Referee report on the paper "Finite type invariants of w-knotted objects II ..." by D. Bar-Natan and S. Dancso

Th paper "Finite type invariants of w-knotted objects II ... " provides an interpretation of the Kashiwara-Vergne problem in Lie theory in terms of circuit algebras of w-tangles and w-foams. This is a very nice result. Diagrammatic interpretations play an important role in Quantum Topology. The classical example is the works of the first author on diagrammatic expansions of knot invariants. The article under review gives an entirely new approach to the Kashiwara-Vergne and Duflo problems in Lie theory. This approach will certainly bring new interesting results in the field, as it is already announced in the next paper by the authors.

As I understand, the main results obtained in the paper are as follows:

1) Theorem 4.8 which establishes a bijection between solutions of the Kashiwara-Vergne (KV) problem and a certain type of homomorphic expansions for w-foams.

2) Theorem 4.9 which is a version of Theorem 4.9 applies to solutions of the KV problem with even Duflo function and to a somewhat modified version for w-foams.

3) Theorem 4.12 establishes a relation between homomorphic expansions for KTGs (knotted trivalent graphs) given by Drinfeld associators and homomorphic expansions for w-foams.

In my view, these results should be better explained (possibly, already in the introductory part). At some points, the authors have to adjust the definitions of w-foams and of KTGs to obtain nicer statements. They should not be shy to clearly state the definitions of the objects that they are working with. At the moment, these definitions are scattered, and one needs to browse through the paper to find them.

On several occasions, the authors highlight the relation of their circuit algebras to 4-dimensional topology. A lot of geometric intuition in the paper comes from this correspondence. However, the authors also state that the precise relation between circuit algebras and 4-dimensional topology is only conjectural. In my view, this should be better highlighted (possibly in the introduction) and better explained. Does this correspondence work at least in one direction? What are the major obstacles to establish the correspondence?

In conclusion, the paper is of great interest in the fields of Quantum Topology and Lie Theory. If the changes and clarifications suggested above are implemented I will be happy to recommend it for publication.

Some small remarks that I encountered while reading the manuscript:

Section 2 is meant to be an informal introduction into the notions of algebraic structures, projectivization etc. While I do not object to the informal style, I believe having a complete treatment of some (possibly vey simple) example would be a great advantage for the reader. This also applies to circuit algebras.

Page 11, definition 3.1

I find it a bit confusing: our generators are positive and negative crossings. Then, what is the neutral (or virtual) crossing which appears in Figure 2 and in the definition of wT? I guess, having a virtual crossing is considered to be a general feature of circuit algebras. In this case, this fact should be emphasized in Definition 3.1

page 23

δ: v-tangles $\to$ Ribbon tubes in R^4
should it be w-tangles?