] & \widetilde{wTF} \arrow[r, "pou"] \arrow[d, "Z^w"] \arrow[dr, phantom, "(4)"] & \widetilde{wTF} \arrow[d, "Z^w"] \\
 & \mathcal{A}^u(\varpi(T)) \arrow[r, "\alpha"] & \mathcal{A}^{sw}(\varpi(T)) \arrow[r, "\kappa\hat{C}"] \arrow[d, phantom, "(1)"] & \mathcal{A}^{sw}(\kappa\varpi(T)) \arrow[r, "pou"] & \mathcal{A}^{sw}(\hat{T}) \arrow[d, "\varphi"] \\
 & & & & \mathcal{A}^{sw}(T)
\end{tikzcd}
$$

Can we use the same labeling strategy here as in Figure 27?

Figure 21. Comparing ξ and Z^w, assuming that Z^w exists.

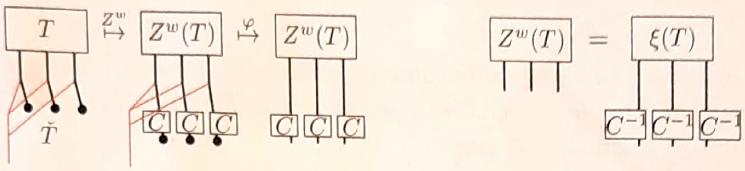


Figure 22. Computing $Z^w(\check{T})$ and the forced definition of Z^w.

values at each capped end, as shown in Figure 22. This can then be interpreted as a value in $\mathcal{A}^{sw}(T)$ via the isomorphism φ of Lemma 2.4. This implies the statement of the Lemma, also illustrated in Figure 22. $\square$

A ↓

As an aside, assuming that Z^w exists, Lemma 3.3 can be used to compute the *even part* of the value of the cap C. Recall that $C \in \mathcal{A}^{sw}(\uparrow)$, hence by Fact 2.2, C is a "power series" consisting of wheels of progressively larger size, all of which commute. Lemma 3.4 and Corollary 3.5 imply that the even wheels part of C is uniquely determined, that is, regardless of the choice of Z^u (Drinfeld associator).

which in itself is equivalent to the choice of a...

Drinfel'd has an apostrophe in a funny place.

Lemma 3.4. *For the un-knotted strand, $\xi(\uparrow) = \alpha(\nu^{1/2})$, where $\nu \in \mathcal{A}^u(\uparrow)$ denotes the Kontsevich integral of the un-knot*[16].

Proof. We apply ϖ to $\uparrow$, as shown in Figure 23. Our first goal is to comupte $Z^u(\varpi(\uparrow))$. In [WKO2, Section 5.2] we give an algorithm for writing any *sKTG* as a "porduct" of generators, and hence expressing its Z^u value in terms of the Z^u-values of the generators. To feed $\varpi(\uparrow)$ into this algorithm, one needs to "curve up" one strand, in this case the strand on the right (this choice doesn't affect the outcome).

The result after applying ϖ and Z^u are shown in Figure 23. The value of Z^u is expressed in terms of:

- the value of the *associator graph*, Φ, a horizontal-chord Drinfel'd associator
- the value of the *twist*, $R = e^{c/2}$, where c is a single chord, and
- the values n and b of the *noose* and *balloon* graphs, respectively.

See [WKO2, Section 4.6] for details.

[16]The value of ν was conjectured in [BGRT] and proven in [BLT]. Note that ν involves wheels only, which is what one would expect of the value C.

A: Dror is too confused by what is going on from here to the end of section 3.3.1, so he hasn't really read it.

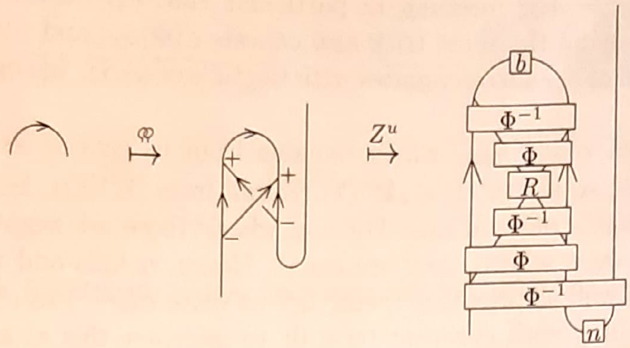


Figure 23. The double tree map composed with Z^u, applied to a single strand.

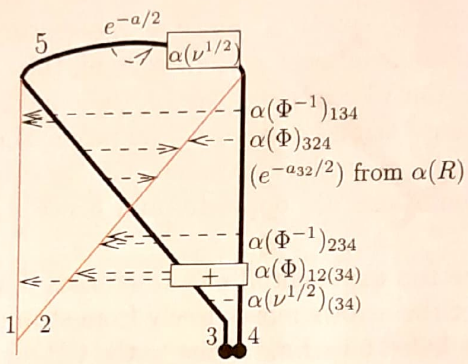


Figure 24. The result after applying α, cap, unzips and puncture.

In $\xi(\uparrow)$, Z^u is followed by α, a cap attachment, unzips and punctures. As explained in [WKO2, Section 4.6], there is a one-parameter uncertainty in the exact values of n and b, but we do know that and $\alpha(b) = e^{a/2}\alpha(\nu)^{1/2}$, and $\alpha(n) = e^{-a/2}\alpha(\nu)^{1/2}$. Note that the exponential part of n cancels by the CP relation once the cap is attached. We "push" most of the arrow diagrams to the middle four strands using the VI relation. The result after α, cap attachment, unzip and puncture is shown in Figure 24 and explained below.

Recall that α maps a chord to the sum of its two possible orientations. However, when one supporting strand is punctured, only one of these orientations survive. Hence, for example, $p_2(\alpha(R_{23})) = (e^{a_{32}/2})$. Figure 24 shows a schematic picture of $puC\alpha Z^u \circledast(\uparrow)$ with exponentials and associators indicated by single arrows. To explain the notation for associators, recall that $\Phi \in \mathcal{A}^{hor}(\uparrow_3)$ can be written as a power series in any two of the three generators of $\mathcal{A}^{hor}(\uparrow_3)$: c_{12}, c_{23} and c_{13}. For each associator above, we chose the presentation in which $\alpha(\Phi)$ is of the simplest form. For example, we write the top associator $\Phi^{-1}_{13(24)}$ of Figure 23 in terms of c_{13} and $c_{1(24)} = c_{12} + c_{14}$, since after the puctures $p_1\alpha(c_{13}) = a_{31}$ and $p_1p_2\alpha(c_{1(24)}) = a_{41}$. This is reflected in Figure 24 in drawing only these two arrows for this associator.

Now observe that the two arguments of $\alpha(\Phi)_{12(34)}$ commute by the TC relation: strands 3 and 4 support only tails in this associator. As mentioned before, the value of an associator in the quotient where its two arguments commute is 1, hence $p\alpha(\Phi)_{12(34)} = 1$. Next, study $p\alpha(\Phi^{-1})_{134}$: the tail of an arrow a_{41} can be "pulled over the top on strand 5" using the VI relation and the fact that $e^{a/2}\alpha(\nu)$ is a local arrow diagram on one strand and hence it is central. So $a_{41} = a_{31}$, meaning in particular that a_{41} commutes with a_{31}, and so $p\alpha(\Phi^{-1})_{134} = 1$. Applying the same trick also cancels $\alpha(\Phi)_{324}$, and $\alpha(\Phi^{-1})_{234}$. In the latter case note that the tail of a_{42} also commutes with the tails of $\alpha(R)$. In summary, all associators cancel.

Next we show that $\alpha(\nu^{1/2})_{(34)}$, which remains from n, cancels as well. Since ν is an exponential of wheels, so is $\alpha(\nu^{1/2}) \in \mathcal{A}^{sw}(\uparrow)$. Recall from [WKO1, Section 3.8] that wheels in $\mathcal{A}^{sw}$ have two possible orientations. For odd wheels these are negatives of each other by the AS relation, for even wheels they are equal. Hence, α kills odd wheels and multiplies even wheels by 2, as well as orienting them. Let us write $\alpha(\nu^{1/2})$ as $e^{w(x)}$, where $w(x)$ is an (even) power series in x with constant term 0: to interpret this as an element of $\mathcal{A}^{sw}(\uparrow)$, expand it and interpret each monomial x^k as a k-wheel on the single strand. Then by the action of unzip, $\alpha(\nu^{1/2})_{(34)} = e^{w(x_3+x_4)}$, where each monomial is interpreted as a cyclic word which is in turn interpreted as a wheel on strands 3 and 4. Now slide this arrow diagram up on strands 3 and 4 to strand 5. Since all associators have canceled, and $\alpha(R)$ has only tails on strand 3, there is no obstruction to doing this. At the junction of strands 3, 1 and 5, we need to apply the VI relation. Tails on the punctured strand 1 are zero (TF relation), so each tail slides onto strand 5, whose orientation is compatible with strand 3. In other words we replace x_3 by x_5 in the expression. On the other side, tails again slide onto strand 5 but now the orientations are opposite, and hence x_4 is replaced by $-x_5$. In summary, $\alpha(\nu^{1/2})_{(34)} = e^{w(x_5-x_5)} = 1$.

Finally, move the top exponential $e^{a/2}$ to strands 3 and 2, using the VI relation at both vertices. The tail of the arrows moves freely from strand 5 to strand 3. The heads commute with $\alpha(\nu)$, they are killed on strand 4 due to the CP relation, so they slide onto strand 2 but acquire a negative sign. Hence, $(e^{a_{55}/2}) = (e^{-a_{32}/2})$, and this cancels $\alpha(R)$. To summarize, $\xi(\uparrow) = \alpha(\nu^{1/2})$, as claimed. $\square$

Recall that the value $C = e^c = Z^w(\uparrow)$ consists of wheels only, meaning $c = \sum_{n=0}^{\infty} w_n$, where w_n denotes the n-wheel and w_0 is the unit arrow diagram (with no arrows). Let $c = c_0 + c_1$, where c_0 denotes the even part of c (sum of all even wheels), and c_1 denotes the odd part, that is, $c = c_0 + c_1$. Let $C_0 = e^{c_0}$, the even part of the value of the cap.

Corollary 3.5. *Let C_0 denote the even part of $C = Z^w(\uparrow)$. Then $C_0 = \alpha(\nu^{1/4})$ for any Z^w.*

Proof. By Lemma 3.3, $Z^w(\uparrow) = C^{-1}\xi(\uparrow)S(C^{-1})$; here S denotes the orientation switch[17]. Note that $S(w_2k) = w_{2k}$ and $S(w_{2k+1}) = -w_{2k+1}$, and hence $S(C) = e^{c_0-c_1}$. Also, by homomorphicity, $Z^w(\uparrow) = 1$. Thus, with the input of Lemma 3.4, $1 = e^{c_0+c_1}\alpha(\nu^{1/2})e^{c_0-c_1}$, and therefore $\alpha(\nu^{1/2}) = e^{2c_0}$, which gives $C_0 = e^{c_0} = \alpha(\nu^{1/4})$. $\square$

Corollary 3.5 shows that the even part of the Z^w-value of the cap is uniquely determined, independent of the choice of Drinfel'd associator and Z^u. Lemma 3.3 shows that the choice of Z^u uniquely determines V, the Z^w-value of the vertex: as the vertex is an element of wTT. In turn, by Section 3.1.2 (the complete proof of Part 1), V uniquely determines C, and thus, Z^w is well-defined. ??

When?

3.3.2. *Z^w is a Z^u-compatible homomorphic expansion.* Having constructed Z^w, A it remains to show that it is indeed a homomorphic expansion of wTF compatible with Z^u. We begin with the easier second statement:

[17] In Lemma 3.3 we assumed by convention that all strands below T were oriented upwards (towards T).

A: It's confusing that Z^w changes status from ~~[illegible]~~ "given, and has all properties" in sec ~~3.3.1~~ 3.3.1 to "constructed, and properties still unproven" in 3.3.2.

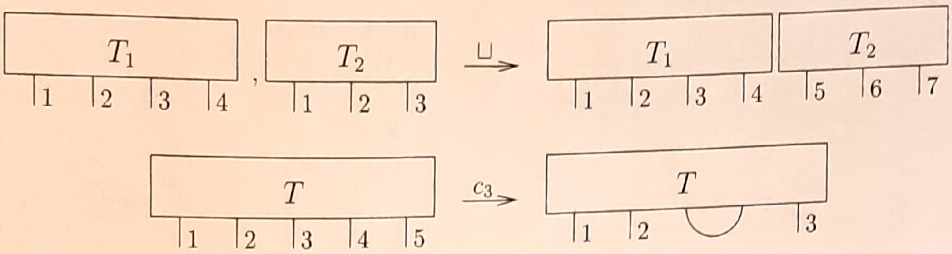


Figure 25. Basic planar algebra operations: disjoint union and contraction.

Proposition 3.6. *The map Z^w defined as in Lemma 3.3 is compatible with Z^u in the sense that $Z^w(K) = \alpha(Z^u(K))$, when $K \in sKTG \overset{a}{\hookrightarrow} \widetilde{wTF}$.*

Proof. Note that $sKTG \subseteq uTT$. ~~The key observation is that when an element~~ we need to show that if $K \in uTT$ is in fact in $sKTG$ (that is, when K is a tangle with two ends), then $Z^w(K) = \alpha(Z^u(K))$. Note that $\varphi(K)$ can be obtained by inserting K into the top strand of $\varphi(\uparrow)$. Since Z^u is compatible with insertions, $Z^u(\varphi(K))$ can be obtained by $Z(K)$ inserted into $Z^u(\varphi(\uparrow))$. Through the sequence of α, capping, puncturing, φ and multiplications by C^{-1}, all of $Z^u(\varphi(\uparrow))$ cancels, as in Lemma 3.4. Note that the cancellations still go through despite the fact that $\alpha(Z^u(K))$ is inserted on the top strand: this follows from the fact that $\alpha(Z^u(K))$ is in the α-image of $\mathcal{A}^u$, and the appropriate "commutativity" property holds in $\mathcal{A}^u$. Hence, $Z^w(K) = Z^w(K)$ as required. □

a pict. of the insertion ?

~~Finally, it~~ It remains to show that Z^w, as constructed via ξ, is indeed a homomorphic expansion of $\widetilde{wTF}$, which boils down to checking that it satisfies the R4, Unitarity, and Cap equations of Fact 2.5. ~~To this end, we show~~ that Z^w it is a planar algebra homomorphism on uTT, which implies the R4 equation immediately. We start with showing that...

Theorem 3.7. *The restriction of Z^w to $a(uTT)$ is a planar algebra map.*

Proof. Planar algebra operations can be written as compositions of two simpler, basic operations: disjoint unions and contractions. For two tangles T_1 and T_2, the disjoint union $T_1 \sqcup T_2$ is the disjoint union of the two tangles where the ends are ordered by declaring that the ordered ends of T_1 come first, followed by the ordered ends of T_2. The contraction operation c_i can be applied to any tangle with at least $i+1$ ends, and acts by joining the i-th and $(i+1)$-st ends of T and re-numbering the rest, resulting in a tangle with two less ends. Both operations are shown in Figure 25.

Thus This, we only need to show that Z^w commutes with these two operations, that is, $Z^w(T_1 \sqcup T_2) = Z^w(T_1) \sqcup Z^w(T_2)$, and $Z^w(c_i(T)) = c_i(Z^w(T))$. Note that the right sides of these equalities make sense: arrow diagrams on uTT skeleta, where Z^w takes its values, also form a planar algebra.

Disjoint unions. We need to compute $\xi(T_1 \sqcup T_2)$, where ~~T_1 and T_2 are uTT-s.~~ $T_1, T_2 \in uTT$. The φ map applied to a disjoint union of T_1 and T_2 is shown in Figure 26. Recall that the trees can be chosen arbitrarily by Lemma 3.1, so we chose the most convenient trees for the proof. Observe that $\varphi(T_1 \sqcup T_2)$ is the same $sKTG$ as $\varphi(T_1)$ and $\varphi(T_2)$ inserted into a simpler $sKTG$, denoted H, as shown in the same figure (up to some orientation switches which don't impact what follows and hence will be ignored ~~for simplicity~~). Hence, $Z^u(\varphi(T_1 \sqcup T_2))$ is given by inserting $Z^u(\varphi(T_1))$ and $Z^u(\varphi(T_2))$ into $Z^u(H)$. One could compute $Z^u(H)$ explicitly using

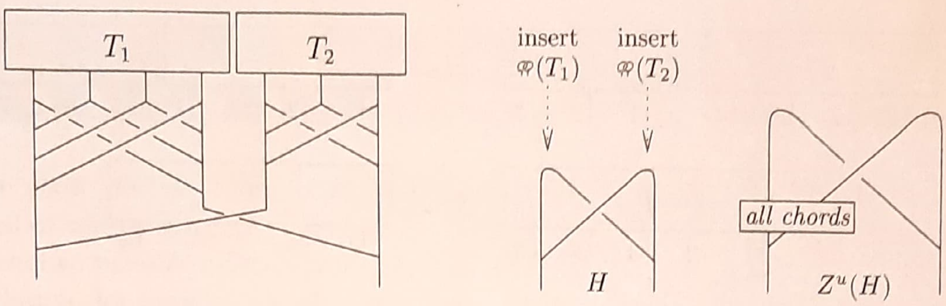


Figure 26. The double tree map applied to a disjoint union of $u\mathcal{T}T$-s is the same as inserting the double tree of each individual $u\mathcal{T}T$ into the *sKTG* H. In $Z^u(H)$ all chords can be pushed into the rectangle shown, using VI relations when necessary.

the same algorithm as before, but we can save ourselves the work, as follows. All chords in $Z^u(H)$ can be assumed to be located in the rectangle shown in Figure 26. After applying α, both supporting strands are punctured, meaning that after punctures $p\alpha(Z^u(H)) = 1$ in $\mathcal{A}^{sw}$. This implies that $\xi(T_1 \sqcup T_2) = \xi(T_1) \sqcup \xi(T_2)$, and it follows immediately that $Z^w(T_1 \sqcup T_2) = Z^w(T_1) \sqcup Z^w(T_2)$.

Contractions. Proving that Z^w commutes with contractions is more involved. By Lemma 3.2, we can assume ~~without loss of generality~~ that the ends contracted are the last (rightmost) two ends ~~of~~ T. Hence we will drop the subscript from c_i and denote this operation simply by c. of the n ends of T

We need to show that $Z^w(cT) = cZ^w(T)$, for any $T \in u\mathcal{T}T$. The homomorphic expansion Z^w is constructed as the composition of many maps, so this can be restated as the commutativity of the perimeter of a large diagram, which can be broken down to the commutativity of its smaller parts: the diagram is shown in Figure 27. To complete the proof, we analyze the commutativity of each numbered square (and one pentagon) of the diagram:

(1) This square plays out at *skeleton level* in $u\mathcal{T}T$ and *sKTG*. It clearly commutes, for example by inspection of the first square of the bottom diagram in Figure 27. In the diagram, the three strands to delete are indicated by dotted lines. Deletion in the consecutive square of the diagram always refers to these same strands.

(2) To show that this square commutes, one needs to understand the properties of Z^u with respect to deleting edges in *sKTG*. When an edge – which ends in a vertex at both ends – is deleted, those two vertices cease to be vertices. The associated graded operation on chord diagrams deletes the skeleton edge, and chord diagrams with any chord endings on the deleted edge are set equal to 0. See [WKO2, Section 4.6.1] for more detail.

The expansion Z^u is not quite homomorphic with respect to such edge deletions. Recall that in [WKO2, Section 4.6.1] Z^u is constructed from an invariant Z^{old} by adding vertex normalisations, which make it commute with unzips. In fact, Z^{old} does commute with edge deletions[18] [Da, Proposition 6.7], so the deletion errors of Z^u depend only on the vertex normalisations implemented in [WKO2].

[18]The reader might wonder, why normalize so that the expansion respects unzips, rather than deletions? The answer is that for finite generation of knotted trivalent graphs, unzips are crucial but deletions are not.

$$\begin{tikzcd}
u\mathit{TT} \arrow[r, "\varphi"] \arrow[d, "c"] \arrow[dr, phantom, "(1)"] & s\mathit{KTG} \arrow[r, "Z^u"] \arrow[d, "d^3"] \arrow[dr, phantom, "(2)"] & A^u \arrow[r, "\alpha"] \arrow[d, "d^3\circ(\cdot e^{-\frac{c}{4}}\nu^{\frac12})"] \arrow[dr, phantom, "(3)"] & A^{sw} \arrow[r, "\cdot C"] \arrow[d, "d^3\circ(\cdot e^{-\frac{a}{2}}\alpha(\nu^{\frac12}))"] \arrow[dr, phantom, "(4)"] & A^{sw} \arrow[r, "u^{n-1}"] \arrow[d, "d^3\circ(\cdot e^{-\frac{a}{2}}\alpha(\nu^{\frac12}))"] \arrow[dr, phantom, "(5)"] & A^{sw} \arrow[r, "p^{n-2}"] \arrow[d, "d^3\circ(\cdot e^{-\frac{a}{2}}\alpha(\nu^{\frac12}))"] \arrow[dr, phantom, "(6)"] & {} \\
u\mathit{TT} \arrow[r, "\varphi"] & s\mathit{KTG} \arrow[r, "Z^u"] & A^u \arrow[r, "\alpha"] & A^{sw} \arrow[r, "\cdot C"] & A^{sw} \arrow[r, "u^{n-1}\ n-3"] & A^{sw} \arrow[r, "p^{n-2}"] & {}
\end{tikzcd}$$

add: diagram continues....

$$\begin{tikzcd}
A^{sw} \arrow[r, "p^2"] \arrow[d, "d^3\circ(\cdot e^{-\frac{a}{2}}\alpha(\nu^{\frac12}))"] \arrow[drr, phantom, "(7)"] & A^{sw} \arrow[r, "\varphi"] & A^{sw} \arrow[r, "\cdot(\alpha(\nu^{-\frac14}))^n"] \arrow[d, "c\circ(\cdot\alpha(\nu^{-\frac12}))"] \arrow[dr, phantom, "(8)"] & A^{sw} \arrow[d, "c"] \\
A^{sw} \arrow[rr, "\varphi"] & & A^{sw} \arrow[r, "\cdot(\alpha(\nu^{-\frac14}))^{n-2}"] & A^{sw}
\end{tikzcd}$$

flush to right

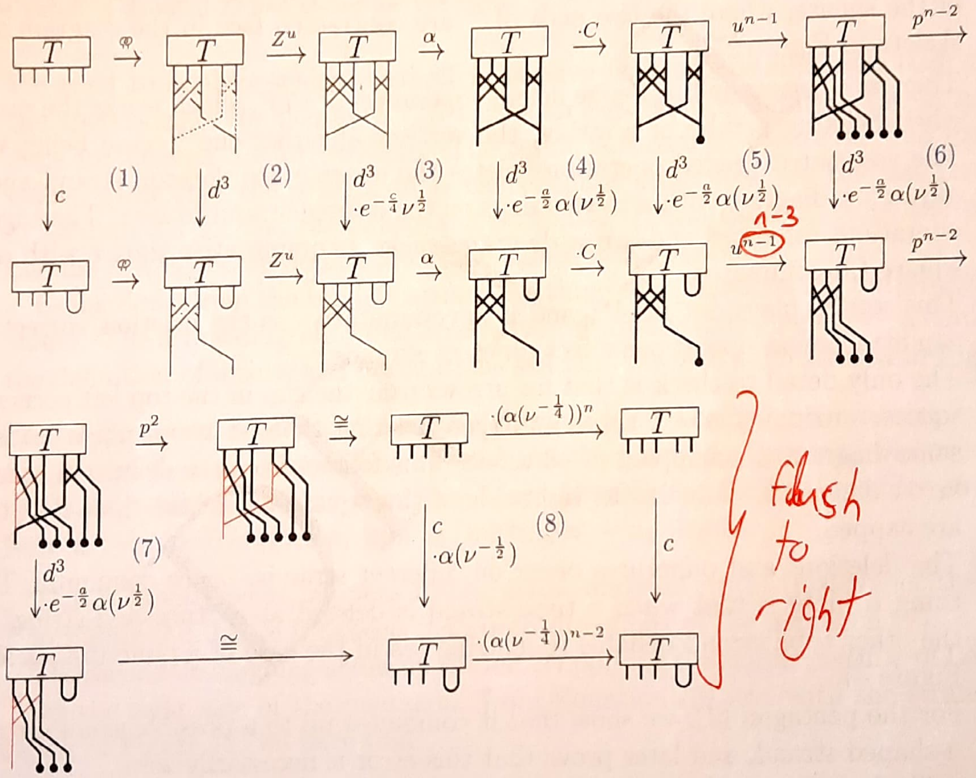


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Figure 27. The top diagram summarises the proof that Z^w commutes with contractions. The bottom diagram follows the skeleton changes of $T \in u\mathit{TT}$ throughout the same diagram. In the first square, the dotted lines indicate which three strands are to be deleted.

fig:Contra

Each vertex of an *sKTG* has one incident edge marked *distinguished*, in our drawings this is the vertical edge. If an edge is not distinguished at either of its ending vertices, then deleting it commutes with Z^u. When deleting a distinguished edge, such as the right vertical edge of square (2), then deletion and Z^u commutes only up to a correction term of $e^{-c/4}\nu^{1/2}$ inserted at the place of the vertex, where c stands for a