FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS II: TANGLES, FOAMS AND THE KASHIWARA-VERGNE PROBLEM

DROR BAR-NATAN AND ZSUZSANNA DANCSO

ABSTRACT. This is the second in a series of papers dedicated to studying w-knots, and more generally, w-knotted objects (w-braids, w-tangles, etc.). These are classes of knotted objects that are wider but weaker than their "usual" counterparts. To get (say) w-knots from usual knots (or u-knots), one has to allow non-planar "virtual" knot diagrams, hence enlarging the the base set of knots. But then one imposes a new relation beyond the ordinary collection of Reidemeister moves, called the "overcrossings commute" relation, making w-knotted objects a bit weaker once again. Satoh [Sa] studied several classes of w-knotted objects (under the name "weakly-virtual") and has shown them to be closely related to certain classes of knotted surfaces in $\mathbb{R}^4$.

In this article we study finite type invariants of w-tangles and w-trivalent graphs (also referred to as w-tangled foams). Much as the spaces $\mathcal{A}$ of chord diagrams for ordinary knotted objects are related to metrized Lie algebras, the spaces $\mathcal{A}^w$ of "arrow diagrams" for w-knotted objects are related to not-necessarily-metrized Lie algebras. Many questions concerning w-knotted objects turn out to be equivalent to questions about Lie algebras. Most notably we find that a homomorphic universal finite type invariant of w-foams is essentially the same as a solution of the Kashiwara-Vergne [KV] conjecture and much of the Alekseev-Torossian [AT] work on Drinfel'd associators and Kashiwara-Vergne can be re-interpreted as a study of w-foams.

CONTENTS

Date: first edition May 5, 2014, this edition Oct. 2, 2015. http://www.math.toronto.edu/drorbn/LOP.html#WKO2; the arXiv:1405.1955 edition may be older.

2010 *Mathematics Subject Classification.* 57M25.

Key words and phrases. virtual knots, w-braids, w-knots, w-tangles, knotted graphs, finite type invariants, Alexander polynomial, Kashiwara-Vergne, associators, free Lie algebras.

This work was partially supported by NSERC grant RGPIN 262178. This paper is part 2 of a 4-part series whose first two parts originally appeared as a combined preprint, [WKO0].

1. Introduction

This is the second in a series of papers on w-knotted objects. In the first paper [WKO1], we took a classical approach to studying finite type invariants of w-braids and w-knots and proved that the universal finite type invariant for w-knots is essentially the Alexander polynomial. In this paper we will study finite type invariants of w-tangles and w-tangled foams from a more algebraic point of view, and prove that "homomorphic" universal finite type invariants of w-tangled foams are in one-to-one correspondence with solutions to the (Alekseev-Torossian version of) the Kashiwara-Vergne problem in Lie theory. Mathematically, this paper does not depend on the results of [WKO1] in any significant way, and the reader familiar with the theory of finite type invariants will have no difficulty reading this paper without having read [WKO1]. However, since this paper starts with an abstract rephrasing of the well-known finite type story in terms of general algebraic structures, readers who need an introduction to finite type invariants may find it more pleasant to read [WKO1] first (especially Sections 1, 2 and 3.1–3.5).

1.1. Motivation and hopes.

Motivation and hopes. This article and its siblings [WKO1] and [WKO3] are efforts towards a larger goal. Namely, we believe many of the difficult algebraic equations in mathematics, especially those that are written in graded spaces, more especially those that are related in one way or another to quantum groups [Dr1], and to the work of Etingof and Kazhdan [EK], can be understood, and indeed would appear more natural, in terms of finite type invariants of various topological objects.

This work was inspired by Alekseev and Torossian's results [AT] on Drinfel'd associators and the Kashiwara-Vergne conjecture, both of which fall into the aforementioned class of "difficult equations in graded spaces". The Kashiwara-Vergne conjecture — proposed in 1978 [KV] and proven in 2006 by Alekseev and Meinrenken [AT] — has strong implications in Lie theory and harmonic analysis, and is a cousin of the Duflo isomorphism, which was shown to be knot-theoretic in [BLT]. We also know that Drinfel'd's theory of associators [Dr2] can be interpreted as a theory of well-behaved universal finite type invariants of parenthesized tangles[1] [LM, BN2], or of knotted trivalent graphs [Da].

In Section 4 we will re-interpret the Kashiwara-Vergne conjecture as the problem of finding a "homomorphic" universal finite type invariant of a class of w-knotted trivalent graphs (more accurately named w-tangled foams). This result fits into a bigger picture incorporating usual, virtual and w-knotted objects and their theories of finite type invariants, connected by the inclusion map from usual to virtual, and the projection from virtual to w-knotted objects. In a sense that will be made precise in Section 2, usual and w-knotted objects with this mapping form a unified algebraic structure, and the mysterious relationship between Drinfel'd associators and the Kashiwara-Vergne conjecture is explained as a theory of finite type invariants for this larger structure. This will be the topic of Section 4.6.

We are optimistic that this paper is a step towards re-interpreting the work of Etingof and Kazhdan [EK] on quantization of Lie bi-algebras as a construction of a well-behaved universal finite type invariant of virtual knots [Ka] or of a similar class of virtually knotted objects. However, w-knotted objects are quite interesting in their own right, both topologically and algebraically: they are related to combinatorial group theory, to groups of movies of flying rings in $\mathbb{R}^3$, and more generally, to certain classes of knotted surfaces in $\mathbb{R}^4$. The references include [BH, FRR, Gol, Mc, Sa].

In [WKO1] we studied the universal finite type invariants of w-braids and w-knots, the latter of which turns out to be essentially the Alexander polynomial. A more thorough introduction about our "hopes and dreams" and the u-v-w big picture can also be found in [WKO1].

1.2. **A brief overview and large-scale explanation.** We are going to start by developing the algebraic ingredients of the paper in Section 2. The general notion of an *algebraic structure* lets us treat spaces of a topological or diagrammatic nature in a unified algebraic manner. All of braids, w-braids, w-knots, w-tangles, etc., and their associated chord- or arrow-diagrammatic counterparts form algebraic structures, and so do any number of these spaces combined, with maps between them.

We then introduce *associated graded structures* with respect to a specific filtration, the machine which in our case takes an algebraic structure of "topological nature" (say, braids with n strands) and produces the corresponding diagrammatic space (for braids, horizontal chord diagrams on n vertical strands). This is done by taking the associated graded space with respect to a given filtration, namely the powers of the augmentation ideal in the algebraic structure.

An *expansion*, sometimes called a universal finite type invariant, is a map from an algebraic structure (in this case one of topological nature) to its associated graded (a structure of combinatorial/diagrammatic nature), with a certain non-degeneracy property. A *homomorphic expansion* is one that is in addition "well behaved" with respect to the *operations* of the algebraic structure (such as composition and strand doubling for braids, for example).

The three main results of the paper are as follows:

(1) As mentioned before, our goal is to provide a topological framework for the Kashiwara-Vergne (KV) problem. The first result in that direction is Theorem 4.9, in which we establish a bijection between certain homomorphic expansions of *w-tangled foams*

[1] "q-tangles" in [LM], "non-associative tangles" in [BN2].

(introduced in Section 4) and solutions of the Kashiwara-Vergne equations. More precisely, "certain" homomorphic expansions means ones that are group-like (a commonly used condition), and subject to another very minor technical condition. Section 3 leads up to this result by studying the simpler case of *w-tangles* and identifying building blocks of its associated graded structure as the spaces which appear in the [AT] formulation of the KV equations.

(2) In Theorem 4.11 we study an unoriented version of w-tangled foams, and prove that homomorphic expansions for this space (group-like and subject to the same minor condition) are in one-to-one correspondence with solutions to the KV problem with *even Duflo function*. This sets the stage for perhaps the most interesting result of the paper:

(3) Section 4.7 marries the theory above with the theory of ordinary (not w-) knotted trivalent graphs (KTGs). For technical reasons explained in Section 4, we work with a signed version of KTGs (sKTG). Roughly speaking, homomorphic expansions for sKTGs are determined by a *Drinfel'd associator*. Furthermore, sKTGs map naturally into w-tangled foams.

In Theorem 4.15 we prove that any homomorphic expansion of sKTGs coming from a *horizontal chord* associator has a compatible homomorphic expansion of w-tangled foams, and furthermore, these expansions are in one-to-one correspondence with *symmetric* solutions of the KV problem. This gives a topological explanation for the relationship between Drinfel'd associators and the KV conjecture.

We note that in [WKO3] we'll further capitalize on these insights to provide a topological proof and interpretation for Alekseev, Enriquez and Torossian's explicit solutions for the KV conjecture in terms of associators [AET].

Several of the structures of a topological nature in this paper (w-tangles and w-foams) are introduced as *Reidemeister theories*. That is, the spaces are built from pictorial generators (such as crossings) which can be connected arbitrarily, and the resulting pictures are then factored out by certain relations ("Reidemeister moves"). Technically speaking, this is done using the framework of *circuit algebras* (similar to planar algebras but without the planarity requirement) which are introduced in Section 2.

One of the fundamental theorems of classical knot theory is Reidemeister's theorem, which states that isotopy classes of knots are in bijection with *knot diagrams* modulo Reidemeister moves. In our case, w-knotted objects have a Reidemeister description and a topological interpretation in terms of ribbon knotted tubes in $\mathbb{R}^4$. However, the analogue of the Reidemeister theorem, i.e. the statement that these two interpretations coincide, is only known for w-braids [Mc, D, BH].

For w-tangles and w-foams (and w-knots as well) there is a map δ from the Reidemeister presentation to the appropriate class of ribbon 2-knotted objects in $\mathbb{R}^4$. In our case this means that all the generators have a local topological interpretation and the relations represent isotopies. The map δ is certainly a surjection, but it is only conjectured to be injective (in other words, it is possible that some relations are missing).

The main difficulty in proving the injectivity of δ lies in the management of the ribbon structure. A ribbon 2-knot is a knotted sphere or long tube in $\mathbb{R}^4$ which admits a filling with only certain types of singularities. While there are Reidemeister theorems for general 2-knots in $\mathbb{R}^4$ [CS], the techniques don't translate well to ribbon 2-knots, mainly because it is not well understood how different ribbon structures (fillings) of the same ribbon 2-knot can be obtained from each other through Reidemeister type moves. The completion of such a theorem would be of great interest. We suspect that even if δ is not injective, the present set of generators and relations describes a set of ribbon-knotted tubes in $\mathbb{R}^4$ with possibly some extra combinatorial information, similarly to how, say, dropping the $R1$ relation in classical knot theory results in a Reidemeister theory for framed knots with rotation numbers.

The paper is organized as follows: we start with a discussion of general algebraic structures, associated graded structures, expansions (universal finite type invariants) and "circuit algebras" in Section 2. In Section 3 we study w-tangles and identify some of the spaces [AT] where the KV conjecture "lives" as the spaces of "arrow diagrams" (the w-analogue of chord diagrams) for certain w-tangles. In Section 4 we study w-tangled foams and we prove the main theorems discussed above. For more detailed information consult the "Section Summary" paragraphs at the beginning of each of the sections. A glossary of notation is on page 55.

1.3. **Acknowledgement.** We wish to thank Anton Alekseev, Jana Archibald, Scott Carter, Karene Chu, Iva Halacheva, Joel Kamnitzer, Lou Kauffman, Peter Lee, Louis Leung, Jean-Baptiste Meilhan, Dylan Thurston, Lucy Zhang and the anonymous referees for comments and suggestions.

2. Algebraic Structures, Expansions, and Circuit Algebras

Section Summary. In this section we introduce the associated graded structure of an "arbitrary algebraic structure" with respect to powers of its augmentation ideal (Sections 2.1 and 2.2) and introduce the notions of "expansions" and "homomorphic expansions" (2.3). Everything is so general that practically anything is an example, yet our main goal is to set the language for the examples of w-tangles and w-tangled foams, which appear later in this paper. Both of these examples are types of "circuit algebras", and hence we end this section with a general discussion of circuit algebras (Sec. 2.4).

2.1. **Algebraic Structures.** An "algebraic structure" $\mathcal{O}$ is some collection $(\mathcal{O}_\alpha)$ of sets of objects of different kinds, where the subscript α denotes the "kind" of the objects in $\mathcal{O}_\alpha$, along with some collection of "operations" ψ_β, where each ψ_β is an arbitrary map with domain some product $\mathcal{O}_{\alpha_1} \times \cdots \times \mathcal{O}_{\alpha_k}$ of sets of objects, and range a single set $\mathcal{O}_{\alpha_0}$ (so operations may be unary or binary or multinary, but they always return a value of some fixed kind). We also allow some named "constants" within some $\mathcal{O}_\alpha$'s (or equivalently, allow some 0-nary operations).[2] The operations may or may not be subject to axioms — an "axiom" is an identity asserting that some composition of operations is equal to some other composition of operations.

Figure 1 illustrates the general notion of an algebraic structure. Here are a few specific examples:

[2]Alternatively define "algebraic structures" using the theory of "multicategories" [Lei]. Using this language, an algebraic structure is simply a functor from some "structure" multicategory $\mathcal{C}$ into the multicategory **Set** (or into **Vect**, if all $\mathcal{O}_i$ are vector spaces and all operations are multi-linear). A "morphism" between two algebraic structures over the same multicategory $\mathcal{C}$ is a natural transformation between the two functors representing those structures.

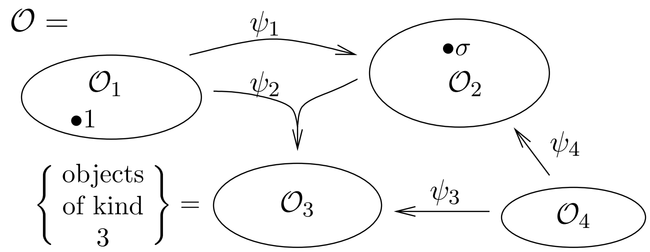


Figure 1. An algebraic structure $\mathcal{O}$ with 4 kinds of objects and one binary, 3 unary and two 0-nary operations (the constants 1 and σ).

- We will use $\langle b\rangle$, the free group on one generator b, as a running example throughout this chapter (of course $\langle b\rangle$ is isomorphic to $\mathbb{Z}$). This is an algebraic structure with one kind of objects, a binary operation "multiplication", a unary operation "inverse", one constant "the identity", and the expected axioms.
- Groups in general: one kind of objects, one binary "multiplication", one unary "inverse", one constant "the identity", and some axioms.
- Group homomorphisms: Two kinds of objects, one for each group. 7 operations — 3 for each of the two groups and the homomorphism itself, going between the two groups. Many axioms.
- A group acting on a set, a group extension, a split group extension and many other examples from group theory.
- A quandle is a set with an operation $\uparrow$, satisfying $(x \uparrow y) \uparrow z = (x \uparrow y) \uparrow (y \uparrow z)$ and some further minor axioms. This is an algebraic structure with one kind of objects and one operation. See [WKO0] for an analysis of quandles from the perspective of this paper.
- Planar algebras as in [Jon] and circuit algebras as in Section 2.4.
- The algebra of knotted trivalent graphs as in [BN4, Da].
- Let $\varsigma\colon B \to S$ be an arbitrary homomorphism of groups (though our notation suggests what we have in mind — B may well be braids, and S may well be permutations). We can consider an algebraic structure $\mathcal{O}$ whose kinds are the elements of S, for which the objects of kind $s \in S$ are the elements of $\mathcal{O}_s := \varsigma^{-1}(s)$, and with the product in B defining operations $\mathcal{O}_{s_1} \times \mathcal{O}_{s_2} \to \mathcal{O}_{s_1 s_2}$.
- W-tangles and w-foams, studied in the following two sections of this paper.
- Clearly, many more examples appear throughout mathematics.

2.2. **Associated Graded Structures.** Any algebraic structure $\mathcal{O}$ has an "especially natural" associated graded structure: that is, we take the associated structure with respect to a specific and natural filtration. This will be a repeating construction throughout the rest of this paper series.

First extend $\mathcal{O}$ to allow formal linear combinations of objects of the same kind (extending the operations in a linear or multi-linear manner), then let $\mathcal{I}$, the "augmentation ideal", be the sub-structure made out of all such combinations in which the sum of coefficients is 0, then let $\mathcal{I}^m$ be the set of all outputs of algebraic expressions (that is, arbitrary compositions of the operations in $\mathcal{O}$) that have at least m inputs in $\mathcal{I}$ (and possibly, further inputs in $\mathcal{O}$), and finally, set

$$\operatorname{grad}\mathcal{O} := \bigoplus_{m\geq 0} \mathcal{I}^m/\mathcal{I}^{m+1}. \tag{1}$$

Clearly, with the operations inherited from $\mathcal{O}$, the associated graded $\operatorname{grad}\mathcal{O}$ is again algebraic structure with the same multi-graph of spaces and operations, but with new objects and with new operations that may or may not satisfy the axioms satisfied by the operations of $\mathcal{O}$. The main new feature in $\operatorname{grad}\mathcal{O}$ is that it is a "graded" structure; we denote the degree m piece $\mathcal{I}^m/\mathcal{I}^{m+1}$ of $\operatorname{grad}\mathcal{O}$ by $\operatorname{grad}_m\mathcal{O}$.

We believe that many of the most interesting graded structures that appear in mathematics are the result of this construction (i.e., as associated graded structures with respect to powers of the augmentation ideal), and that many of the interesting graded equations that appear in mathematics arise when one tries to find "expansions", or "universal finite type invariants", which are also morphisms[3] $Z\colon \mathcal{O} \to \operatorname{grad}\mathcal{O}$ (see Section 2.3) or when one studies "automorphisms" of such expansions[4]. Indeed, the paper you are reading now is really the study of the associated graded structures of various algebraic structures associated with w-knotted objects. We would like to believe that much of the theory of quantum groups (at "generic" $\hbar$) will eventually be shown to be a study of the associatead graded structures of various algebraic structures associated with v-knotted objects.

Example 2.1. We compute the associated graded structuture of the running example $\langle b\rangle$. Allowing formal $\mathbb{Q}$-linear combinations of elements we get $\mathbb{Q}\langle b\rangle = \mathbb{Q}[b, b^{-1}]$. The augmentation ideal $\mathcal{I}$ is generated by differences $(b^n - 1)$ as a vector space (where $1 = b^0$), and generated by $(b-1)$ as an ideal.

We claim that $\operatorname{grad}\langle b\rangle \cong \mathbb{Q}[[c]]$, the algebra of power series in one variable. To show this, consider the map $\pi : \mathbb{Q}[[c]] \to \operatorname{grad}\langle b\rangle$ by setting $\pi(c) = [b-1] \pmod{\mathcal{I}^2}$. It is easy to show explicitly that π is surjective. For example, in degree 1, we need to show that $b-1$ generates $\mathcal{I}/\mathcal{I}^2$. indeed, $(b^n-1) - n(b-1)$ has a double zero at $b = 1$, and hence $f = \frac{(b^n-1)-n(b-1)}{(b-1)^2}$ is a polynomial, and $b^n - 1 = n(b-1) + f(b-1)^2$. So modulo $(b-1)^2 \in \mathcal{I}^2$, $b^n - 1 = n(b-1)$. A similar argument works to show that $(b-1)^k$ generates $\mathcal{I}^k/\mathcal{I}^{k+1}$.

Note that $\langle b\rangle$ can also be thought of as the pure braid group on two strands: b would be a "full twist" and c can be represented as a single "horizontal chord". In other knot theoretic settings, it is generally relatively easy to find a "candidate associated graded" and a map π, which can be shown to be surjective by explicit means.

To show that π is injective we are going to use the machinery of "expansions" which is the tool we use to accomplish similar tasks in the later sections of this paper.

We end this section with two more examples of computing associated graded structures: the proof of Proposition 2.2 is an exercise; for the proof of Proposition 2.3 see [WKO0].

Proposition 2.2. *If G is a group,* $\operatorname{grad} G$ *is a graded associative algebra with unit. Similarly, the associated graded structure of a group homomorphism is a homomorphism of graded associative algebras.* $\square$

Proposition 2.3. *If Q is a unital quandle,* $\operatorname{grad}_0 Q$ *is one-dimensional and* $\operatorname{grad}_{>0} Q$ *is a graded right Leibniz algebra*[5] *generated by* $\operatorname{grad}_1 Q$*.*

---

[3]Indeed, if $\mathcal{O}$ is finitely presented then finding such a morphism $Z\colon \mathcal{O} \to \operatorname{grad}\mathcal{O}$ amounts to finding its values on the generators of $\mathcal{O}$, subject to the relations of $\mathcal{O}$. Thus it is equivalent to solving a system of equations written in some graded spaces.

[4]The Drinfel'd graded Grothendieck-Teichmuller group *GRT* is an example of such an automorphism group. See [Dr3, BN3].

[5]A Leibniz algebra is a Lie algebra without anti-commutativity, as defined by Loday in [Lod].

2.3. **Expansions and Homomorphic Expansions.** We start with the definition. Given an algebraic structure $\mathcal{O}$ let fil $\mathcal{O}$ denote the filtered structure of linear combinations of objects in $\mathcal{O}$ (respecting kinds), filtered by the powers $(\mathcal{I}^m)$ of the augmentation ideal $\mathcal{I}$. Recall also that any graded space $G = \bigoplus_m G_m$ is automatically filtered, by $\left(\bigoplus_{n\geq m} G_n\right)_{m=0}^{\infty}$.

Definition 2.4. An "expansion" Z for $\mathcal{O}$ is a map $Z\colon \mathcal{O} \to \operatorname{grad}\mathcal{O}$ that preserves the kinds of objects and whose linear extension (also called Z) to fil $\mathcal{O}$ respects the filtration of both sides, and for which $(\operatorname{gr} Z) : (\operatorname{gr} \operatorname{fil} \mathcal{O} = \operatorname{grad}\mathcal{O}) \to (\operatorname{gr} \operatorname{grad}\mathcal{O} = \operatorname{grad}\mathcal{O})$ is the identity map of $\operatorname{grad}\mathcal{O}$; we refer to this as the "universality property".

In practical terms, this is equivalent to saying that Z is a map $\mathcal{O} \to \operatorname{grad}\mathcal{O}$ whose restriction to $\mathcal{I}^m$ vanishes in degrees less than m (in $\operatorname{grad}\mathcal{O}$) and whose degree m piece is the projection $\mathcal{I}^m \to \mathcal{I}^m/\mathcal{I}^{m+1}$.

We come now to what is perhaps the most crucial definition in this paper.

Definition 2.5. A "homomorphic expansion" is an expansion which also commutes with all the algebraic operations defined on the algebraic structure $\mathcal{O}$.

Why Bother with Homomorphic Expansions? Primarily, for two reasons:

- Often $\operatorname{grad}\mathcal{O}$ is simpler to work with than $\mathcal{O}$; for one, it is graded and so it allows for finite "degree by degree" computations, whereas often times, such as in many topological examples, anything in $\mathcal{O}$ is inherently infinite. Thus it can be beneficial to translate questions about $\mathcal{O}$ to questions about $\operatorname{grad}\mathcal{O}$. A simplistic example would be, "is some element $a \in \mathcal{O}$ the square (relative to some fixed operation) of an element $b \in \mathcal{O}$?". Well, if Z is a homomorphic expansion and by a finite computation it can be shown that $Z(a)$ is not a square already in degree 7 in $\operatorname{grad}\mathcal{O}$, then we've given a conclusive negative answer to the example question. Some less simplistic and more relevant examples appear in [BN4].
- Often $\operatorname{grad}\mathcal{O}$ is "finitely presented", meaning that it is generated by some finitely many elements $g_1, \ldots, g_k \in \mathcal{O}$, subject to some relations $R_1 \ldots R_n$ that can be written in terms of $g_1, \ldots, g_k$ and the operations of $\mathcal{O}$. In this case, finding a homomorphic expansion Z is essentially equivalent to guessing the values of Z on $g_1, \ldots, g_k$, in such a manner that these values $Z(g_1), \ldots, Z(g_k)$ would satisfy the $\operatorname{grad}\mathcal{O}$ versions of the relations $R_1 \ldots R_n$. So finding Z amounts to solving equations in graded spaces. It is often the case (as will be demonstrated in this paper; see also [BN2, BN3]) that these equations are very interesting for their own algebraic sake, and that viewing such equations as arising from an attempt to solve a problem about $\mathcal{O}$ sheds further light on their meaning.

In practice, often the first difficulty in searching for an expansion (or a homomorphic expansion) $Z\colon \mathcal{O} \to \operatorname{grad}\mathcal{O}$ is that its would-be target space $\operatorname{grad}\mathcal{O}$ is hard to identify. It is typically easy to make a suggestion $\mathcal{A}$ for what $\operatorname{grad}\mathcal{O}$ could be. It is typically easy to come up with a reasonable generating set $\mathcal{D}_m$ for $\mathcal{I}^m$ (keep some knot theoretic examples in mind, or $\mathbb{Z}$ in Example 2.1). It is a bit harder but not exceedingly difficult to discover some relations $\mathcal{R}$ satisfied by the elements of the image of $\mathcal{D}$ in $\mathcal{I}^m/\mathcal{I}^{m+1}$ (4T, $\overrightarrow{4T}$, and more in knot theory, there are no relations for $\mathbb{Z}$). Thus we set $\mathcal{A} := \mathcal{D}/\mathcal{R}$; but it is often very hard to be sure that we found everything that ought to go in $\mathcal{R}$; so perhaps our suggestion $\mathcal{A}$ is still too big? Finding 4T for example was actually not *that* easy. Could we have missed some further relations that are hiding in $\mathcal{A}$?

The notion of an $\mathcal{A}$-expansion, defined below, solves two problems at once. Once we find an $\mathcal{A}$-expansion we know that we've identified $\operatorname{grad}\mathcal{O}$ correctly, and we automatically get what we really wanted, a $(\operatorname{grad}\mathcal{O})$-valued expansion.

Definition 2.6. A "candidate assoctaed graded structure" for an algebraic structure $\mathcal{O}$ is a graded structure $\mathcal{A}$ with the same operations as $\mathcal{O}$ along with a homomorphic surjective graded map $\pi\colon \mathcal{A} \to \operatorname{grad}\mathcal{O}$. An "$\mathcal{A}$-expansion" is a kind and filtration respecting map $Z_{\mathcal{A}}\colon \mathcal{O} \to \mathcal{A}$ for which $(\operatorname{gr} Z_{\mathcal{A}}) \circ \pi\colon \mathcal{A} \to \mathcal{A}$ is the identity. One can similarly define "homomorphic $\mathcal{A}$-expansions".

$$\begin{tikzcd} & \mathcal{A} \arrow[d, shift right, "\pi"'] \\ \mathcal{O} \arrow[ur, "Z_{\mathcal{A}}"] \arrow[r, "Z"'] & \operatorname{grad}\mathcal{O} \arrow[u, shift right, "\operatorname{gr} Z_{\mathcal{A}}"'] \end{tikzcd}$$

Proposition 2.7. *If $\mathcal{A}$ is a candidate associated graded of $\mathcal{O}$ and $Z_{\mathcal{A}}\colon \mathcal{O} \to \mathcal{A}$ is a homomorphic $\mathcal{A}$-expansion, then $\pi : \mathcal{A} \to \operatorname{grad}\mathcal{O}$ is an isomorphism and $Z := \pi \circ Z_{\mathcal{A}}$ is a homomorphic expansion. (Often in this case, $\mathcal{A}$ is identified with* $\operatorname{grad}\mathcal{O}$ *and $Z_{\mathcal{A}}$ is identified with Z).*

Proof. π is surjective by birth. Since $(\operatorname{gr} Z_{\mathcal{A}}) \circ \pi$ is the identity, π it is also injective and hence it is an isomorphism. The rest is immediate. □

Example 2.8. Back to $\langle b\rangle$, in Example 2.1 we found a candidate associated graded structure $\mathcal{A} = \mathbb{Q}[[c]]$ and a map $\pi : c \mapsto [b-1]$. According to Proposition 2.7, it is enough to find a homomorphic $\mathcal{A}$-expansion, that is, an algebra homomorphism $Z_{\mathcal{A}} : \mathbb{Q}\langle b\rangle \to \mathbb{Q}[[c]]$ such that $\operatorname{gr} Z_{\mathcal{A}} \circ \pi$ is the identity of $\mathbb{Q}[[c]]$. It is a straightforward calculation to check that any algebra map defined by $Z_{\mathcal{A}}(b) = 1 + c + \{\text{higher order terms}\}$ satisfies this property. If one seeks a "group-like" homomorphic expansion then $Z_{\mathcal{A}}(b) = e^c$ is the only solution. In either case, exhibiting $Z_{\mathcal{A}}$ proves that π is injective and hence $\mathcal{A}$ is the associated graded structure of $\langle b\rangle$.

2.4. **Circuit Algebras.** "Circuit algebras" are so common and everyday, and they make such a useful language (definitely for the purposes of this paper, but also elsewhere), we find it hard to believe they haven't made it into the standard mathematical vocabulary[6]. People familiar with planar algebras [Jon] may note that circuit algebras are just the same as planar algebras, except with the planarity requirement dropped from the "connection diagrams" (and all colourings are dropped as well).

In our context, the main utility of circuit algebras is that they allow for a much simpler presentation of v(irtual)- and w-tangles. The planar algebra presentation of classical tangles (generated by crossings and factored out by Reidemeister moves) can be modified to obtain a planar algebra presentation of, say, v-tangles, by adding an extra generator (the "virtual crossing") and a number of extra relations (the "virtual" and "mixed" Reidemeister moves). Switching from planar algebras to circuit algebras however renders these extra generators and relations unnecessary: the "virtual crossing" becomes merely a circuit algebra artifact, and the new Reidemeister moves are implied by the circuit algebra structure (see Definition 3.1 and Remark 3.3).

The everyday intuition for circuit algebras comes from electronic circuits, whose components can be wired together in many, not necessarily planar, ways, and it is not important to know how these wires are embedded in space. For details and more motivation see Section 5.1. We start formalizing this image by defining "wiring diagrams", the abstract analogs

[6]Or have they, and we have been looking the wrong way?

of printed circuit boards. Let $\mathbb{N}$ denote the set of natural numbers including 0, and for $n \in \mathbb{N}$ let $\underline{n}$ denote some fixed set with n elements, say $\{1, 2, \ldots, n\}$.

Definition 2.9. Let $k, n, n_1, \ldots, n_k \in \mathbb{N}$ be natural numbers. A "wiring diagram" D with inputs $\underline{n_1}, \ldots \underline{n_k}$ and outputs $\underline{n}$ is an unoriented compact 1-manifold whose boundary is $\underline{n} \amalg \underline{n_1} \amalg \cdots \amalg \underline{n_k}$, regarded up to homeomorphism. In strictly combinatorial terms, it is a pairing[7] of the elements of the set $\underline{n} \amalg \underline{n_1} \amalg \cdots \amalg \underline{n_k}$ along with a single further natural number that counts closed circles. If $D_1; \ldots; D_m$ are wiring diagrams with inputs $\underline{n_{11}}, \ldots, \underline{n_{1k_1}}; \ldots; \underline{n_{m1}}, \ldots, \underline{n_{mk_m}}$ and outputs $\underline{n_1}; \ldots; \underline{n_m}$ and D is a wiring diagram with inputs $\underline{n_1}; \ldots; \underline{n_m}$ and outputs $\underline{n}$, there is an obvious "composition" $D(D_1, \ldots, D_m)$ (obtained by gluing the corresponding 1-manifolds, and also describable in completely combinatorial terms) which is a wiring diagram with inputs $(\underline{n_{ij}})_{1 \leq i \leq k_j, 1 \leq j \leq m}$ and outputs $\underline{n}$ (note that closed circles may be created in $D(D_1, \ldots, D_m)$ even if none existed in D and in $D_1; \ldots; D_m$).

A circuit algebra is an algebraic structure (in the sense of Section 2.2) whose operations are parametrized by wiring diagrams. Here's a formal definition:

Definition 2.10. A circuit algebra consists of the following data:

- For every natural number $n \geq 0$ a set (or a $\mathbb{Z}$-module) C_n "of circuits with n legs".
- For any wiring diagram D with inputs $\underline{n_1}, \ldots \underline{n_k}$ and outputs $\underline{n}$, an operation (denoted by the same letter) $D\colon C_{n_1} \times \cdots \times C_{n_k} \to C_n$ (or linear $D\colon C_{n_1} \otimes \cdots \otimes C_{n_k} \to C_n$ if we work with $\mathbb{Z}$-modules).

We insist that the obvious "identity" wiring diagrams with $\underline{n}$ inputs and $\underline{n}$ outputs act as the identity of C_n, and that the actions of wiring diagrams be compatible in the obvious sense with the composition operation on wiring diagrams.

A silly but useful example of a circuit algebra is the circuit algebra $\mathcal{S}$ of empty circuits, or in our context, of "skeletons". The circuits with n legs for $\mathcal{S}$ are wiring diagrams with n outputs and no inputs; namely, they are 1-manifolds with boundary $\underline{n}$ (so n must be even).

More generally one may pick some collection of "basic components" (analogous to logic gates and junctions for electronic circuits as in Figure 21) and speak of the "free circuit algebra" generated by these components; even more generally we can speak of circuit algebras given in terms of "generators and relations". (In the case of electronics, our relations may include the likes of De Morgan's law $\neg(p \vee q) = (\neg p) \wedge (\neg q)$ and the laws governing the placement of resistors in parallel or in series.) We feel there is no need to present the details here, yet many examples of circuit algebras given in terms of generators and relations appear in this paper, starting with the next section. We will use the notation $C = \mathrm{CA}\langle G \mid R \rangle$ to denote the circuit algebra generated by a collection of elements G subject to some collection R of relations.

People familiar with electric circuits know that connectors sometimes come in "male" and "female" versions, and that you can't plug a USB cable into a headphone jack. Thus one may define "directed circuit algebras" in which the wiring diagrams are oriented, the circuit sets C_n get replaced by $C_{n_1 n_2}$ for "circuits with n_1 incoming wires and n_2 outgoing wires" and only orientation preserving connections are ever allowed. Likewise there is a "coloured" version of everything, in which the wires may be coloured by the elements of some given set

[7]We mean "pairing" in the sense of combinatorics, not in the sense of linear algebra. That is, an involution without fixed point.

X (which may include among its members the elements "USB" and "audio") and in which connections are allowed only if the colour coding is respected. We will leave the formal definitions of directed and coloured circuit algebras, as well as the definitions of directed and coloured analogues of the skeletons algebra $\mathcal{S}$ and generators and relations for directed and coloured algebras, as an exercise.

Note that there is an obvious notion of "a morphism between two circuit algebras" and that circuit algebras (directed or not, coloured or not) form a category. We feel that a precise definition is not needed. A lovely example is the "implementation morphism" of logic circuits in the style of Figure 21 in Section 5 into more basic circuits made of transistors and resistors.

Perhaps the prime mathematical example of a circuit algebra is tensor algebra. If t_1 is an element (a "circuit") in some tensor product of vector spaces and their duals, and t_2 is the same except in a possibly different tensor product of vector spaces and their duals, then once an appropriate pairing D (a "wiring diagram") of the relevant vector spaces is chosen, t_1 and t_2 can be contracted ("wired together") to make a new tensor $D(t_1, t_2)$. The pairing D must pair a vector space with its own dual, and so this circuit algebra is coloured by the set of vector spaces involved, and directed, by declaring (say) that some vector spaces are of one gender and their duals are of the other. We have in fact encountered this circuit algebra in [WKO1, Section 3.5].

Let G be a group. A G-graded algebra A is a collection $\{A_g \colon g \in G\}$ of vector spaces, along with products $A_g \otimes A_h \to A_{gh}$ that induce an overall structure of an algebra on $A := \bigoplus_{g \in G} A_g$. In a similar vein, we define the notion of an $\mathcal{S}$-graded circuit algebra:

Definition 2.11. An $\mathcal{S}$-graded circuit algebra, or a "circuit algebra with skeletons", is an algebraic structure C with spaces C_β, one for each element β of the circuit algebra of skeletons $\mathcal{S}$, along with composition operations $D_{\beta_1,\dots,\beta_k} \colon C_{\beta_1} \times \cdots \times C_{\beta_k} \to C_\beta$, defined whenever D is a wiring diagram and $\beta = D(\beta_1, \dots, \beta_k)$, so that with the obvious induced structure, $\coprod_\beta C_\beta$ is a circuit algebra. A similar definition can be made if/when the skeletons are taken to be directed or coloured.

Loosely speaking, a circuit algebra with skeletons is a circuit algebra in which every element T has a well-defined skeleton $\varsigma(T) \in \mathcal{S}$. Yet note that as an algebraic structure a circuit algebra with skeletons has more "spaces" than an ordinary circuit algebra, for its spaces are enumerated by skeleta and not merely by integers. The prime examples for circuit algebras with skeletons appear in the next section.

3. w-Tangles

Section Summary. In Sec. 3.1 we introduce v-tangles and w-tangles, the obvious v- and w- counterparts of the standard knot-theoretic notion of "tangles", and briefly discuss their finite type invariants and their associated spaces of "arrow diagrams", $\mathcal{A}^v(\uparrow_n)$ and $\mathcal{A}^w(\uparrow_n)$. We then construct a homomorphic expansion Z, or a "well-behaved" universal finite type invariant for w-tangles. The only algebraic tool we need to use is $\exp(a) := \sum a^n/n!$ (Sec. 3.1 is in fact a routine extension of parts of [WKO1, Section 3]). In Sec. 3.2 we show that $\mathcal{A}^w(\uparrow_n) \cong \mathcal{U}(\mathfrak{a}_n \oplus \mathfrak{tder}_n \ltimes \mathfrak{tr}_n)$, where $\mathfrak{a}_n$ is an Abelian algebra of rank n and where $\mathfrak{tder}_n$ and $\mathfrak{tr}_n$, two of the primary spaces used by Alekseev and Torossian [AT], have simple descriptions in terms of cyclic words and free Lie algebras. We also show that some functionals studied in [AT], div and j, have a natural interpretation in our language. In 3.3 we discuss a subclass of w-tangles called “special” w-tangles, and relate them by similar means to Alekseev and Torossian’s $\mathfrak{sder}_n$ and to “tree level” ordinary Vassiliev theory. Some conventions are described in Sec. 3.4 and the uniqueness of Z is studied in Sec. 3.5.

3.1. **v-Tangles and w-Tangles.** With the task of defining circuit algebras completed in Section 2.4, the definition of v-tangles and w-tangles is simple.

Definition 3.1. The ($\mathcal{S}$-graded) circuit algebra vT of v-tangles is the $\mathcal{S}$-graded directed circuit algebra generated by two generators in $C_{2,2}$ called the “positive crossing” and the “negative crossing”, modulo the R1^s, R2 and R3 moves as depicted in Figure 2 (these relations clearly make sense as circuit algebra relations between our two generators), with the obvious meaning for their skeleta. The circuit algebra wT of w-tangles is the same, except we also mod out by the OC relation of Figure 2 (note that each side in that relation involves only two generators, with the apparent third “virtual” crossing being merely a circuit algebra artifact). In fewer words, $\mathit{vT} := \mathrm{CA}\langle\ ,\ \mid\ =\ ,\ =\ ,\ =\ \rangle$, and $\mathit{wT} := \mathit{vT}/\ =\ $.

Example 3.2. An example of a w-tangle (or v-tangle, as these only differ in the relations they are subject to) is shown the left side of Figure 3. The right side of the same figure shows the skeleton of this w-tangle. Let S be a given skeleton: an element of the circuit algebra of skeleta as defined in Section 2.4. Let us denote by $\mathit{vT}(S)$ and $\mathit{wT}(S)$, respectively, the v- and w-tangles with skeleton S: that is, the pre-image of S under the skeleton map ς (see after Definition 2.11). Note that in our case the skeleton map is “forgetting topology”, in other words, forgetting the under/over information of crossings. This results in “only virtual crossings”, which is precisely what an empty circuit means. With this notation, $\mathit{wT}(\uparrow)$, the set of w-tangles whose skeleton is a single line, is exactly the set of w-knots discussed in [WKO1, Section 3]. Note also that $\mathit{wT}(\uparrow_n)$, the set of w-tangles whose skeleton is n lines,

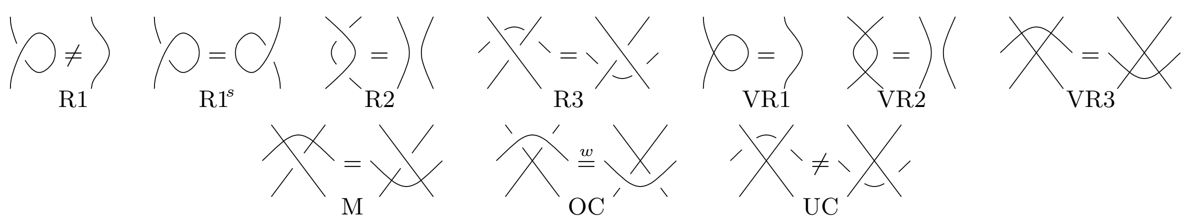


Figure 2. The relations (“Reidemeister moves”) defining v and w-tangles, and some relations that are not imposed.

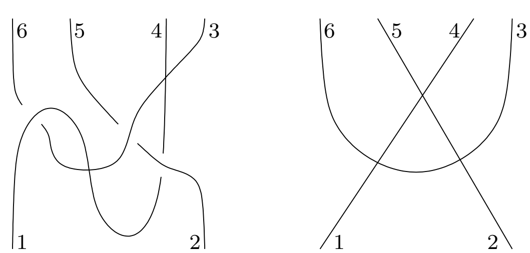


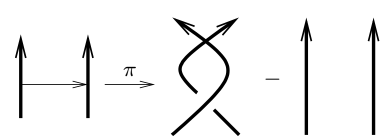
Figure 3. A w-tangle (or v-tangle) and its skeleton. Note that other than here, we usually suppress the circuit algebra leg numbering (1–6, here).

includes w-braids ([WKO1, Section 2]) but it is more general. Neither w-knots nor w-braids are circuit algebras.

Remark 3.3. One may also define v-tangles and w-tangles using the language of planar algebras, except then another generator is required (the "virtual crossing") and also a few further relations shown in Figure 2 (VR1–VR3, M), and some of the operations (non-planar wirings) become less elegant to define. In our context "virtual crossings" are automatically present (but unimportant) as part of the circuit algebra structure, and the "virtual Reidemeister moves" VR1–VR3 and M are also automatically true. This is the main advantage we gain from the language of circuit algebras: it simplifies the presentation significantly and, in our opinion, makes it more natural.

Remark 3.4. Since we do not mod out by the R1 relation, only by its weak (or "spun") version $R1^s$, it is more appropriate to call our class of v/w-tangles *framed* v/w-tangles. (Recall that framed u-tangles are characterized as the planar algebra generated by the positive and negative crossings modulo the $R1^s$, R2 and R3 relations.) However, since we are for the most part interested in studying the framed theories (cf. Comment 4.4), we will reserve the unqualified name for the framed case, and will explicitly write "unframed v/w-tangles" if we wish to mod out by R1. For a more detailed explanation of framings and R1 moves, see [WKO1, Remark 3.5].

Our next task is to study the associated graded structures $\operatorname{grad} v\mathcal{T}$ and $\operatorname{grad} w\mathcal{T}$ of $v\mathcal{T}$ and $w\mathcal{T}$. These are "arrow diagram spaces on tangle skeletons": directed analogues of the chord diagram spaces of ordinary finite type invariant theory, and even more similar to the arrow diagram spaces for braids and knots discussed in [WKO1]. Our convention for figures will be to show skeletons as thick lines with thin arrows (directed chords). Again, the language of circuit algebras makes defining these spaces exceedingly simple.



Definition 3.5. The ($\mathcal{S}$-graded) circuit algebra $\mathcal{D}^v = \mathcal{D}^w$ of arrow diagrams is the graded and $\mathcal{S}$-graded directed circuit algebra generated by a single degree 1 generator a in $C_{2,2}$ called "the arrow" as shown on the right, with the obvious meaning for its skeleton. There are morphisms $\pi\colon \mathcal{D}^v \to v\mathcal{T}$ and $\pi\colon \mathcal{D}^w \to w\mathcal{T}$ defined by mapping the arrow to an overcrossing minus a no-crossing. (On the right some virtual crossings were added to make the skeleta match). Let $\mathcal{A}^v$ be $\mathcal{D}^v/6T$, let $\mathcal{A}^w := \mathcal{A}^v/TC = \mathcal{D}^w/(\overrightarrow{4T}, TC)$, and let $\mathcal{A}^{sv} := \mathcal{A}^v/RI$ and $\mathcal{A}^{sw} := \mathcal{A}^w/RI$ as usual, with RI, $6T$, $\overrightarrow{4T}$, and TC being the relations shown in Figures 4 and 5. Note that the pair of relations $(\overrightarrow{4T}, TC)$ is equivalent to the pair $(6T, TC)$, as discussed in [WKO1, Section 2.3.1].

Proposition 3.6. *The maps π above induce surjections $\pi\colon \mathcal{A}^{sv} \to \operatorname{grad} v\mathcal{T}$ and $\pi\colon \mathcal{A}^{sw} \to \operatorname{grad} w\mathcal{T}$. Hence in the language of Definition 2.6, $\mathcal{A}^{sv}$ and $\mathcal{A}^{sw}$ are candidate associated graded structures of $v\mathcal{T}$ and $w\mathcal{T}$.*

Proof. Proving that π is well-defined amounts to checking directly that the RI and 6T or RI, $\overrightarrow{4T}$ and TC relations are in the kernel of π. (Just like in the finite type theory of virtual knots and braids.) Thanks to the circuit algebra structure, it is enough to verify the surjectivity of π in degree 1. We leave this as an exercise for the reader. ∎

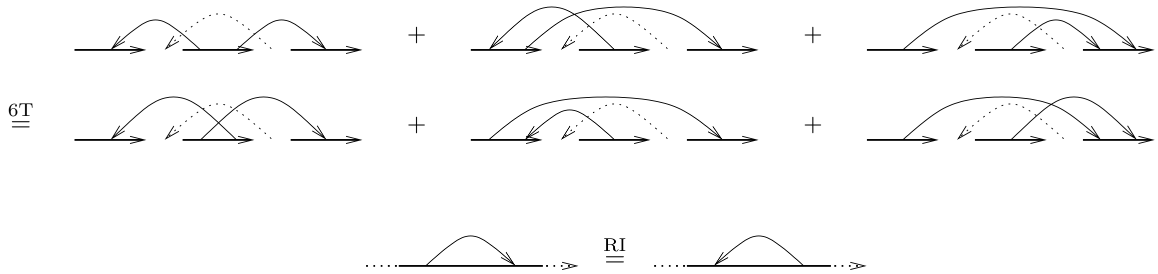


Figure 4. Relations for v-arrow diagrams on tangle skeletons. Skeleta parts that are not connected can lie on separate skeleton components; and the dotted arrow that remains in the same position means "all other arrows remain the same throughout".

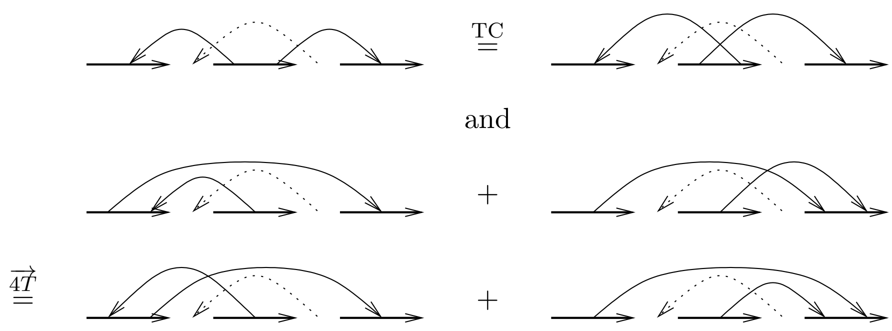


Figure 5. Relations for w-arrow diagrams on tangle skeletons.

We do not know if $\mathcal{A}^{sv}$ is indeed the associated graded of $v\mathcal{T}$ (also see [BHLR]). Yet in the w case, the picture is simple:

Theorem 3.7. *The assignment* $\overcrossing \mapsto e^a$ *(with* e^a *denoting the exponential of a single arrow from the over strand to the under strand, interpreted via its power series) extends to a well defined* $Z\colon w\mathcal{T} \to \mathcal{A}^{sw}$*. The resulting map* Z *is a homomorphic* $\mathcal{A}^{sw}$*-expansion, and in particular,* $\mathcal{A}^{sw} \cong \operatorname{grad} w\mathcal{T}$ *and* Z *is a homomorphic expansion.*

Proof. The proof is essentially the same as the proof of [WKO1, Theorem 2.15], and follows [BP, AT]. One needs to check that Z satisfies the Reidemeister moves and the OC relation. R1^s follows easily from RI, R2 is obvious, TC implies OC. For R3, let $\mathcal{A}^{sw}(\uparrow_n)$ denote the space of "arrow diagrams on n vertical strands". We need to verify that $R := e^a \in \mathcal{A}^{sw}(\uparrow_2)$ satisfies the Yang-Baxter equation

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}, \quad \text{in } \mathcal{A}^{sw}(\uparrow_3),$$

where $R^{ij} = e^{a_{ij}}$ means "place R on strands i and j". By $4T$ and TC relations, both sides of the equation can be reduced to $e^{a_{12}+a_{13}+a_{23}}$, proving the Reidemeister invariance of Z.

Z is by definition a circuit algebra homomorphism. Hence to show that Z is an $\mathcal{A}^{sw}$-expansion we only need to check the universality property in degree one, where it is very easy. The rest follows from Proposition 2.7. □

Remark 3.8. Note that the restriction of Z to w-knots and w-braids (in the sense of Example 3.2) recovers the expansions constructed in [WKO1]. Note also that the filtration and associated graded structure for w-braids fits into the general algebraic framework of Section 2

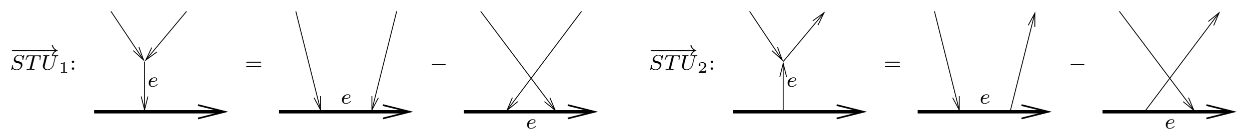


Figure 6. The $\overrightarrow{STU}$ relations for arrow diagrams, with their "central edges" marked e for easier memorization.

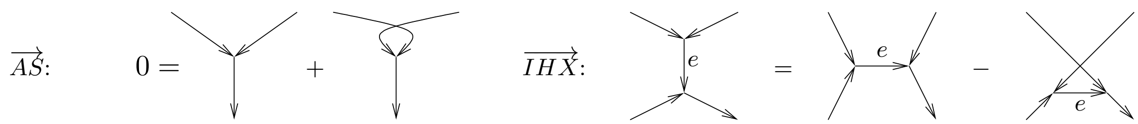


Figure 7. The $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations.

by applying the machinery to the skeleton-graded group of w-braids instead the circuit algebra of w-tangles. (The skeleton of a w-braid is the permutation it represents.) However, as w-knots do not form a finitely presented algebraic structure in the sense of Section 2, the "finite type" filtration used in [WKO1] does not arise as powers of any augmentation ideal. This captures the reason why w-knots are "the wrong objects to study", as we have mentioned at the beginning of Section 3 of [WKO1].

In a similar spirit to [WKO1, Definition 3.12], one may define a "w-Jacobi diagram" on an arbitrary skeleton:

Definition 3.9. A "w-Jacobi diagram on a tangle skeleton"[8] is a graph made of the following ingredients:

- An oriented "skeleton" consisting of long lines and circles (i.e., an oriented one-manifold). In figures we draw the skeleton lines thicker.
- Other directed edges, usually called "arrows".
- Trivalent "skeleton vertices" in which an arrow starts or ends on the skeleton line.
- Trivalent "internal vertices" in which two arrows end and one arrow begins. The internal vertices are cyclically oriented; in figures the assumed orientation is always counterclockwise unless marked otherwise. Furthermore, all trivalent vertices must be connected to the skeleton via arrows (but not necessarily following the direction of the arrows).

Note that we allow multiple and loop arrow edges, as long as trivalence and the two-in-one-out rule is respected.

Formal linear combinations of (w-Jacobi) arrow diagrams form a circuit algebra. We denote by $\mathcal{A}^{wt}$ the quotient of the circuit algebra of arrow diagrams modulo the $\overrightarrow{STU}_1$, $\overrightarrow{STU}_2$ relations of Figure 6, and the TC relation. We denote $\mathcal{A}^{wt}$ modulo the RI relation by $\mathcal{A}^{swt}$. We then have the following "bracket-rise" theorem:

Theorem 3.10. *The obvious inclusion of arrow diagrams (with no internal vertices) into w-Jacobi diagrams descends to a map $\bar{\iota} : \mathcal{A}^w \to \mathcal{A}^{wt}$, which is a circuit algebra isomorphism. Furthermore, the $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations of Figure 7 hold in $\mathcal{A}^{wt}$. Consequently, it is also true that $\mathcal{A}^{sw} \cong \mathcal{A}^{swt}$.*

[8]We usually short this to "w-Jacobi diagram", or sometimes "arrow diagram" or just "diagram".

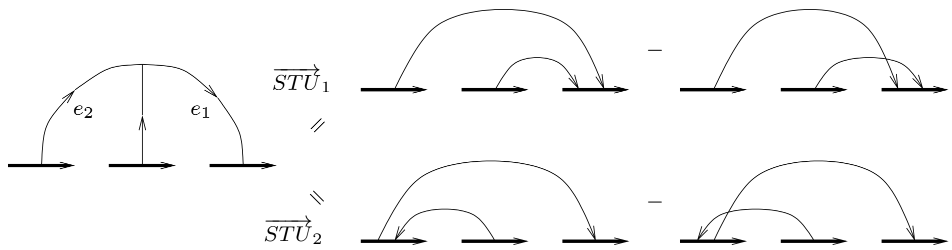


Figure 8. Applying $\overrightarrow{STU}_1$ and $\overrightarrow{STU}_2$ to the diagram on the left, we get the two sides of $\overrightarrow{4T}$.

Figure 9. A 4-wheel and the RI relation re-phrased.

Proof. In the proof of [WKO1, Theorem 3.14] we showed this for long w-knots (i.e., tangles whose skeleton is a single long line). That proof applies here verbatim, noting that it does not make use of the connectivity of the skeleton.

In short, to check that $\bar{\iota}$ is well-defined, we need to show that the $\overrightarrow{STU}$ relations imply the $\overrightarrow{4T}$ relation. This is shown in Figure 8. To show that $\bar{\iota}$ is an isomorphism, we construct an inverse $\mathcal{A}^{wt} \to \mathcal{A}^w$, which "eliminates all internal vertices" using a sequence of $\overrightarrow{STU}$ relations. Checking that this is well-defined requires some case analysis; the fact that it is an inverse to $\bar{\iota}$ is obvious. Verifying that the $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations hold in $\mathcal{A}^{wt}$ is an easy exercise. $\square$

Given the above theorem, we no longer keep the distinction between $\mathcal{A}^w$ and $\mathcal{A}^{wt}$ and between $\mathcal{A}^{sw}$ and $\mathcal{A}^{swt}$.

We recall from [WKO1] that a "k-wheel", sometimes denoted w_k, is a an arrow diagram consisting of an oriented cycle of arrows with k incoming "spokes", the tails of which rest on the skeleton. An example is shown in Figure 9. In this language, the RI relation can be rephrased using the $\overrightarrow{STU}$ relation to say that all one-wheels are 0, or $w_1 = 0$.

Remark 3.11. Note that if T is an arbitrary w tangle, then the equality on the left side of the figure below always holds, while the one on the right generally doesn't:

(2)

The arrow diagram version of this statement is that if D is an arbitrary arrow diagram in $\mathcal{A}^w$, then the left side equality in the figure below always holds (we will sometimes refer to this as the "head-invariance" of arrow diagrams), while the right side equality ("tail-invariance") generally fails.

(3)

We leave it to the reader to ascertain that Equation (2) implies Equation (3). There is also a direct proof of Equation (3) which we also leave to the reader, though see an analogous statement and proof in [BN2, Lemma 3.4]. Finally note that a restricted version of tail-invariance does hold — see Section 3.3.

3.2. $\mathcal{A}^w(\uparrow_n)$ and the Alekseev-Torossian Spaces.

Definition 3.12. Let $\mathit{vT}(\uparrow_n)$ (likewise $\mathit{wT}(\uparrow_n)$) be the set of v-tangles (w-tangles) whose skeleton is the disjoint union of n directed lines. Likewise let $\mathcal{A}^v(\uparrow_n)$ be the part of $\mathcal{A}^v$ in which the skeleton is the disjoint union of n directed lines, with similar definitions for $\mathcal{A}^w(\uparrow_n)$, $\mathcal{A}^{sv}(\uparrow_n)$, and $\mathcal{A}^{sw}(\uparrow_n)$.

Theorem 3.13. (Diagrammatic PBW Theorem.) Let $\mathcal{B}^w_n$ denote the space of uni-trivalent diagrams[9] with symmetrized ends coloured with colours in some n-element set (say $\{x_1, \dots, x_n\}$), modulo the $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations of Figure 7. Then there is an isomorphism $\mathcal{A}^w(\uparrow_n) \cong \mathcal{B}^w_n$.

Proof sketch. Readers familiar with the diagrammatic PBW theorem [BN1, Theorem 8] will note that the proof carries through almost verbatim. There is a map $\chi : \mathcal{B}^w_n \to \mathcal{A}^w(\uparrow_n)$, which sends each uni-trivalent diagram to the average of all ways of attaching their univalent ends to the skeleton of n lines, so that ends of colour x_i are attached to the strand numbered i. I.e., a diagram with k_i uni-valent vertices of colour x_i is sent to a sum of $\prod_i k_i!$ terms, divided by $\prod_i k_i!$.

The goal is to show that χ is an isomorphism by constructing an inverse for it. The image of χ are *symmetric* sums of diagrams, that is, sums of diagrams that are invariant under permuting arrow endings on the same skeleton component. One can show that in fact any arrow diagram D in $\mathcal{A}^w(\uparrow_n)$ is equivalent via $\overrightarrow{STU}$ and TC relations to a symmetric sum. The obvious candidate is its "symmetrization" $Sym(D)$: the average of all ways of permuting the arrow endings on each skeleton component of D. It is not true that each diagram is equivalent to its symmetrization (hence, the "simply delete the skeleton" map is not an inverse for χ), but it is true that $D - Sym(D)$ has fewer skeleton vertices (lower degree) than D, hence we can construct χ^{-1} inductively. The fact that this inductive procedure is well-defined requires a proof; that proof is essentially the same as the proof of the corresponding fact in [BN1, Theorem 8]. □

Both $\mathcal{A}^w(\uparrow_n)$ and $\mathcal{B}^w_n$ have a natural bi-algebra structure. In $\mathcal{A}^w(\uparrow_n)$ multiplication is given by stacking. For a diagram $D \in \mathcal{A}^w(\uparrow_n)$, the co-product $\Delta(D)$ is given by the sum of all ways of dividing D between a "left co-factor" and a "right cofactor" so that the connected components of $D - S$ are kept intact, where S is the skeleton of D. In $\mathcal{B}^w_n$ multiplication is given by disjoint union, and Δ is the sum of all ways of dividing the connected components of a diagram between two co-factors (here there is no skeleton). Note that the isomorphism χ above is a co-algebra isomorphism, but not an algebra homomorphism.

The primitives $\mathcal{P}^w_n$ of $\mathcal{B}^w_n$ are the connected diagrams (and hence the primitives of $\mathcal{A}^w(\uparrow_n)$ are the diagrams that remain connected even when the skeleton is removed). Given the "two in one out" rule for internal vertices, the diagrams in $\mathcal{P}^w_n$ can only be trees (diagrams with no cycles) or wheels (a single oriented cycle with a number of "spokes", or leaves, attached

[9] Oriented graphs with vertex degrees either 1 or 3, where trivalent vertices must have two edges incoming and one edge outgoing and are cyclically oriented.

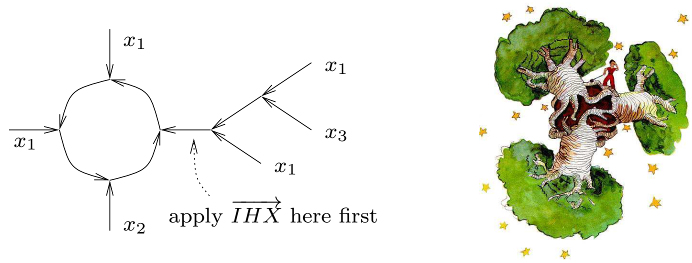


Figure 10. A wheel of trees can be reduced to a combination of wheels, and a wheel of trees with a Little Prince.

to it). "Wheels of trees" can be reduced to simple wheels by repeatedly using $\overrightarrow{IHX}$, as in Figure 10.

Thus as a vector space $\mathcal{P}_n^w$ is easy to identify. It is a direct sum $\mathcal{P}_n^w = \langle\text{trees}\rangle \oplus \langle\text{wheels}\rangle$. The wheels part is simply the graded vector space generated by all cyclic words in the letters $x_1, \dots, x_n$. Alekseev and Torossian [AT] denote the space of cyclic words by $\mathfrak{tr}_n$, and so shall we. The trees in $\mathcal{P}_n^w$ have leafs coloured $x_1, \dots, x_n$. Modulo $\overrightarrow{AS}$ and $\overrightarrow{IHX}$, they correspond to elements of the free Lie algebra $\mathfrak{lie}_n$ on the generators $x_1, \dots, x_n$. But the root of each such tree also carries a label in $\{x_1, \dots, x_n\}$, hence there are n types of such trees as separated by their roots, and so $\mathcal{P}_n^w$ is isomorphic to the direct sum $\mathfrak{tr}_n \oplus \bigoplus_{i=1}^n \mathfrak{lie}_n$.

Note that with $\mathcal{B}_n^{sw}$ and $\mathcal{P}_n^{sw}$ defined in the analogous manner (i.e., factoring out by one-wheels, as in the RI relation), we can also conclude that $\mathcal{P}_n^{sw} \cong \mathfrak{tr}_n /(\deg 1) \oplus \bigoplus_{i=1}^n \mathfrak{lie}_n$.

By the Milnor-Moore theorem [MM], $\mathcal{A}^w(\uparrow_n)$ is isomorphic to the universal enveloping algebra $\mathcal{U}(\mathcal{P}_n^w)$, with $\mathcal{P}_n^w$ identified as the subspace $\mathcal{P}^w(\uparrow_n)$ of primitives of $\mathcal{A}^w(\uparrow_n)$ using the PBW symmetrization map $\chi\colon \mathcal{B}_n^w \to \mathcal{A}^w(\uparrow_n)$. Thus in order to understand $\mathcal{A}^w(\uparrow_n)$ as an associative algebra, it is enough to understand the Lie algebra structure induced on $\mathcal{P}_n^w$ via the commutator bracket of $\mathcal{A}^w(\uparrow_n)$.

Our goal is to identify $\mathcal{P}^w(\uparrow_n)$ as the Lie algebra $\mathfrak{tr}_n \rtimes (\mathfrak{a}_n \oplus \mathfrak{tder}_n)$, which in itself is a combination of the Lie algebras $\mathfrak{a}_n$, $\mathfrak{tder}_n$ and $\mathfrak{tr}_n$ studied by Alekseev and Torossian [AT]. Here are the relevant definitions:

Definition 3.14. Let $\mathfrak{a}_n$ denote the vector space with basis $x_1, \dots, x_n$, also regarded as an Abelian Lie algebra of dimension n. As before, let $\mathfrak{lie}_n = \mathfrak{lie}(\mathfrak{a}_n)$ denote the free Lie algebra on n generators, now identified as the basis elements of $\mathfrak{a}_n$. Let $\mathfrak{der}_n = \mathfrak{der}(\mathfrak{lie}_n)$ be the (graded) Lie algebra of derivations acting on $\mathfrak{lie}_n$, and let

$$\mathfrak{tder}_n = \{D \in \mathfrak{der}_n \colon \forall i\ \exists a_i \text{ s.t. } D(x_i) = [x_i, a_i]\}$$

denote the subalgebra of "tangential derivations". A tangential derivation D is determined by the a_i's for which $D(x_i) = [x_i, a_i]$, and determines them up to the ambiguity $a_i \mapsto a_i + \alpha_i x_i$, where the α_i's are scalars. Thus as vector spaces, $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_{i=1}^n \mathfrak{lie}_n$.

Definition 3.15. Let $\mathrm{Ass}_n = \mathcal{U}(\mathfrak{lie}_n)$ be the free associative algebra "of words", and let Ass_n^+ be the degree > 0 part of Ass_n. As before, we let $\mathfrak{tr}_n = \mathrm{Ass}_n^+ /(x_{i_1}x_{i_2}\cdots x_{i_m} = x_{i_2}\cdots x_{i_m}x_{i_1})$ denote "cyclic words" or "(coloured) wheels". Ass_n, Ass_n^+, and $\mathfrak{tr}_n$ are $\mathfrak{tder}_n$-modules and there is an obvious equivariant "trace" $\mathrm{tr}\colon \mathrm{Ass}_n^+ \to \mathfrak{tr}_n$.

Proposition 3.16. *There is a split short exact sequence of Lie algebras*

$$0 \longrightarrow \mathfrak{tr}_n \xrightarrow{\iota} \mathcal{P}^w(\uparrow_n) \xrightarrow{\pi} \mathfrak{a}_n \oplus \mathfrak{tder}_n \longrightarrow 0.$$

Proof. The inclusion ι is defined the natural way: $\mathfrak{tr}_n$ is spanned by coloured "floating" wheels, and such a wheel is mapped into $\mathcal{P}^w(\uparrow_n)$ by attaching its ends to their assigned strands in arbitrary order. Note that this is well-defined: wheels have only tails, and tails commute.

As vector spaces, the statement is already proven: $\mathcal{P}^w(\uparrow_n)$ is generated by trees and wheels (with the all arrow endings fixed on n strands). When factoring out by the wheels, only trees remain. Trees have one head and many tails. All the tails commute with each other, and commuting a tail with a head on a strand costs a wheel (by $\overrightarrow{STU}$), thus in the quotient the head also commutes with the tails. Therefore, the quotient is the space of floating (coloured) trees, which we have previously identified with $\bigoplus_{i=1}^n \mathfrak{lie}_n \cong \mathfrak{a}_n \oplus \mathfrak{tder}_n$.

It remains to show that the maps ι and π are Lie algebra maps as well. For ι this is easy: the Lie algebra $\mathfrak{tr}_n$ is commutative, and is mapped to the commutative (due to TC) subalgebra of $\mathcal{P}^w(\uparrow_n)$ generated by wheels.

To show that π is a map of Lie algebras we give two proofs, first a "hands-on" one, then a "conceptual" one.

Hands-on argument. $\mathfrak{a}_n$ is the image of single arrows on one strand. These commute with everything in $\mathcal{P}^w(\uparrow_n)$, and so does $\mathfrak{a}_n$ in the direct sum $\mathfrak{a}_n \oplus \mathfrak{tder}_n$.

It remains to show that the bracket of $\mathfrak{tder}_n$ works the same way as commuting trees in $\mathcal{P}^w(\uparrow_n)$. Let D and D' be elements of $\mathfrak{tder}_n$ represented by $(a_1, \dots, a_n)$ and $(a'_1, \dots, a'_n)$, meaning that $D(x_i) = [x_i, a_i]$ and $D'(x_i) = [x_i, a'_i]$ for $i = 1, \dots, n$. Let us compute the commutator of these elements:

$$\begin{aligned}[D, D'](x_i) &= (DD' - D'D)(x_i) = D[x_i, a'_i] - D'[x_i, a_i] = \\ &= [[x_i, a_i], a'_i] + [x_i, Da'_i] - [[x_i, a'_i], a_i] - [x_i, D'a_i] = [x_i, Da'_i - D'a_i + [a_i, a'_i]].\end{aligned}$$

Now let T and T' be two trees in $\mathcal{P}^w(\uparrow_n)/\mathfrak{tr}_n$, their heads on strands i and j, respectively (i may or may not equal j). Let us denote by a_i (resp. a'_j) the element in $\mathfrak{lie}_n$ given by forming the appropriate commutator of the colours of the tails of T's (resp. T'). In $\mathfrak{tder}_n$, let $D = \pi(T)$ and $D' = \pi(T')$. D and D' are determined by $(0, \dots, a_i, \dots, 0)$, and $(0, \dots, a'_j, \dots 0)$, respectively. (In each case, the i-th or the j-th is the only non-zero component.) The commutator of these elements is given by $[D, D'](x_i) = [Da'_i - D'a_i + [a_i, a'_i], x_i]$, and $[D, D'](x_j) = [Da'_j - D'a_j + [a_j, a'_j], x_j]$. Note that unless $i = j$, $a_j = a'_i = 0$.

In $\mathcal{P}^w(\uparrow_n)/\mathfrak{tr}_n$, all tails commute, as well as a head of a tree with its own tails. Therefore, commuting two trees only incurs a cost when commuting a head of one tree over the tails of the other on the same strand, and the two heads over each other, if they are on the same strand.

If $i \neq j$, then commuting the head of T over the tails of T' by $\overrightarrow{STU}$ costs a sum of trees given by Da'_j, with heads on strand j, while moving the head of T' over the tails of T costs exactly $-D'a_i$, with heads on strand i, as needed.

If $i = j$, then everything happens on strand i, and the cost is $(Da'_i - D'a_i + [a_i, a'_i])$, where the last term happens when the two heads cross each other.

Conceptual argument. There is an action of $\mathcal{P}^w(\uparrow_n)$ on $\mathfrak{lie}_n$, as follows: introduce and extra strand on the right. An element L of $\mathfrak{lie}_n$ corresponds to a tree with its head on the extra strand. Its commutator with an element of $\mathcal{P}^w(\uparrow_n)$ (considered as an element of $\mathcal{P}^w(\uparrow_{n+1})$ by the obvious inclusion) is again a tree with head on strand $(n+1)$, defined to be the result of the action.

Since L has only tails on the first n strands, elements of $\mathfrak{tr}_n$, which also only have tails, act trivially. So do single (local) arrows on one strand ($\mathfrak{a}_n$). It remains to show that trees act as $\mathfrak{tder}_n$, and it is enough to check this on the generators of $\mathfrak{lie}_n$ (as the Leibniz rule is obviously satisfied). The generators of $\mathfrak{lie}_n$ are arrows pointing from one of the first n strands, say strand i, to strand $(n+1)$. A tree with head on strand i acts on this element, according $\overrightarrow{STU}$, by forming the commutator, which is exactly the action of $\mathfrak{tder}_n$.

To identify $\mathcal{P}^w(\uparrow_n)$ as the semidirect product $\mathfrak{tr}_n \rtimes (\mathfrak{a}_n \oplus \mathfrak{tder}_n)$, it remains to show that the short exact sequence of the Proposition splits. This is indeed the case, although not canonically. Two —of the many— splitting maps $u, l\colon \mathfrak{tder}_n \oplus \mathfrak{a}_n \to \mathcal{P}^w(\uparrow_n)$ are described as follows: $\mathfrak{tder}_n \oplus \mathfrak{a}_n$ is identified with $\bigoplus_{i=1}^n \mathfrak{lie}_n$, which in turn is identified with floating (coloured) trees. A map to $\mathcal{P}^w(\uparrow_n)$ can be given by specifying how to place the legs on their specified strands. A tree may have many tails but has only one head, and due to TC, only the positioning of the head matters. Let u (for *upper*) be the map placing the head of each tree above all its tails on the same strand, while l (for *lower*) places the head below all the tails. It is obvious that these are both Lie algebra maps and that $\pi \circ u$ and $\pi \circ l$ are both the identity of $\mathfrak{tder}_n \oplus \mathfrak{a}_n$. This makes $\mathcal{P}^w(\uparrow_n)$ a semidirect product. $\square$

Remark 3.17. Let $\mathfrak{tr}_n^s$ denote $\mathfrak{tr}_n$ mod out by its degree one part (one-wheels). Since the RI relation is in the kernel of π, there is a similar split exact sequence

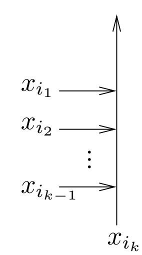
$$0 \to \mathfrak{tr}_n^s \xrightarrow{\bar{\iota}} \mathcal{P}^{sw} \xrightarrow{\bar{\pi}} \mathfrak{a}_n \oplus \mathfrak{tder}_n \,.$$

Definition 3.18. For any $D \in \mathfrak{tder}_n$, $(l-u)D$ is in the kernel of π, therefore is in the image of ι, so $\iota^{-1}(l-u)D$ makes sense. We call this element $\operatorname{div} D$.

Definition 3.19. In [AT] div is defined as follows: $\operatorname{div}(a_1, \ldots, a_n) := \sum_{k=1}^n \operatorname{tr}((\partial_k a_k) x_k)$, where ∂_k picks out the words of a sum which end in x_k and deletes their last letter x_k, and deletes all other words (the ones which do not end in x_k).

Proposition 3.20. *The div of Definition 3.18 and the div of* [AT] *are the same.*

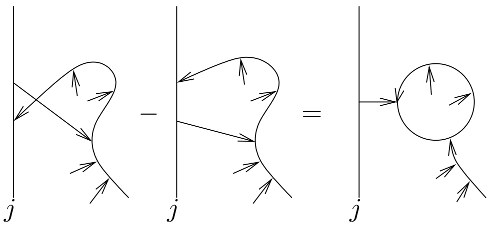
Proof. It is enough to verify the claim for the linear generators of $\mathfrak{tder}_n$, namely, elements of the form $(0, \ldots, a_j, \ldots, 0)$, where $a_j \in \mathfrak{lie}_n$ or equivalently, single (floating, coloured) trees, where the colour of the head is j. By the Jacobi identity, each a_j can be written in a form $a_j = [x_{i_1}, [x_{i_2}, [\ldots, x_{i_k}] \ldots]$. Equivalently, by $\overrightarrow{IHX}$, each tree has a standard "comb" form, as shown on the picture on the right.



For an associative word $Y = y_1 y_2 \ldots y_l \in \mathrm{Ass}_n^+$, we introduce the notation $[Y] := [y_1, [y_2, [\ldots, y_l] \ldots]$. The div of [AT] picks out the words that end in x_j, forgets the rest, and considers these as cyclic words. Therefore, by interpreting the Lie brackets as commutators, one can easily check that for a_j written as above,

$$\operatorname{div}((0, \ldots, a_j, \ldots, 0)) = \sum_{\alpha\colon i_\alpha = x_j} -x_{i_1} \ldots x_{i_{\alpha-1}} [x_{i_{\alpha+1}} \ldots x_{i_k}] x_j. \tag{4}$$

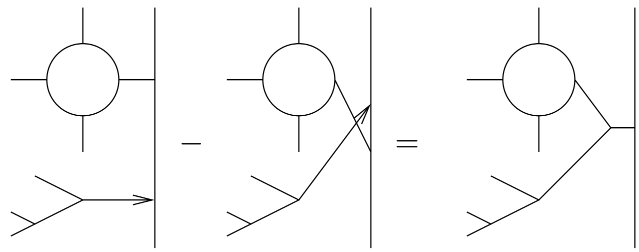
In Definition 3.18, div of a tree is the difference between attaching its head on the appropriate strand (here, strand j) below all of its tails and above. As shown in the figure on the right, moving the head across each of the tails on strand j requires an $\overrightarrow{STU}$ relation, which "costs" a wheel (of trees, which is equivalent to a sum of honest wheels). Namely, the head gets connected to the tail in question. So div of the tree represented by a_j is given by



$$\sum_{\alpha\colon x_{i_\alpha}=j} \text{“connect the head to the } \alpha \text{ leaf”}.$$

This in turn gets mapped to the formula above via the correspondence between wheels and cyclic words. □

Remark 3.21. There is an action of $\mathfrak{tder}_n$ on $\mathfrak{tr}_n$ as follows. Represent a cyclic word $w \in \mathfrak{tr}_n$ as a wheel in $\mathcal{P}^w(\uparrow_n)$ via the map ι. Given an element $D \in \mathfrak{tder}_n$, $u(D)$, as defined above, is a tree in $\mathcal{P}^w(\uparrow_n)$ whose head is above all of its tails. We define $D \cdot w := \iota^{-1}(u(D)\iota(w) - \iota(w)u(D))$. Note that $u(D)\iota(w) - \iota(w)u(D)$ is in the image of ι, i.e., a linear combination of wheels, for the following reason. The wheel $\iota(w)$ has only tails. As we commute the tree $u(D)$ across the wheel, the head of the tree is commuted across tails of the wheel on the same strand. Each time this happens the cost, by the $\overrightarrow{STU}$ relation, is a wheel with the tree attached to it, as shown on the right, which in turn (by $\overrightarrow{IHX}$ relations, as Figure 10 shows) is a sum of wheels. Once the head of the tree has been moved to the top, the tails of the tree commute up for free by TC. Note that the alternative definition, $D \cdot w := \iota^{-1}(l(D)\iota(w) - \iota(w)l(D))$ is in fact equal to the definition above.



Definition 3.22. In [AT], the group TAut_n is defined as $\exp(\mathfrak{tder}_n)$. Note that $\mathfrak{tder}_n$ is positively graded, hence it integrates to a group. Note also that TAut_n is the group of "basis-conjugating" automorphisms of $\mathfrak{lie}_n$, i.e., for $g \in \mathrm{TAut}_n$, and any x_i, $i = 1, \dots, n$ generator of $\mathfrak{lie}_n$, there exists an element $g_i \in \exp(\mathfrak{lie}_n)$ such that $g(x_i) = g_i^{-1} x_i g_i$.

The action of $\mathfrak{tder}_n$ on $\mathfrak{tr}_n$ lifts to an action of TAut_n on $\mathfrak{tr}_n$, by interpreting exponentials formally, in other words e^D acts as $\sum_{n=0}^{\infty} \frac{D^n}{n!}$. The lifted action is by conjugation: for $w \in \mathfrak{tr}_n$ and $e^D \in \mathrm{TAut}_n$, $e^D \cdot w = \iota^{-1}(e^{uD}\iota(w)e^{-uD})$.

Recall that in Section 5.1 of [AT] Alekseev and Torossian construct a map $j\colon \mathrm{TAut}_n \to \mathfrak{tr}_n$ which is characterized by two properties: the cocycle property

$$j(gh) = j(g) + g \cdot j(h), \tag{5}$$

where in the second term multiplication by g denotes the action described above; and the condition

$$\frac{d}{ds} j(\exp(sD))|_{s=0} = \operatorname{div}(D). \tag{6}$$

Now let us interpret j in our context.

Definition 3.23. The adjoint map $*\colon \mathcal{A}^w(\uparrow_n) \to \mathcal{A}^w(\uparrow_n)$ acts by "flipping over diagrams and negating arrow heads on the skeleton". In other words, for an arrow diagram D,

$$D^* := (-1)^{\#\{\text{tails on skeleton}\}} S(D),$$

where S denotes the map which switches the orientation of the skeleton strands (i.e. flips the diagram over), and multiplies by $(-1)^{\#\text{skeleton vertices}}$.

Proposition 3.24. *For $D \in \mathfrak{tder}_n$, define a map $J\colon \mathrm{TAut}_n \to \exp(\mathfrak{tr}_n)$ by $J(e^D) := e^{uD}(e^{uD})^*$. Then*

$$\exp(j(e^D)) = J(e^D).$$

Proof. Note that $(e^{uD})^* = e^{-lD}$, due to "Tails Commute" and the fact that a tree has only one head.

Let us check that $\log J$ satisfies properties (5) and (6). Namely, with $g = e^{D_1}$ and $h = e^{D_2}$, and using that $\mathfrak{tr}_n$ is commutative, we need to show that

$$J(e^{D_1}e^{D_2}) = J(e^{D_1})\big(e^{uD_1} \cdot J(e^{D_2})\big), \tag{7}$$

where $\cdot$ denotes the action of $\mathfrak{tder}_n$ on $\mathfrak{tr}_n$; and that

$$\frac{d}{ds} J(e^{sD})|_{s=0} = \operatorname{div} D. \tag{8}$$

Indeed, with $\mathrm{BCH}(D_1, D_2) = \log e^{D_1}e^{D_2}$ being the standard Baker–Campbell–Hausdorff formula,

$$\begin{aligned} J(e^{D_1}e^{D_2}) &= J(e^{\mathrm{BCH}(D_1,D_2)}) = e^{u(\mathrm{BCH}(D_1,D_2)}e^{-l(\mathrm{BCH}(D_1,D_2)} = e^{\mathrm{BCH}(uD_1,uD_2)}e^{-\mathrm{BCH}(lD_1,lD_2)} \\ &= e^{uD_1}e^{uD_2}e^{-lD_2}e^{-lD_1} = e^{uD_1}(e^{uD_2}e^{-lD_2})e^{-uD_1}e^{uD_1}e^{lD_1} = (e^{uD_1} \cdot J(D_2))J(D_1), \end{aligned}$$

as needed.

As for condition (6), a direct computation of the derivative yields

$$\frac{d}{ds} J(e^{sD})|_{s=0} = uD - lD = \operatorname{div} D,$$

as desired. □

3.3. **The Relationship with u-Tangles.** Let $u\!T$ be the planar algebra of classical, or "*u*sual" tangles. There is a map $a\colon u\!T \to w\!T$ of u-tangles into w-tangles: algebraically, it is defined in the obvious way on the planar algebra generators of $u\!T$. (It can also be interpreted topologically as Satoh's tubing map, see [WKO1, Section 3.1.1], where a u-tangle is a tangle drawn on a sphere. However, it is only conjectured that the circuit algebra presented here is a Reidemeister theory for "tangled ribbon tubes in $\mathbb{R}^4$".) The map a induces a corresponding map $\alpha\colon \mathcal{A}^u \to \mathcal{A}^{sw}$, which maps an ordinary Jacobi diagram (i.e., unoriented chords with internal trivalent vertices modulo the usual AS, IHX and STU relations) to the sum of all possible orientations of its chords (many of which are zero in $\mathcal{A}^{sw}$ due to the "two in one out" rule).

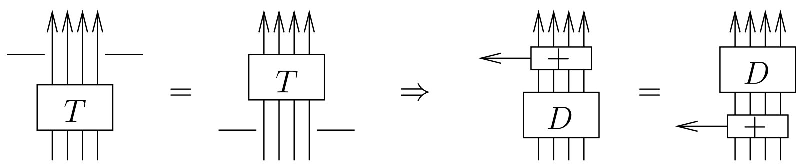
$$\begin{tikzcd} u\!T \arrow[r, dotted, "Z^u"] \arrow[d, "a"] & \mathcal{A}^u \arrow[d, "\alpha"] \\ w\!T \arrow[r, "Z^w"] & \mathcal{A}^{sw} \end{tikzcd}$$

It is tempting to ask whether the square on the left commutes. Unfortunately, this question hardly makes sense, as there is no canonical choice for the dotted line in it. Similarly to the braid case of [WKO1, Section 2.5.5], the definition of the homomorphic expansion (Kontsevich integral) for u-tangles typically depends on various choices of "parenthesizations". Choosing parenthesizations, this square becomes commutative up to some fixed corrections. The details are in Proposition 4.18.

Yet already at this point we can recover something from the existence of the map $a\colon uT \to wT$, namely an interpretation of the Alekseev-Torossian [AT] space of special derivations,

$$\mathfrak{sder}_n := \{D \in \mathfrak{tder}_n\colon D(\sum_{i=1}^{n} x_i) = 0\}.$$

Recall from Remark 3.11 that in general it is not possible to slide a strand under an arbitrary w-tangle. However, it is possible to slide strands freely under tangles *in the image of* a, and thus by reasoning similar to the reasoning in Remark 3.11, diagrams D in the image of α respect "tail-invariance":



(9)

Let $\mathcal{P}^u(\uparrow_n)$ denote the primitives of $\mathcal{A}^u(\uparrow_n)$, that is, Jacobi diagrams that remain connected when the skeleton is removed. Remember that $\mathcal{P}^w(\uparrow_n)$ stands for the primitives of $\mathcal{A}^w(\uparrow_n)$. Equation (9) readily implies that the image of the composition

$$\mathcal{P}^u(\uparrow_n) \xrightarrow{\alpha} \mathcal{P}^w(\uparrow_n) \xrightarrow{\pi} \mathfrak{a}_n \oplus \mathfrak{tder}_n$$

is contained in $\mathfrak{a}_n \oplus \mathfrak{sder}_n$. Even better is true.

Theorem 3.25. *The image of $\pi\alpha$ is precisely $\mathfrak{a}_n \oplus \mathfrak{sder}_n$.*

This theorem was first proven by Drinfel'd (Lemma after Proposition 6.1 in [Dr3]), but the proof we give here is due to Levine [Lev].

Proof. Let $\mathfrak{lie}_n^d$ denote the degree d piece of $\mathfrak{lie}_n$. Let V_n be the vector space with basis $x_1, x_2, \ldots, x_n$. Note that

$$V_n \otimes \mathfrak{lie}_n^d \cong \bigoplus_{i=1}^{n} \mathfrak{lie}_n^d \cong (\mathfrak{tder}_n \oplus \mathfrak{a}_n)^d,$$

where $\mathfrak{tder}_n$ is graded by the number of tails of a tree, and $\mathfrak{a}_n$ is contained in degree 1.

The bracket defines a map $\beta\colon V_n \otimes \mathfrak{lie}_n^d \to \mathfrak{lie}_n^{d+1}$: for $a_i \in \mathfrak{lie}_n^d$ where $i = 1, \ldots, n$, the "tree" $D = (a_1, a_2, \ldots, a_n) \in (\mathfrak{tder}_n \oplus \mathfrak{a}_n)^d$ is mapped to

$$\beta(D) = \sum_{i=1}^{n} [x_i, a_i] = D\left(\sum_{i=1}^{n} x_i\right),$$

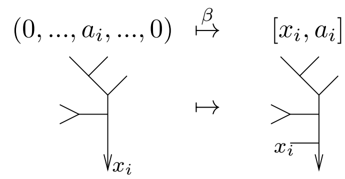
where the first equality is by the definition of tensor product and the bracket, and the second is by the definition of the action of $\mathfrak{tder}_n$ on $\mathfrak{lie}_n$.

Since $\mathfrak{a}_n$ is contained in degree 1, by definition $\mathfrak{sder}_n^d = (\ker\beta)^d$ for $d \geq 2$. In degree 1, $\mathfrak{a}_n$ is obviously in the kernel, hence $(\ker\beta)^1 = \mathfrak{a}_n \oplus \mathfrak{sder}_n^1$. So overall, $\ker\beta = \mathfrak{a}_n \oplus \mathfrak{sder}_n$.

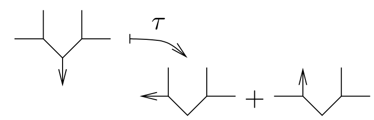
We want to study the image of the map $\mathcal{P}^u(\uparrow^n) \xrightarrow{\pi\alpha} \mathfrak{a}_n \oplus \mathfrak{tder}_n$. Under α, all connected Jacobi diagrams that are not trees or wheels go to zero, and under π so do all wheels.

Furthermore, π maps trees that live on n strands to "floating" trees with univalent vertices coloured by the strand they used to end on. So for determining the image, we may replace $\mathcal{P}^u(\uparrow^n)$ by the space $\mathcal{T}_n$ of connected *un*oriented "floating trees" (uni-trivalent graphs), the ends (univalent vertices) of which are coloured by the $\{x_i\}_{i=1,..,n}$. We denote the degree d piece of $\mathcal{T}_n$, i.e., the space of trees with $d+1$ ends, by $\mathcal{T}_n^d$. Abusing notation, we shall denote the map induced by $\pi\alpha$ on $\mathcal{T}_n$ by $\alpha\colon \mathcal{T}_n \to \mathfrak{a}_n \oplus \mathfrak{tder}_n$. Since choosing a "head" determines the entire orientation of a tree by the two-in-one-out rule, α maps a tree in $\mathcal{T}_n^d$ to the sum of $d+1$ ways of choosing one of the ends to be the "head".

We want to show that $\ker\beta = \operatorname{im}\alpha$. This is equivalent to saying that $\bar\beta$ is injective, where $\bar\beta\colon V_n \otimes \mathfrak{lie}_n \,/\, \operatorname{im}\alpha \to \mathfrak{lie}_n$ is map induced by β on the quotient by $\operatorname{im}\alpha$.

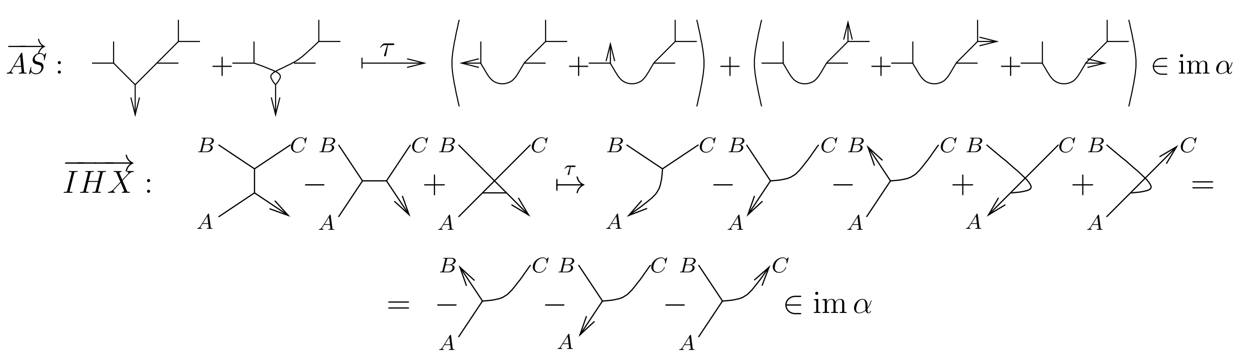


The degree d piece of $V_n \otimes \mathfrak{lie}_n$, in the pictorial description, is generated by floating trees with d tails and one head, all coloured by x_i, $i = 1,\dots,n$. This is mapped to $\mathfrak{lie}_n^{d+1}$, which is isomorphic to the space of floating trees with $d+1$ tails and one head, where only the tails are coloured by the x_i. The map β acts as shown on the picture on the right.



We show that $\bar\beta$ is injective by exhibiting a map $\tau\colon\ \mathfrak{lie}_n^{d+1} \to V_n \otimes \mathfrak{lie}_n^d \,/\, \operatorname{im}\alpha$ so that $\tau\bar\beta = I$. τ is defined as follows: given a tree with one head and $d+1$ tails τ acts by deleting the head and the arc connecting it to the rest of the tree and summing over all ways of choosing a new head from one of the tails on the left half of the tree relative to the original placement of the head (see the picture on the right). As long as we show that τ is well-defined, it follows from the definition and the pictorial description of β that $\tau\bar\beta = I$.

For well-definedness we need to check that the images of $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations under τ are in the image of α. This we do in the picture below. In both cases it is enough to check the case when the "head" of the relation is the head of the tree itself, as otherwise an $\overrightarrow{AS}$ or $\overrightarrow{IHX}$ relation in the domain is mapped to an $\overrightarrow{AS}$ or $\overrightarrow{IHX}$ relation, thus zero, in the image.



In the $\overrightarrow{IHX}$ picture, in higher degrees A, B and C may denote an entire tree. In this case, the arrow at A (for example) means the sum of all head choices from the tree A. $\square$

Comment 3.26. In view of the relation between the right half of Equation (9) and the special derivations $\mathfrak{sder}$, it makes sense to call w-tangles that satisfy the condition in the left half of Equation (9) "special". The a images of u-tangles are thus special. We do not know if the global version of Theorem 3.25 holds true. Namely, we do not know whether every special w-tangle is the a-image of a u-tangle.

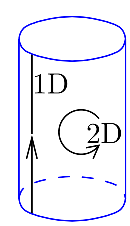
Figure 11. A virtual crossing corresponds to non-interacting tubes, while a crossing means that the tube corresponding to the under strand "goes through" the tube corresponding to the over strand.

3.4. **The local topology of w-tangles.** So far throughout this section we have presented w-tangles as a Reidemeister theory: a circuit algebra given by generators and relations. There is a topological intuition behind this definition: we can interpret the strings of a w-tangle diagram as oriented tubes in $\mathbb{R}^4$, as shown in Figure 11. Each tube has a 3-dimensional "filling", and each crossings represents a ribbon intersection between the tubes where the one corresponding to the under-strand intersects the filling of the over-strand. (For an explanation of ribbon intersections see [WKO1, Section 2.2.2].) In Figure 11 we use the drawing conventions of [CS]: we draw surfaces as if projected from $\mathbb{R}^4$ to $\mathbb{R}^3$, and cut them open when they are "hidden" by something with a higher 4-th coordinate.

Note that w-braids can also be thought of in terms of flying rings, with "time" being the fourth dimension; this is equivalent to the tube interpretation in the obvious way. In this language a crossing represents a ring (the under strand), flying through another (the over strand). This is described in detail in [WKO1, Section 2.2.1].

The assignment of tangled ribbon tubes in $\mathbb{R}^4$ to w-tangles is well-defined (the Reidemeister and OC relations are satisfied), and after Satoh [Sa] we call it the tubing map and denote it by $\delta\colon \{\text{w-tangles}\} \to \{\text{Ribbon tubes in } \mathbb{R}^4\}$. It is natural to expect that δ is an isomorphism, and indeed it is a surjection. However, the injectivity of δ remains unproven even for long w-knots. Nonetheless, ribbon tubes in $\mathbb{R}^4$ will serve as the topological motivation and local topological interpretation behind the circuit algebras presented in this paper. In [WKO1, Section 3.1.1] we present a topological construction for δ. We will mention that construction occasionally in this paper, but only for motivational purposes.

We observe that the ribbon tubes in the image of δ are endowed with two orientations, we will call these the 1- and 2-dimensional orientations. The one dimensional orientation is the direction of the tube as a "strand" of the tangle. In other words, each tube has a "core"[10]: a distinguished line along the tube, which is oriented as a 1-dimensional manifold. Furthermore, the tube as a 2-dimensional surface is oriented as given by δ. An example is shown on the right.



Next we wish to understand the topological meaning of crossing signs. Recall that a tube in $\mathbb{R}^4$ has a "filling": a solid (3-dimensional) cylinder embedded in $\mathbb{R}^4$, with boundary the tube, and the 2D orientation of the tube induces an orientation of its filling as a 3-dimensional manifold. At a (non-virtual) crossing the core of one tube intersects the filling of another transversely. Due to the complementary dimensions, the intersection is a single point, and the 1D orientation of the core along with the 3D orientation of the filling it passes through determines an orientation of the ambient space. We say that the crossing is positive

[10] The core of Lord Voldemort's wand was made of a phoenix feather.

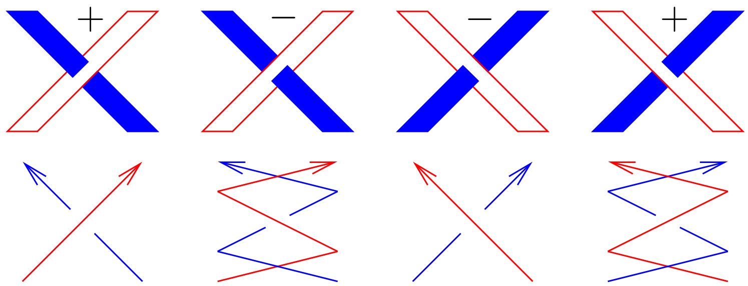
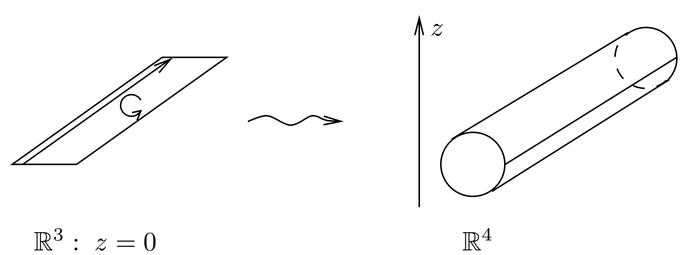


Figure 12. Crossings and crossing signs in band notation.

if this agrees with the standard orientation of $\mathbb{R}^4$, and negative otherwise. Hence, there are four types of crossings, given by whether the core of tube A intersects the filling of B or vice versa, and two possible signs in each case. In the flying ring interpretation, the 1D orientation of the tube is the direction of the flow of time. The 2D and 1D orientations of the tube together induce an orientation of the flying ring which is a cross-section of the tube at each moment. Hence, saying "below" and "above" the ring makes sense, and there are four types of crossings: ring A flies through ring B from below or from above; and ring B flies through ring A from below or from above (cf. [WKO1, Exercise 2.7]). A crossing is positive if the inner ring comes from below, and negative otherwise.



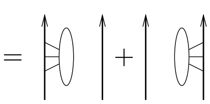
We take the opportunity here to introduce another notation, to be called the "band notation", which is more suggestive of the 4D topology than the strand notation we have been using so far. We represent a tube in $\mathbb{R}^4$ by a picture of an oriented band in $\mathbb{R}^3$. By "oriented band" we mean that it has two orientations: a 1D direction (for example an orientation of one of the edges), and a 2D orientation as a surface. To interpret the 3D picture of a band as an tube in $\mathbb{R}^4$, we add an extra coordinate. Let us refer to the $\mathbb{R}^3$ coordinates as x, y and t, and to the extra coordinate as z. Think of $\mathbb{R}^3$ as being embedded in $\mathbb{R}^4$ as the hyperplane $z=0$, and think of the band as being made of a thin double membrane. Push the membrane up and down in the z direction at each point as far as the distance of that point from the boundary of the band, as shown on the right. Furthermore, keep the 2D orientation of the top membrane (the one being pushed up), but reverse it on the bottom. This produces an oriented tube embedded in $\mathbb{R}^4$.

In band notation, the four possible crossings appear as in Figure 12, where underneath each crossing we indicate the corresponding strand picture. The signs for each type of crossing are also shown. Note that the sign of a crossing depends on the 2D orientation of the over-strand, as well as the 1D direction of the under-strand. Hence, switching only the direction (1D orientation) of a strand changes the sign of the crossing if and only if the strand involved is the under strand. However, fully changing the orientation (both 1D and 2D) always switches the sign of the crossing. Note that switching the strand direction in the strand notation corresponds to the complete (both 1D and 2D) orientation switch.

3.5. **Good properties and uniqueness of the homomorphic expansion.** In much the same way as in the case of braids [WKO1, Section 2.5.1], Z has a number of good properties with respect to various tangle operations: it is group-like[11]; it commutes with adding an inert strand (note that this is a circuit algebra operation, hence it doesn't add anything beyond homomorphicity); and it commutes with deleting a strand and with strand orientation reversals. All but the last of these were explained in the context of braids and the explanations still hold. Orientation reversal $S_k\colon \mathit{wT} \to \mathit{wT}$ is the operation which reverses the orientation of the k-th component. Note that in the world of topology (via Satoh's tubing map) this means reversing both the 1D and the 2D orientations. The induced diagrammatic operation $S_k\colon \mathcal{A}^w(T) \to \mathcal{A}^w(S_k(T))$, where T denotes the skeleton of a given w-tangle, acts by multiplying each arrow diagram by (-1) raised to the power the number of arrow endings (both heads and tails) on the k-th strand, as well as reversing the strand orientation. Saying that "Z commutes with S_k" means that the appropriate square commutes.

The following theorem asserts that a well-behaved homomorphic expansion of w-tangles is unique:

Theorem 3.27. *The only homomorphic expansion satisfying the good properties described above is the Z defined in Section 3.1.*

$\rho =$ 

Proof. We first prove the following claim: Assume, by contradiction, that Z' is a different homomorphic expansion of w-tangles with the good properties described above. Let $R' = Z'(\nwarrow\!\!\!\!\nearrow)$ and $R = Z(\nwarrow\!\!\!\!\nearrow)$, and denote by ρ the lowest degree homogeneous non-vanishing term of $R' - R$. (Note that R' determines Z', so if $Z' \neq Z$, then $R' \neq R$.) Suppose ρ is of degree k. Then we claim that $\rho = \alpha_1 w_k^1 + \alpha_2 w_k^2$ is a linear combination of w_k^1 and w_k^2, where w_k^i denotes a k-wheel living on strand i, as shown on the right.

Before proving the claim, note that it leads to a contradiction. Let d_i denote the operation "delete strand i". Then up to degree k, we have $d_1(R') = \alpha_2 w_k^1$ and $d_2(R') = \alpha_1 w_k^2$, but Z' is compatible with strand deletions, so $\alpha_1 = \alpha_2 = 0$. Hence Z is unique, as stated.

On to the proof of the claim, note that Z' being an expansion determines the degree 1 term of R' (namely, the single arrow a^{12} from strand 1 to strand 2, with coefficient 1). So we can assume that $k \geq 2$. Note also that since both R' and R are group-like, ρ is primitive. Hence ρ is a linear combination of connected diagrams, namely trees and wheels.

Both R and R' satisfy the Reidemeister 3 relation:

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}, \qquad R'^{12}R'^{13}R'^{23} = R'^{23}R'^{13}R'^{12}$$

where the superscripts denote the strands on which R is placed (compare with the proof of Theorem 3.7). We focus our attention on the degree $k+1$ part of the equation for R', and use that up to degree $k+1$. We can write $R' = R + \rho + \mu$, where μ denotes the degree $k+1$ homogeneous part of $R' - R$. Thus, up to degree $k+1$, we have

$$(R^{12}+\rho^{12}+\mu^{12})(R^{13}+\rho^{13}+\mu^{13})(R^{23}+\rho^{23}+\mu^{23}) = (R^{23}+\rho^{23}+\mu^{23})(R^{13}+\rho^{13}+\mu^{13})(R^{12}+\rho^{12}+\mu^{12}).$$

The homogeneous degree $k+1$ part of this equation is a sum of some terms which contain ρ and some which don't. The diligent reader can check that those which don't involve ρ cancel on both sides, either due to the fact that R satisfies the Reidemeister 3 relation, or

[11] In practice this simply means that the value of the crossing is an exponential.

by simple degree counting. Rearranging all the terms which do involve ρ to the left side, we get the following equation, where a^{ij} denotes an arrow pointing from strand i to strand j:

$$[a^{12},\rho^{13}]+[\rho^{12},a^{13}]+[a^{12},\rho^{23}]+[\rho^{12},a^{23}]+[a^{13},\rho^{23}]+[\rho^{13},a^{23}]=0. \tag{10}$$

The third and fifth terms sum to $[a^{12}+a^{13},\rho^{23}]$, which is zero due to the "head-invariance" of diagrams, as in Remark 3.11.

We treat the tree and wheel components of ρ separately. Let us first assume that ρ is a linear combination of trees. Recall that the space of trees on two strands is isomorphic to $\mathfrak{lie}_2 \oplus \mathfrak{lie}_2$, the first component given by trees whose head is on the first strand, and the second component by trees with their head on the second strand. Let $\rho=\rho_1+\rho_2$, where ρ_i is the projection to the i-th component for $i=1,2$.

Note that due to TC, we have $[a^{12},\rho_2^{13}]=[\rho_2^{12},a^{13}]=[\rho_1^{12},a^{23}]=0$. So Equation (10) reduces to

$$[a^{12},\rho_1^{13}]+[\rho_1^{12},a^{13}]+[\rho_2^{12},a^{23}]+[\rho_1^{13},a^{23}]+[\rho_2^{13},a^{23}]=0$$

The left side of this equation lives in $\bigoplus_{i=1}^{3}\mathfrak{lie}_3$. Notice that only the first term lies in the second direct sum component, while the second, third and last terms live in the third one, and the fourth term lives in the first. This in particular means that the first term is itself zero. By $\overrightarrow{STU}$, this implies

$$0=[a^{12},\rho_1^{13}]=-[\rho_1,x_1]_2^{13},$$

where $[\rho_1,x_1]_2^{13}$ means the tree defined by the element $[\rho_1,x_1]\in\mathfrak{lie}_2$, with its tails on strands 1 and 3, and head on strand 2. Hence, $[\rho_1,x_1]=0$, so ρ_1 is a multiple of x_1. The tree given by $\rho_1=x_1$ is a degree 1 element, a possibility we have eliminated, so $\rho_1=0$.

Equation (10) is now reduced to

$$[\rho_2^{12},a^{23}]+[\rho_2^{13},a^{23}]=0.$$

Both terms are words in $\mathfrak{lie}_3$, but notice that the first term does not involve the letter x_3. This means that if the second term involves x_3 at all, i.e., if ρ_2 has tails on the second strand, then both terms have to be zero individually. Assuming this and looking at the first term, ρ_2^{12} is a Lie word in x_1 and x_2, which does involve x_2 by assumption. We have $[\rho_2^{12},a^{23}]=[x_2,\rho_2^{12}]=0$, which implies ρ_2^{12} is a multiple of x_2, in other words, ρ is a single arrow on the second strand. This is ruled out by the assumption that $k\geq 2$.

On the other hand if the second term does not involve x_3 at all, then ρ_2 has no tails on the second strand, hence it is of degree 1, but again $k\geq 2$. We have proven that the "tree part" of ρ is zero.

So ρ is a linear combination of wheels. Wheels have only tails, so the first, second and fourth terms of (10) are zero due to the tails commute relation. What remains is $[\rho^{13},a^{23}]=0$. We assert that this is true if and only if each linear component of ρ has all of its tails on one strand.

To prove this, recall each wheel of ρ^{13} represents a cyclic word in letters x_1 and x_3. The map $r\colon \rho^{13}\mapsto[\rho^{13},a^{23}]$ is a map $\mathfrak{tr}_2\to\mathfrak{tr}_3$, which sends each cyclic word in letters x_1 and x_3 to the sum of all ways of substituting $[x_2,x_3]$ for one of the x_3's in the word. Note that if we expand the commutators, then all terms that have x_2 between two x_3's cancel. Hence all remaining terms will be cyclic words in x_1 and x_3 with a single occurrence of x_2 in between an x_1 and an x_3.

We construct an almost-inverse r' to r: for a cyclic word w in $\mathfrak{tr}_3$ with one occurrence of x_2, let r' be the map that deletes x_2 from w and maps it to the resulting word in $\mathfrak{tr}_2$ if x_2 is followed by x_3 in w, and maps it to 0 otherwise. On the rest of $\mathfrak{tr}_3$ the map r' may be defined to be 0.

The composition $r'r$ takes a cyclic word in x_1 and x_3 to itself multiplied by the number of times a letter x_3 follows a letter x_1 in it. The kernel of this map can consist only of cyclic words that do not contain the sub-word x_3x_1, namely, these are the words of the form x_3^k or x_1^k. Such words are indeed in the kernel of r, so these make up exactly the kernel of r. This is exactly what needed to be proven: all wheels in ρ have all their tails on one strand.

This concludes the proof of the claim, and the proof of the theorem. □

4. w-Tangled Foams

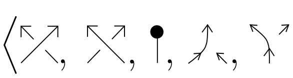
Section Summary. In this section we add "foam vertices" to w-tangles (and a few lesser things as well) and ask the same questions we asked before; primarily, "is there a homomorphic expansion?". As we shall see, in the current context this question is equivalent to the Alekseev-Torossian [AT] version of the Kashiwara-Vergne [KV] problem and explains the relationship between these topics and Drinfel'd's theory of associators.

4.1. The Circuit Algebra of w-Tangled Foams.

In the same manner as we did for tangles, we will present the circuit algebra of w-tangled foams via its Reidemeister-style diagrammatic description accompanied by a local topological interpretation.

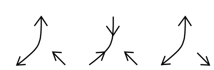
Definition 4.1. Let *wTF^o* (where o stands for "orientable", to be explained in Section 4.5) be the algebraic structure

$$wTF^o = \text{CA}\left\langle \quad \right. $$



$$\left. \middle|\, \begin{array}{c}\text{w-relations as in}\\ \text{Section 4.1.2}\end{array} \,\middle|\, \begin{array}{c}\text{w-operations as}\\ \text{in Section 4.1.3}\end{array} \right\rangle .$$

Hence wTF^o is the circuit algebra generated by the generators listed above and described below, modulo the relations described in Section 4.1.2, and augmented with several "auxiliary operations", which are a part of the algebraic structure of wTF^o but are not a part of its structure as a circuit algebra, as described in Section 4.1.3. To be more precise, wTF^o is skeleton-graded where the circuit algebra of skeleta $\mathcal{S}^o$ is a version of the $\mathcal{S}$ introduced in Section 2.4, but with vertices and caps included (as opposed to only empty circuits).

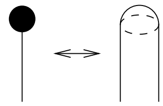


To be completely precise, we have to admit that wTF^o as a circuit algebra has more generators than shown above. The last two generators are "foam vertices", as will be explained shortly, and exist in all possible orientations of the three strands. Some examples are shown on the right. However, in Section 4.1.3 we will describe the operation "orientation switch" which allows switching the orientation of any given strand. In the algebraic structure which includes this extra operation in addition to the circuit algebra structure, the generators of the definition above are enough.

4.1.1. *The generators of wTF^o.*

There is topological meaning to each of the generators of wTF^o: they each stand for a certain local feature of framed knotted ribbon tubes in $\mathbb{R}^4$. As in Section 3.4, the tubes are oriented as 2-dimensional surfaces, and also have a distinguished core with a 1-dimensional orientation (direction).

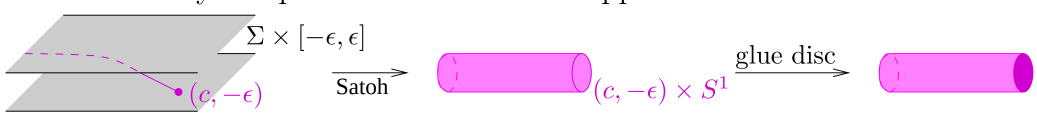
The crossings are as explained in Section 3.4: the under-strand denotes the ring flying through, or the "thin" tube. Recall that there really are four kinds of crossings, but in the circuit algebra the two not shown are obtained from the two that are shown by adding virtual crossings (see Figures 11 and 12).

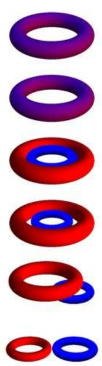


A bulleted end denotes a cap on the tube, or a flying ring that shrinks to a point, as in the figure on the right. For further motivation, in terms of the topological construction of Satoh's tubing map [WKO1, Section 3.1.1], the cap means that "the string is attached to the bottom of the thickened surface", as shown in the figure below. We Recall that the tubing map is the composition

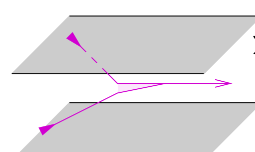
$$\gamma \times S^1 \hookrightarrow \Sigma \times [-\epsilon, \epsilon] \hookrightarrow \mathbb{R}^4.$$

Here γ is a trivalent tangle "drawn on the virtual surface Σ", with caps ending on $\Sigma \times \{-\epsilon\}$. The first embedding above is the product of this "drawing" with an S^1, while the second arises from the unit normal bundle of Σ in $\mathbb{R}^4$. For each cap $(c, -\epsilon)$ the tube resulting from Satoh's map has a boundary component $\partial_c = (c, -\epsilon) \times S^1$. Follow the tubing map by gluing a disc to this boundary component to obtain the capped tube mentioned above.





The last two generators denote singular "foam vertices". As the notation suggests, a vertex can be thought of as "half of a crossing". To make this precise using the flying rings interpretation, the first singular vertex represents the movie shown on the left: the ring corresponding to the right strand approaches the ring represented by the left strand from below, flies inside it, and then the two rings fuse (as opposed to a crossing where the ring coming from the right would continue to fly out to above and to the left of the other one). The second vertex is the movie where a ring splits radially into a smaller and a larger ring, and the small one flies out to the right and below the big one.



The vertices can also be interpreted topologically via a natural extension of Satoh's tubing map. For the first vertex, imagine the broken right strand approaching the continuous left strand directly from below in a thickened surface, as shown.

The reader might object that there really are four types of vertices (as there are four types of crossings), and each of these can be viewed as a "fuse" or a "split" depending on the strand directions, as shown in Figure 13. However, looking at the fuse vertices for example, observe that the last two of these can be obtained from the first two by composing with virtual crossings, which always exist in a circuit algebra.

The sign of a vertex can be defined the same way as the sign of a crossing (see Section 3.4). We will sometimes refer to the first generator vertex as "the positive vertex" and to the second one as "the negative vertex". We use the band notation for vertices the same way we do for crossings: the fully coloured band stands for the thin (inner) ring.

4.1.2. *The relations of wTF^o.* In addition to the usual $R1^s$, R2, R3, and OC moves of Figure 2, we more relations are added to describe the behaviour of the additional features.

Comment 4.2. As before, the relations have local topological explanations, and we conjecture that they provide a Reidemeister theory for "w-tangled foams", that is, knotted ribbon tubes

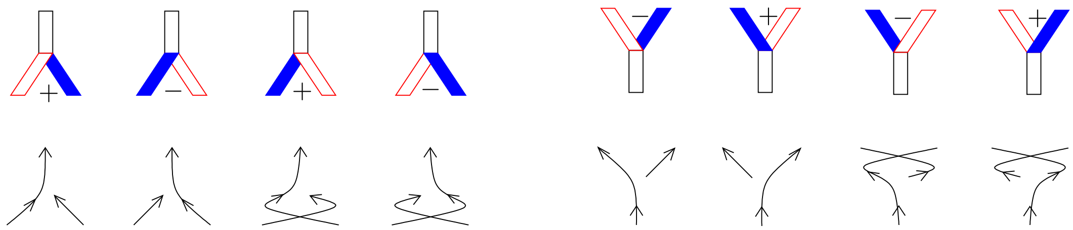
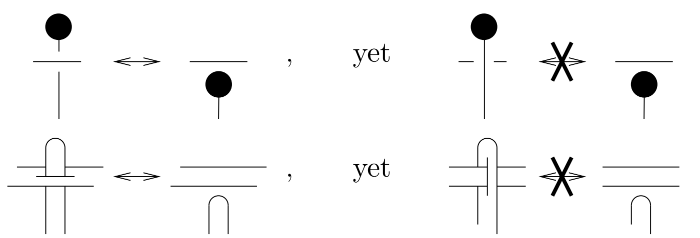


Figure 13. Vertex types in wTF^o.

with foam vertices in $\mathbb{R}^4$. In this section we list the relations along with the topological reasoning behind them. However, for any rigorous purposes below, wTF^o is studied as a circuit algebra given by the declared generators and relations, with topology serving only as intuition.

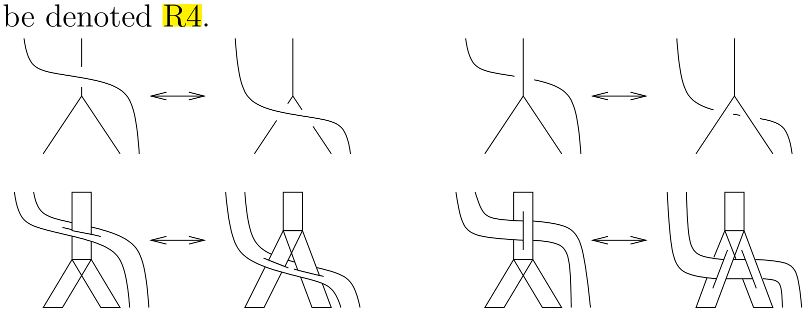
Recall that topologically, a cap represents a capped tube or equivalently, flying ring shrinking to a point. Hence, a cap on the thin (or under) strand can be "pulled out" from a crossing, but the same is not true for a cap on the thick (or over) strand, as shown below. This is the case for any orientation of the strands. We denote this relation by CP, for Cap Pull-out.

CP :



The Reidemeister 4 relations assert that a strand can be moved under or over a crossing, as shown in the picture below. The ambiguously drawn vertices in the picture denote a vertex of any kind (as described in Section 4.1.1), and the strands can be oriented arbitrarily. The local topological (tube or flying ring) interpretations can be read from the pictures below. These relations will be denoted R4.

$R4$:



4.1.3. *The auxiliary operations of* wTF^o. The circuit algebra wTF^o is equipped with several extra operations.

The first of these is the familiar orientation switch. We will, as mentioned in Section 3.4, distinguish between switching both the 2D and 1D orientations, or just the strand (1D) direction.

Topologically *orientation switch*, denoted S_e, is the switch of both orientations of the strand e. Diagrammatically (and this is the definition) S_e is the operation which reverses the orientation of a strand in a wTF^o diagram. The reader can check that when applying

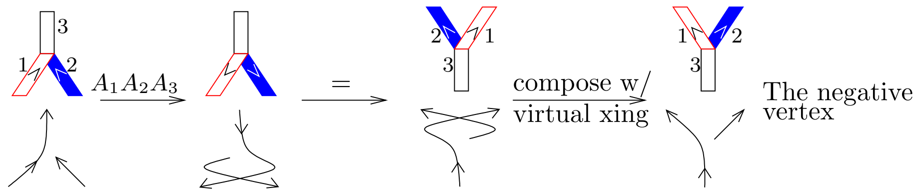
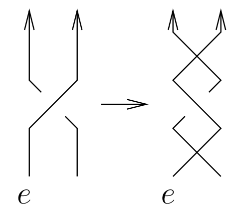


Figure 14. Switching strand orientations at a vertex. The adjoint operation only switches the tube direction, hence in the *band picture* only the arrows change. To express this vertex in terms of the negative generating vertex in strand notation, we use a virtual crossing (see Figure 13).

Satoh's tubing map, this amounts to reversing both the direction and the 2D orientation of the tube arising from the strand.



The operation which, in topology world, reverses a tube's direction but not its 2D orientation is called *"adjoint"*, and denoted by A_e. This is slightly more intricate to define rigorously in terms of diagrams. In addition to reversing the direction of the strand e of the wTF^o diagram, A_e also locally changes each crossing of e *over* another strand by adding two virtual crossings, as shown on the right. We recommend for the reader to convince themselves that this indeed represents a direction switch in topology after reading Section 4.5.

Remark 4.3. As an example, let us observe how the negative generator vertex can be obtained from the positive generator vertex by adjoint operations and composition with virtual crossings, as shown in Figure 14. Note that also all other vertices can be obtained from the positive vertex via orientation switch and adjoint operations and composition by virtual crossings.

As a small exercise, it is worthwhile to convince ourselves of the effect of orientation switch operations on the *band picture*. For example, replace $A_1A_2A_3$ by $S_1S_2S_3$ in figure 14. In the strand diagram, this will only reverse the direction of the strands. The reader can check that in the band picture not only the arrows will reverse but also the blue band will switch to be on top of the red band.

Comment 4.4. Framings were discussed in Section 3.4, but have not played a significant role so far, except to explain the lack of a Reidemeister 1 relation. We now need to discuss framings in order to provide a topological explanation for the unzip (tube doubling) operation.

In the local topological interpretation of wTF^o, strands represent ribbon-knotted tubes with foam vertices, which are also equipped with a framing, arising from the blackboard framing of the strand diagrams via Satoh's tubing map. Strand doubling is the operation of doubling a tube by "pushing it off itself slightly" in the framing direction, as shown in Figure 15.

Recall that ribbon knotted tubes have a "filling", with only "ribbon" self-intersections [WKO1, Section 2.2.2]. When we double a tube, we want this ribbon property to be preserved. This is equivalent to saying that the ring obtained by pushing off any given girth of

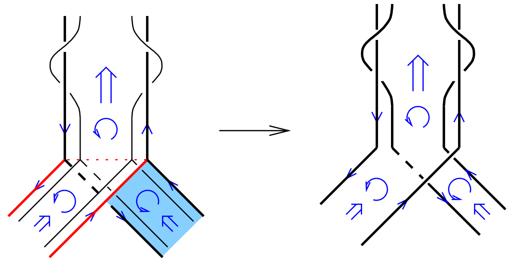
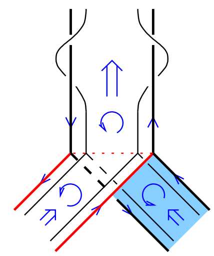


Figure 15. Unzipping a tube, in band notation with orientations and framing marked.

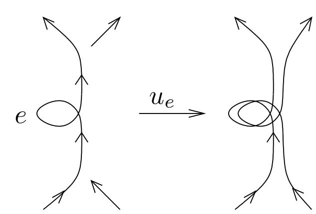
the tube in the framing direction is not linked with the original tube, which is indeed the case.

Framings arising from the blackboard framing of strand diagrams via Satoh's tubing map always match at the vertices, with the normal vectors pointing either directly towards or away from the center of the singular ring. Note that the orientations of the three tubes may or may not match. An example of a vertex with the orientations and framings shown is on the right. Note that the framings on the two sides of each band are mirror images of each other, as they should be.

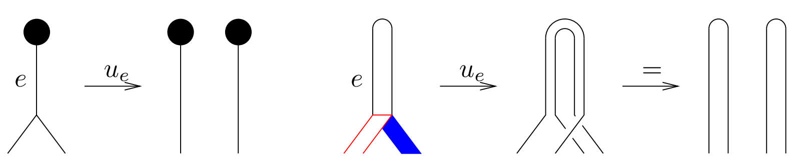


Unzip, or tube doubling is perhaps the most interesting of the auxiliary *wTF*^{o} operations. As mentioned above, topologically this means pushing the tube off itself slightly in the framing direction. At each of the vertices at the two ends of the doubled tube there are two tubes to be attached to the doubled tube. At each end, the normal vectors pointed either directly towards or away from the center, so there is an "inside" and an "outside" ending ring. The two tubes to be attached also come as an "inside" and an "outside" one, which defines which one to attach to which. An example is shown in Figure 15. Unzip can only be done if the 1D and 2D orientations match at both ends.

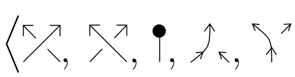
To define unzip rigorously, we must talk only of strand diagrams. The natural definition is to let u_e double the strand e using the blackboard framing, and then attach the ends of the doubled strand to the connecting ones, as shown on the right. We restrict unzip to strands whose two ending vertices are of different signs. This is a somewhat artificial condition which we impose to get equations equivalent to the [AT] equations.



A related operation, *disk unzip*, is unzip done on a capped strand, pushing the tube off in the direction of the framing (in diagrammatic world, in the direction of the blackboard framing), as before. An example in the line and band notations (with the framing suppressed) is shown below.

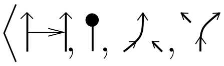


Finally, we allow deletion d_e of "long linear" strands, meaning strands that do not end in a vertex on either side. To summarize,

$$wTF^o = \mathrm{CA}\left\langle \right.$$  $$\left.\middle|\ \begin{matrix}\mathrm{R1}^s,\ \mathrm{R2},\ \mathrm{R3},\ \mathrm{R4},\\ \mathrm{OC},\ \mathrm{CP}\end{matrix}\ \middle|\ S_e, A_e, u_e, d_e \right\rangle.$$

The goal, as before, is to construct a homomorphic expansion for wTF^o. However, first we need to understand its target space, the associated graded structure grad wTF^o.

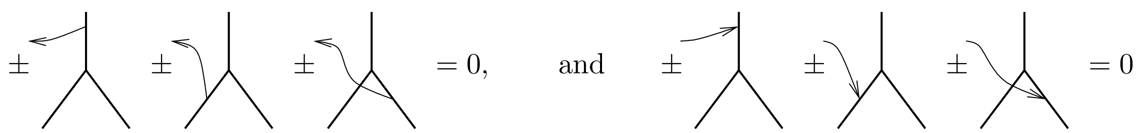
4.2. **The Associated Graded Structure.** Mirroring the previous section, we describe the associated graded $\mathcal{A}^{sw}$ of wTF^o and its "full version" $\mathcal{A}^w$ as circuit algebras on certain generators modulo a number of relations. From now on we will write $\mathcal{A}^{(s)w}$ to mean "$\mathcal{A}^w$ and/or $\mathcal{A}^{sw}$".

$$\mathcal{A}^{(s)w} = \mathrm{CA}\left\langle \right.$$  $$\left.\middle|\ \begin{matrix}\text{relations as in}\\ \text{Section 4.2.1}\end{matrix}\ \middle|\ \begin{matrix}\text{operations as in}\\ \text{Section 4.2.2}\end{matrix} \right\rangle.$$

In other words, $\mathcal{A}^{(s)w}$ are the circuit algebras of arrow diagrams on trivalent (or foam) skeletons with caps; that is, skeleta are elements of $\mathcal{S}^o$ as in Definition 4.1. Note that all but the first of the generators are skeleton features (of degree 0), and that the single arrow is the only generator of degree 1. As for the generating vertices, the same remark applies as in Definition 4.1, that is, there are more vertices with all possible strand orientations needed to generate $\mathcal{A}^{(s)w}$ as circuit algebras.

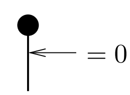
4.2.1. *The relations of $\mathcal{A}^{(s)w}$.* In addition to the usual $\overrightarrow{4T}$ and TC relations (see Figure 5), as well as RI in the case of $\mathcal{A}^{sw} = \mathcal{A}^w/RI$, diagrams in $\mathcal{A}^{(s)w}$ satisfy the following additional relations:

Vertex invariance, denoted by VI, are relations arising the same way as $\overrightarrow{4T}$ does, but with the participation of a vertex as opposed to a crossing:



The other end of the arrow is in the same place throughout the relation, somewhere outside the picture shown. The signs are positive whenever the strand on which the arrow ends is directed towards the vertex, and negative when directed away. The ambiguously drawn vertex means any kind of vertex, but the same one throughout.

The CP relation (a cap can be pulled out from under a strand but not from over, Section 4.1.2) implies that arrow heads near a cap are zero, as shown on the right. Denote this relation also by CP. (Also note that a tail near a cap is not set to zero.)



As in the previous sections, and in particular in Definition 3.9, we define a "w-Jacobi diagram" (or just "arrow diagram") on a foam skeleton by allowing trivalent chord vertices. Denote the circuit algebra of formal linear combinations of arrow diagrams by $\mathcal{A}^{(s)wt}$. We have the following bracket-rise theorem:

Theorem 4.5. *The obvious inclusion of diagrams induces a circuit algebra isomorphism $\mathcal{A}^{(s)w} \cong \mathcal{A}^{(s)wt}$. Furthermore, the $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations of Figure 7 hold in $\mathcal{A}^{(s)wt}$.*

Proof. Same as the proof of Theorem 3.10. $\square$

As in Section 3.1, the primitive elements of $\mathcal{A}^{(s)w}$ are connected diagrams (that is, connected even with the skeleton removed), namely trees and wheels. Before moving on to the auxiliary operations of $\mathcal{A}^{(s)w}$, let us make two useful observations:

Lemma 4.6. *$\mathcal{A}^w(\uparrow)$, the part of $\mathcal{A}^w$ with skeleton $\uparrow$, is isomorphic as a vector space to the completed polynomial algebra freely generated by wheels w_k with $k \geq 1$. Likewise $\mathcal{A}^{sw}(\uparrow)$, except here $k \geq 2$.*

Proof. Any arrow diagram with an arrow head at its top is zero by the Cap Pull-out (CP) relation. If D is an arrow diagram that has a head somewhere on the skeleton but not at the top, then one can use repeated $\overrightarrow{STU}$ relations to commute the head to the top at the cost of diagrams with one fewer skeleton head.

Iterating this procedure, we can get rid of all arrow heads, and hence write D as a linear combination of diagrams having no heads on the skeleton. All connected components of such diagrams are wheels.

To prove that there are no relations between wheels in $\mathcal{A}^{(s)w}(\uparrow)$, let $S_L\colon \mathcal{A}^{(s)w}(\uparrow_1) \to \mathcal{A}^{(s)w}(\uparrow_1)$ (resp. S_R) be the map that sends an arrow diagram to the sum of all ways of dropping one left (resp. right) arrow (on a vertical strand, left means down and right means up). Define

$$F := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} D_R^k (S_L + S_R)^k,$$

where D_R is a short right arrow. We leave it as an exercise for the reader to check that F is a bi-algebra homomorphism that kills diagrams with an arrow head at the top (i.e., CP is in the kernel of F), and F is injective on wheels. This concludes the proof. $\square$

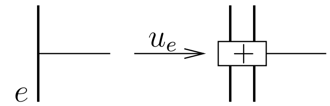
Lemma 4.7. *$\mathcal{A}^{(s)w}(Y) = \mathcal{A}^{(s)w}(\uparrow_2)$, where $\mathcal{A}^{(s)w}(Y)$ stands for the space of arrow diagrams whose skeleton is a Y-graph with any orientation of the strands, and as before $\mathcal{A}^{(s)w}(\uparrow_2)$ is the space of arrow diagrams on two strands.*

Proof. We can use the vertex invariance (VI) relation to push all arrow heads and tails from the "trunk" of the vertex to the other two strands. $\square$

4.2.2. *The auxiliary operations of $\mathcal{A}^{(s)w}$.* Recall from Section 3.4 that the orientation switch S_e (i.e. changing both the $1D$ and $2D$ orientations of a strand) always changes the sign of a crossing involving the strand e. Hence, letting S denote any foam (trivalent) skeleton, the induced arrow diagrammatic operation is a map $S_e\colon \mathcal{A}^{(s)w}(S) \to \mathcal{A}^{(s)w}(S_e(S))$ which acts by multiplying each arrow diagram by (-1) raised to the number of arrow endings on e (counting both heads and tails).

The adjoint operation A_e (i.e. switching only the strand direction), on the other hand, only changes the sign of a crossing when the strand being switched is the under- (or through) strand. (See section 3.4 for pictures and explanation.) Therefore, the arrow diagrammatic A_e acts by switching the direction of e and multiplying each arrow diagram by (-1) raised to the number of *arrow heads* on e. Note that in $\mathcal{A}^{(s)w}(\uparrow_n)$ taking the adjoint on every strand gives the adjoint map of Definition 3.23.

The arrow diagram operations induced by unzip and disc unzip (both to be denoted u_e, and interpreted appropriately according to whether the strand e is capped) are maps $u_e\colon \mathcal{A}^{(s)w}(S) \to \mathcal{A}^{(s)w}(u_e(S))$, where each arrow ending (head or tail) on e is mapped to a sum of two arrows, one ending on each of the new strands, as shown on the right. In other words, if in an arrow diagram D there are k arrow ends on e, then $u_e(D)$ is a sum of 2^k arrow diagrams.



The operation induced by deleting the long linear strand e is the map $d_e\colon \mathcal{A}^{(s)w}(S) \to \mathcal{A}^{(s)w}(d_e(S))$ which kills arrow diagrams with any arrow ending (head or tail) on e, and leaves all else unchanged, except with e removed.

Definition 4.8. In summary,

$$\mathcal{A}^{(s)w} = \mathrm{CA}\left\langle \uparrow\!\!\rightarrow\!\!\uparrow,\ \uparrow\!\!\bullet,\ \nearrow_{\nwarrow},\ {}^{\nwarrow}\!\nearrow \;\middle|\; \begin{matrix}\overrightarrow{4T},\ \mathrm{TC},\ \mathrm{VI},\ \mathrm{CP},\\ (\mathrm{RI\ for\ }\mathcal{A}^{sw})\end{matrix} \;\middle|\; S_e, A_e, u_e, d_e \right\rangle.$$

4.3. **The homomorphic expansion.** The following is one of the main theorems of this paper:

Theorem 4.9 (Proof in Section 4.4)**.** *There exists a group-like*[12] *homomorphic expansion for wTF^o, i.e. a group-like expansion $Z\colon wTF^o \to \mathcal{A}^{sw}$ which is a map of circuit algebras and also intertwines the auxiliary operations of wTF^o with their arrow diagrammatic counterparts. In fact, there is a bijection between the set of solutions (F, a) of the generalized KV problem (see Section 4.4) and the set of homomorphic expansions for wTF^o which do not contain local arrows*[13] *in the value V of the vertex.*

Since both wTF^o and $\mathcal{A}^{sw}$ are circuit algebras defined by generators and relations, when looking for a suitable Z all we have to do is to find values for each of the generators of wTF^o so that these satisfy (in $\mathcal{A}^{sw}$) the equations which arise from the relations in wTF^o and the homomorphicity requirement. In this section we derive these equations and in the next section we show that they are equivalent to the Alekseev-Torossian version of the Kashiwara-Vergne equations [AT]. In [AET] Alekseev Enriquez and Torossian construct explicit solutions to these equations using associators. In [WKO3] we will interpret these results in our context of homomorphic expansions for w-tangled foams.

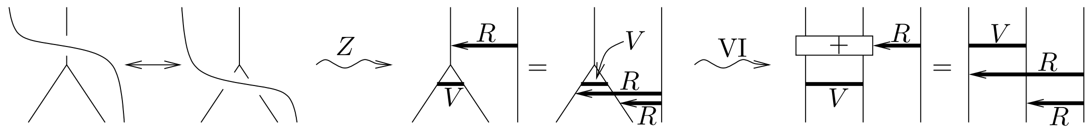
Let $R := Z(\overcrossing) \in \mathcal{A}^{sw}(\uparrow_2)$. It follows from the Reidemeister 2 relation that $Z(\undercrossing) = (R^{-1})^{21}$. As discussed in Sections 3.1 and 3.5, Reidemeister 3 with group-likeness and homomorphicity implies that $R = e^a$, where a is a single arrow pointing from the over to the under strand. Let $C := Z(\uparrow\!\bullet) \in \mathcal{A}^{sw}(\uparrow\!\bullet)$. By Lemma 4.6, we know that C is made up of wheels only. Finally, let $V = V^+ := Z(\nearrow_{\nwarrow}) \in \mathcal{A}^{sw}(\nearrow_{\nwarrow}) \cong \mathcal{A}^{sw}(\uparrow_2)$, and $V^- := Z({}^{\nwarrow}\!\nearrow) \in \mathcal{A}^{sw}({}^{\nwarrow}\!\nearrow) \cong \mathcal{A}^{sw}(\uparrow_2)$.

Before we translate each of the relations of Section 4.1.2 to equations let us slightly extend the notation used in Section 3.5. Recall that R^{23}, for instance, meant "R placed on strands 2 and 3". In this section we also need notation such as $R^{(23)1}$, which means "R with its first strand doubled, placed on strands 2, 3 and 1".

Now on to the relations, note that Reidemeister 2 and 3 and Overcrossings Commute have already been dealt with. Of the two Reidemeister 4 relations, the first one induces an equation that is automatically satisfied. Pictorially, the equation looks as follows:

[12] The formal definition of the group-like property is along the lines of [WKO1, Section 2.5.1.2]. In practice, it means that the Z-values of the vertices, crossings, and cap (denoted V, R and C below) are exponentials of linear combinations of connected diagrams.

[13] For a detailed explanation of this minor point see the third paragraph of the proof.



In other words, we obtained the equation

$$V^{12}R^{3(12)} = R^{32}R^{31}V^{12}.$$

However, observe that by the "head-invariance" property of arrow diagrams (Remark 3.11) V^{12} and $R^{3(12)}$ commute on the left hand side. Hence the left hand side equals $R^{3(12)}V^{12} = R^{32}R^{31}V^{12}$. Also, $R^{3(12)} = e^{a^{31}+a^{32}} = e^{a^{32}}e^{a^{31}} = R^{32}R^{31}$, where the second step is due to the fact that a^{31} and a^{32} commute. Therefore, the equation is true independently of the choice of V.

We have no such luck with the second Reidemeister 4 relation, which, in the same manner as in the paragraph above, translates to the equation

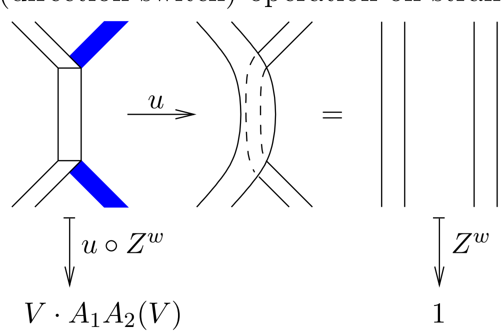
$$V^{12}R^{(12)3} = R^{23}R^{13}V^{12}. \tag{11}$$

There is no "tail invariance" of arrow diagrams, so V and R do not commute on the left hand side; also, $R^{(12)3} \neq R^{23}R^{13}$. As a result, this equation puts a genuine restriction on the choice of V.

The Cap Pull-out (CP) relation translates to the equation $R^{12}C^2 = C^2$. This is true independently of the choice of C: by head-invariance, $R^{12}C^2 = C^2R^{12}$. Now R^{12} is just below the cap on strand 2, and the cap "kills heads", in other words, every term of R^{12} with an arrow head at the top of strand 2 is zero. Hence, the only surviving term of R^{12} is 1 (the empty diagram), which makes the equation true.

The homomorphicity of the orientation switch operation was used to prove the uniqueness of R in Theorem 3.27. The homomorphicity of the adjoint leads to the equation $V_- = A_1A_2(V)$ (see Figure 14), eliminating V_- as an unknown. Note that we also silently assumed these homomorphicity properties when we did not introduce 32 different values of the vertex depending on the strand orientations.

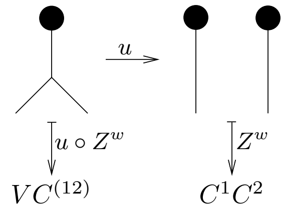
Homomorphicity of the (annular) unzip operation leads to an equation for V, which we are going to refer to as "unitarity". This is illustrated in the figure below. Recall that A_1 and A_2 denote the adjoint (direction switch) operation on strand 1 and 2, respectively.



Reading off the equation, we have

$$V \cdot A_1A_2(V) = 1. \tag{12}$$

Homomorphicity of the disk unzip leads to an equation for C which we will refer to as the "cap equation". The translation from homomorphicity to equation is shown in the figure on the right. C, as we introduced before, denotes the Z-value of the cap. Hence, the cap equation reads



$$V^{12}C^{(12)} = C^1C^2 \qquad \text{in} \quad \mathcal{A}^{sw}(\uparrow_2) \tag{13}$$

The homomorphicity of deleting long strands does not lead to an equation on its own, however it was used to prove the uniqueness of R (Theorem 3.27).

To summarize, we have reduced the problem of finding a homomorphic expansion Z to finding the Z-values of the (positive) vertex and the cap, denoted V and C, subject to three equations: the "hard Reidemeister 4" equation (11); "unitarity of V" equation (12); and the "cap equation" (13).

4.4. **The equivalence with the Alekseev-Torossian equations.** First let us recall Alekseev and Torossian's formulation of the generalized Kashiwara-Vergne problem (see [AT, Section 5.3]):

Generalized KV problem: Find an element $F \in \mathrm{TAut}_2$ with the properties

$$F(x+y) = \log(e^x e^y), \text{ and } j(F) \in \mathrm{im}(\tilde{\delta}). \tag{14}$$

Here $\tilde{\delta}\colon \mathfrak{tr}_1 \to \mathfrak{tr}_2$ is defined by $(\tilde{\delta}a)(x,y) = a(x) + a(y) - a(\log(e^x e^y))$, where $\mathfrak{tr}_2$ is generated by cyclic words in the letters x and y. (See [AT], Equation (8)). Note that an element of $\mathfrak{tr}_1$ is a power series in one variable with no constant term. In other words, the second condition says that there exists $a \in \mathfrak{tr}_1$ such that $jF = a(x) + a(y) - a(\log(e^x e^y))$.

Proof of Theorem 4.9. We have reduced the problem of finding a homomorphic expansion to finding group-like solutions V and C to the hard Reidemeister 4 equation (11), the unitarity equation (12), and the cap equation (13).

Suppose we have found such solutions and write $V = e^b e^{uD}$, where $b \in \mathrm{tr}_2^s$, $D \in \mathfrak{tder}_2 \oplus \mathfrak{a}_2$, and where u is the map $u\colon \mathfrak{tder}_2 \to \mathcal{A}^{sw}(\uparrow_2)$ which plants the head of a tree above all of its tails, as introduced in Section 3.2. V can be written in this form without loss of generality because wheels can always be commuted to the bottom of a diagram (at the possible cost of more wheels). Furthermore, V is group-like and hence it can be written in exponential form. Similarly, write $C = e^c$ with $c \in \mathfrak{tr}_1^s$.

Note that $u(\mathfrak{a}_2)$ is central in $\mathcal{A}^{sw}(\uparrow_2)$ and that replacing a solution (V, C) by $(e^{u(a)}V, C)$ for any $a \in \mathfrak{a}_2$ does not interfere with any of the equations (11), (12) or (13). Hence we may assume that D does not contain any single arrows, that is, $D \in \mathfrak{tder}_2$. Also, a solution (V, C) in $\mathcal{A}^{sw}$ can be lifted to a solution in $\mathcal{A}^w$ by simply setting the degree one terms of b and c to be zero. It is easy to check that this $b \in \mathfrak{tr}_2$ and $c \in \mathfrak{tr}_1$ along with D still satisfy the equations. (In fact, in $\mathcal{A}^w$ (12) and (13) respectively imply that b is zero in degree 1, and that the degree 1 term of c is arbitrary, so we may as well assume it to be zero.) In light of this we declare that $b \in \mathfrak{tr}_2$ and $c \in \mathfrak{tr}_1$.

The hard Reidemeister 4 equation (11) reads $V^{12}R^{(12)3} = R^{23}R^{13}V^{12}$. Denote the arrow from strand 1 to strand 3 by x, and the arrow from strand 2 to strand 3 by y. Substituting the known value for R and rearranging, we get

$$e^b e^{uD} e^{x+y} e^{-uD} e^{-b} = e^y e^x.$$

Equivalently, $e^{uD}e^{x+y}e^{-uD} = e^{-b}e^ye^xe^b$. Now on the right side there are only tails on the first two strands, hence e^b commutes with e^ye^x, so $e^{-b}e^b$ cancels. Taking logarithm of both sides we obtain $e^{uD}(x+y)e^{-uD} = \log e^ye^x$. Now for notational alignment with [AT] we switch strands 1 and 2, which exchanges x and y so we obtain:

$$e^{uD^{21}}(x+y)e^{-uD^{21}} = \log e^xe^y. \tag{15}$$

The unitarity of V (Equation (12)) translates to $1 = e^be^{uD}(e^be^{uD})^*$, where $*$ denotes the adjoint map (Definition 3.23). Note that the adjoint switches the order of a product and acts trivially on wheels. Also, $e^{uD}(e^{uD})^* = J(e^D) = e^{j(e^D)}$, by Proposition 3.24. So we have $1 = e^be^{j(e^D)}e^b$. Multiplying by e^{-b} on the right and by e^b on the left, we get $1 = e^{2b}e^{j(e^D)}$, and again by switching strand 1 and 2 we arrive at

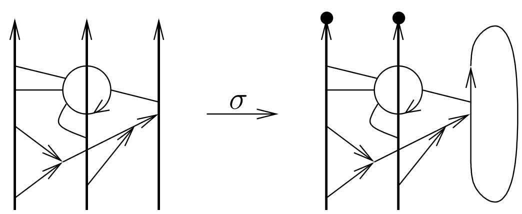
$$1 = e^{2b^{21}}e^{j(e^{D^{21}})}. \tag{16}$$

As for the cap equation, if $C^1 = e^{c(x)}$ and $C^2 = e^{c(y)}$, then $C^{12} = e^{c(x+y)}$. Note that wheels on different strands commute, hence $e^{c(x)}e^{c(y)} = e^{c(x)+c(y)}$, so the cap equation reads

$$e^be^{uD}e^{c(x+y)} = e^{c(x)+c(y)}.$$

As this equation lives in the space of arrow diagrams on two *capped* strands, we can multiply the left side on the right by e^{-uD}: uD has its head at the top, so it is 0 by the Cap relation, hence $e^{uD} = 1$ near the cap. Hence,

$$e^be^{uD}e^{c(x+y)}e^{-uD} = e^{c(x)+c(y)}.$$



On the right side of the equation above $e^{uD}e^{c(x+y)}e^{-uD}$ reminds us of Equation (15), however we cannot use (15) directly as we are working in a different space now. In particular, x there meant an arrow from strand 1 to strand 3, while here it means a one-wheel on (capped) strand 1, and similarly for y. Fortunately, there is a map $\sigma\colon \mathcal{A}^{sw}(\uparrow_3) \to \mathcal{A}^{sw}(\overset{\bullet}{\uparrow}_2)$, where σ "closes the third strand and turns it into a chord (or internal) strand, and caps the first two strands", as shown on the right. This map is well defined (in fact, it kills almost all relations, and turns one $\overrightarrow{STU}$ into an $\overrightarrow{IHX}$). Under this map, using our abusive notation, $\sigma(x) = x$ and $\sigma(y) = y$.

Now we can apply Equation (15) to get $e^{uD}e^{c(x+y)}e^{-uD} = e^{c(\log e^ye^x)}$. Substituting this into the cap equation we obtain $e^be^{c(\log e^ye^x)} = e^{c(x)+c(y)}$, which, using that tails commute, implies $b = c(x) + c(y) - c(\log e^ye^x)$. Switching strands 1 and 2, we obtain

$$b^{21} = c(x) + c(y) - c(\log e^xe^y) \tag{17}$$

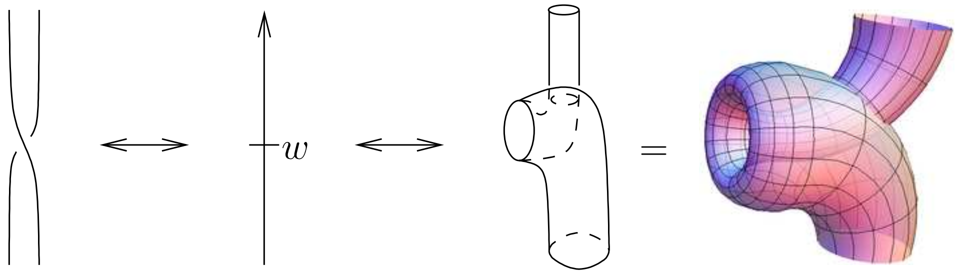
In summary, we can use (V, C) to produce $F := e^{D^{21}}$ (sorry[14]) and $a := -2c$ which satisfy the Alekseev-Torossian equations (14), as follows: $e^{D^{21}}$ acts on $\mathfrak{lie}_2$ by conjugation by $e^{uD^{21}}$, so the first part of (14) is implied by (15). The second half of (14) is true due to (16) and (17).

[14] We apologize for the annoying $2 \leftrightarrow 1$ transposition in this equation, which makes some later equations, especially (22), uglier than they could have been. There is no depth here, just mis-matching conventions between us and Alekseev-Torossian.

On the other hand, suppose that we have found $F \in \mathrm{TAut}_2$ and $a \in \mathrm{tr}_1$ satisfying (14). Then set $D^{21} := \log F$, $b^{21} := \frac{-j(e^{D^{21}})}{2}$, and $c \in \tilde{\delta}^{-1}(b^{21})$, in particular $c = -\frac{a}{2}$ works. Then $V = e^b e^{uD}$ and $C = e^c$ satisfy the equations for homomorphic expansions (11), (12) and (13).

Furthermore, the two maps between solutions of the KV problem and homomorphic expansions for $w\mathit{TF}^o$ defined in the last two paragraphs are obviously inverses of each other, and hence they provide a bijection between these sets as stated. $\square$

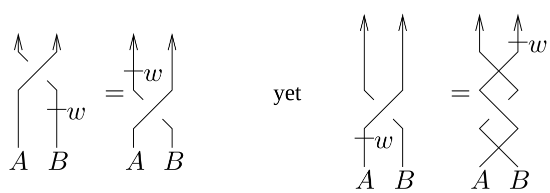
4.5. **The wen.** A topological feature of w-tangled foams which we excluded from the theory so far is the wen w. The wen is a Klein bottle cut apart (as mentioned in [WKO1, Section 2.5.4]); in other words it amounts to changing the 2D orientation of a tube, as shown in the picture below:



In this section we study the circuit algebra of w-Tangled Foams with the wen included as a generator, and denote this space by $w\mathit{TF}$. The wen is also added to the circuit algebra of skeleta. We will find that homomorphic expansions for $w\mathit{TF}$ are in bijection with solutions to the KV problem with "even Duflo function", as explained below.

4.5.1. *The relations and auxiliary operations of* $w\mathit{TF}$. Adding the wen as a generator means we have to impose additional relations involving the wen to keep our topological heuristics intact, as follows:

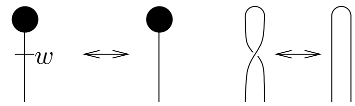
The interaction of a wen and a crossing is described by the following relation (cf. [WKO1, Section 2.5.4]):

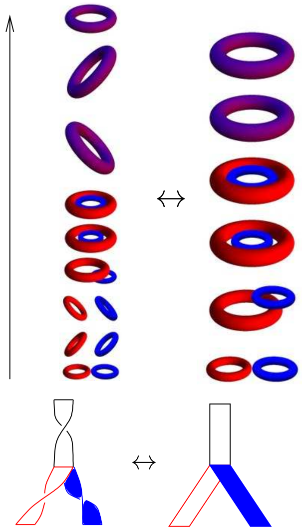


To explain this relation note that in flying ring language, a wen is a ring that flips over. It does not matter whether ring B flips first and then flies through ring A or vice versa. However, the movies in which ring A first flips and then ring B flies through it, or B flies through A first and then A flips differ in the fly-through direction of B through A, which is cancelled by virtual crossings, as in the figure above. We will refer to these relations as the Flip Relations, and abbreviate them by FR.

A double flip is homotopic to no flip, in other words two consecutive wens equal no wen. Let us denote this relation by W^2, for Wen squared. Note that this relation explains why there are no "left and right wens".

A cap can slide through a wen, hence a capped wen disappears, as shown on the right, to be denoted CW.

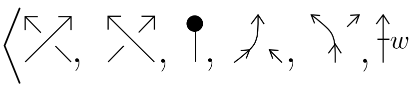




The last wen relation describes the interaction of wens and vertices. Recall that there are four types of vertices with the same strand orientation: among the bottom two bands (in the pictures on the left) there is a non-filled and a filled band (corresponding to over/under in the strand diagrams), meaning the "large" ring and the "small" one which flies into it before they merge. Furthermore, there is a top and a bottom band (among these bottom two, with apologies for the ambiguity in overusing the word bottom): this denotes the fly-in direction (flying in from below or from above). Conjugating a vertex by three wens switches the top and bottom bands, as shown in the figure on the left: if both rings flip, then merge, and then the merged ring flips again, this is homotopic to no flips, except the fly-in direction (from below or from above) has changed. We are going to denote this relation by TV, for "twisted vertex".

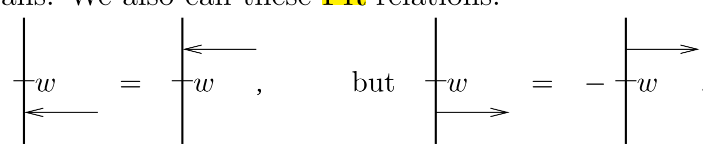
The auxiliary operations are the same as for $w\mathit{TF}^o$: orientation switches, adjoints, deletion of long linear strands, cap unzips and unzips[15].

Definition 4.10. Summarizing the above, we have

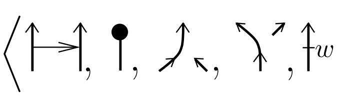
$w\mathit{TF} = \mathrm{CA}\Big\langle$  $\Big|$ R1^s, R2, R3, R4, OC, CP, FR, W^2, CW, TV $\Big|$ $S_e, A_e, u_e, d_e \Big\rangle$.

4.5.2. *The Associated Graded Structure.* The associated graded structure of $w\mathit{TF}$ (still denoted $\mathcal{A}^{sw}$) is the same as the associated graded for $w\mathit{TF}^o$ but with the wen added as a generator (a degree 0 skeleton feature), and with extra relations describing the behaviour of the wen. Of course, the relations describing the interaction of wens with the other skeleton features (W^2, TV, and CW) still apply, as well as the old $\overrightarrow{4T}$, TC, VI, CP, and RI relations.

In addition, the Flip Relations FR imply that wens "commute" with arrow heads, but "anti-commute" with tails. We also call these FR relations:

FR: 

That is,

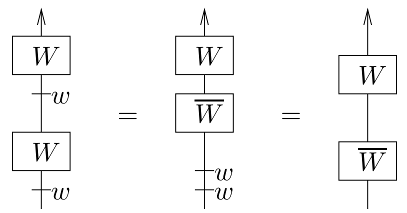
$\mathcal{A}^{sw} = \mathrm{CA}\Big\langle$  $\Big|$ W^2, TW, CW, $\overrightarrow{4T}$, TC, VI, CP, RI, FR $\Big|$ $S_e, A_e, u_e, d_e \Big\rangle$.

4.5.3. *The homomorphic expansion.* The goal of this section is to prove that there exists a homomorphic expansion Z for $w\mathit{TF}$. This involves solving a similar system of equations to Section 4.3, but with an added unknown for the value of the wen, as well as added equations arising from the wen relations. Let $W \in \mathcal{A}^{sw}(\uparrow_1)$ denote the Z-value of the wen, and let us

[15]We need not specify how to unzip an edge e that carries a wen. To unzip such e, first use the TV relation to slide the wen off e.

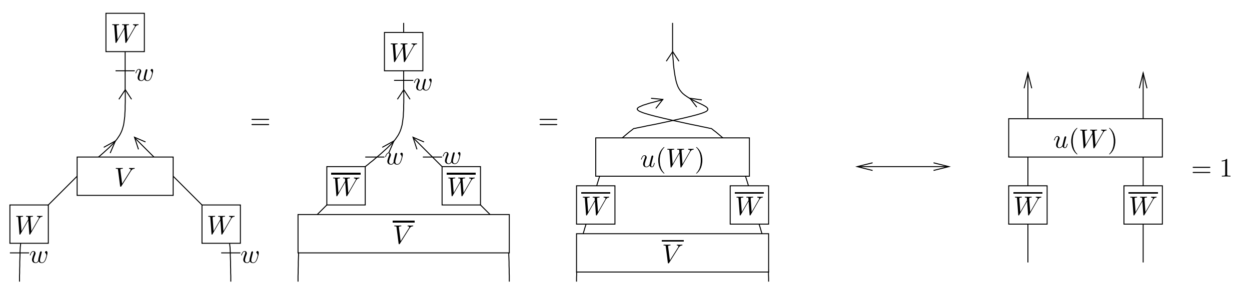
agree that the arrow diagram W always appears just above the wen on the skeleton. In fact, we are going to show that $W = 1$ for any homomorphic expansion.

As two consecutive wens on the skeleton cancel, we obtain the equation shown in the picture and explained below:



The Z-value of two consecutive wens on a strand is a skeleton wen followed by W followed by a skeleton wen and another W. Sliding the bottom W through the skeleton wen "multiplies each tail by (-1)". Let us denote this operation by "bar", i.e. for an arrow diagram D, $\overline{D} = D \cdot (-1)^{\#\text{ of tails in } D}$. Cancelling the two skeleton wens, we obtain $\overline{W}W = 1$. Recall that $\mathcal{A}^{sw}(\uparrow_1)$ consists only of wheels and single arrows. Since we are looking for a group-like Z, we can assume that $W = e^w$ and $\overline{W}W = 1$ means that w is a linear combination of odd wheels and possibly single arrows.

Now recall the Twisted Vertex relation of Section 4.5.1. Note that the Z-value of the negative vertex on the right hand side of the relation can be written as $S_1S_2A_1A_2(V) = \overline{V}$ (cf Remark 4.3). On the other hand, applying Z to the left hand side of the relation we obtain:



Hence, using that $\overline{W} = W^{-1}$, the twisted vertex relation induces the equation $W^1W^2 = W^{(12)}$ in $\mathcal{A}^{sw}(\uparrow_2)$. One can verify degree by degree, using that W can be written as an exponential, that the only solution to this equation is $W = 1$.

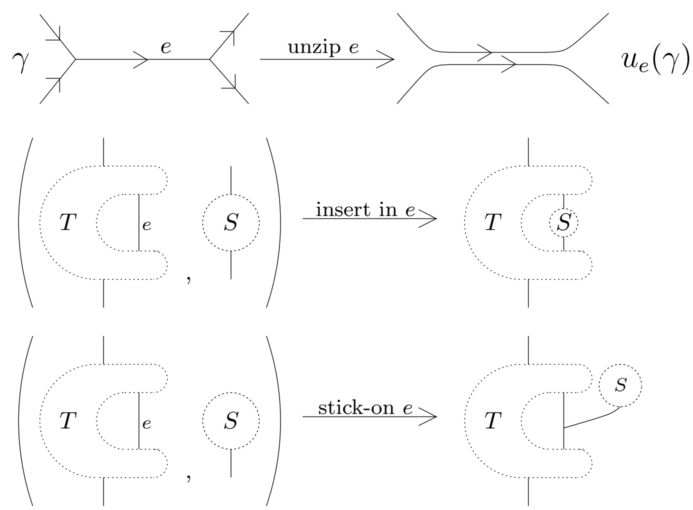
We have yet to verify that the CW relation (i.e., a cap can slide through a wen) can be satisfied with $W = 1$. Keep in mind that the wen as a skeleton feature anti-commutes with tails (this is the Flip Relation of Section 4.2.1). The value of the cap C is a combination of only wheels (Lemma 4.6), hence the CW relation translates to the equation $\overline{C} = C$, which is equivalent to saying that C consists only of even wheels.

The fact that Z can be chosen to have this property follows from Proposition 6.2 of [AT]: the value of the cap is $C = e^c$, where c can be set to $c = -\frac{a}{2}$, as explained in the proof of Theorem 4.9. Here a is such that $\tilde{\delta}(a) = jF$ as in Equation (14). A power series f so that $a = \operatorname{tr} f$ (where tr is the trace which turns words into cyclic words) is called the Duflo function of F. In Proposition 6.2 Alekseev and Torossian show that the even part of f is $\frac{1}{2}\frac{\log(e^{x/2}-e^{-x/2})}{x}$, and that for any f with this even part there is a corresponding solution F of the generalized KV problem. In particular, f can be assumed to be even, namely the power series above, and hence it can be guaranteed that C consists of even wheels only. Thus we have proven the following:

Theorem 4.11. *Group-like homomorphic expansions $Z : w\mathit{TF} \to \mathcal{A}^{sw}$ (with no local arrows in the value of V) are in one-to one correspondence with solutions to the KV problem with an even Duflo function.* $\square$

4.6. **Interlude: u-Knotted Trivalent Graphs.** The "$u$sual", or classical knot-theoretical objects corresponding to $w\mathit{TF}$ are loosely speaking Knotted Trivalent Graphs, or KTGs. We give a brief introduction/review of this structure before studying the relationship between their homomorphic expansions and homomorphic expansions for $w\mathit{TF}$. The last goal of this paper is to show that the topological relationship between the two spaces explains the relationship between the KV problem and Drinfel'd associators.

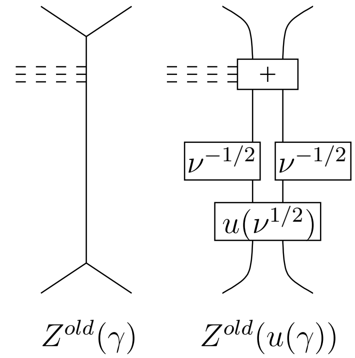
A trivalent graph is a graph with three edges meeting at each vertex, equipped with a cyclic orientation of the three half-edges at each vertex. KTGs are framed embeddings of trivalent graphs into $\mathbb{R}^3$, regarded up to isotopies. The skeleton of a KTG is the trivalent graph (as a combinatorial object) behind it. For a detailed introduction to KTGs see for example [BND]. Here we only recall the most important facts. The reader might recall that in Section 3, the w-knot section, of [WKO1] we only dealt with long w-knots, as the w-theory of round knots is essentially trivial (see [WKO1, Theorem 3.17]). A similar issue arises with "w-knotted trivalent graphs". Hence, the space we are really interested in is "long KTGs", meaning, trivalent tangles with 1 or 2 ends.



Long KTGs form an algebraic structure with operations as follows. *Orientation switch* reverses the orientation of a specified edge. *Edge unzip* doubles a specified edge as shown on the right. *Tangle insertion* is inserting a small copy of a $(1,1)$-tangle S into the middle of some specified strand of a tangle T, as shown in the second row on the right (tangle composition is a special case of this). The *stick-on* operation "sticks a 1-tangle S onto a specified edge of another tangle T", as shown. (In the figures T is a 2-tangle, but this is irrelevant.) *Disjoint union* of two 1-tangles produces a 2-tangle. Insertion, disjoint union and stick-on are a slightly weaker set of operations than the connected sum of [BND].

The associated graded structure of the algebraic structure of long KTGs is the graded space $\mathcal{A}^u$ of chord diagrams on trivalent graph skeleta, modulo the $4T$ and vertex invariance (VI) relations. The induced operations on $\mathcal{A}^u$ are as expected: orientation switch multiplies a chord diagram by (-1) to the number of chord endings on the edge. The edge unzip u_e maps a chord diagram with k chord endings on the edge e to a sum of 2^k diagrams where each chord ending has a choice between the two daughter edges. Finally, tangle insertion, stick-on and disjoint union induces the insertion, sticking on and disjoint union of chord diagrams, respectively.

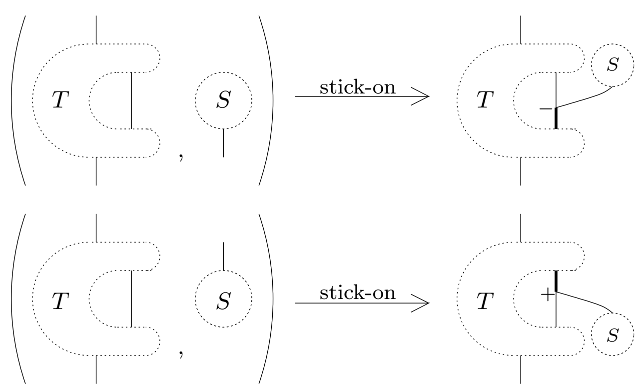
In [BND] the authors prove that there is no *homomorphic* expansion for KTGs. This theorem, as well as the proof, applies to long KTGs with slight modifications. However there is a well-known — and nearly homomorphic — expansion constructed by extending the Kontsevich integral to KTGs and renormalizing at the vertices. There are several constructions that do this ([MO], [CL], [Da]). For now, let us denote any one of these expansions by Z^{old}. All the Z^{old} are almost homomorphic: they intertwine every operation except for edge unzip with their chord-diagrammatic counterparts; but commutativity with unzip fails by a controlled amount, as shown on the right. Here ν denotes the "invariant of the unknot", the value of which was conjectured in [BGRT] and proven in [BLT].



In [BND] the authors fix this anomaly by slightly changing the space of KTGs and adding some extra combinatorics ("dots" on the edges), and construct a homomorphic expansion for this new space by a slight adjustment of Z^{old}. Here we are going to use a similar but different adjustment of the space of trivalent 1- and 2-tangles. Namely we break the symmetry of the vertices and restrict the set of allowed unzips.

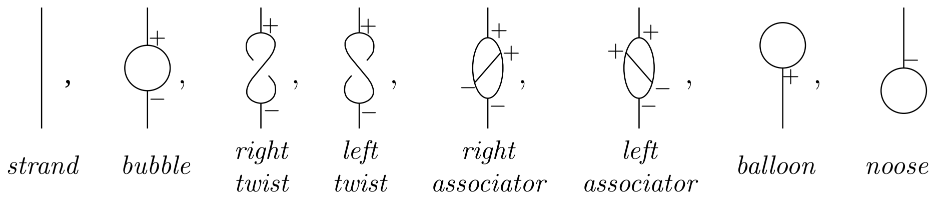
Definition 4.12. A "signed KTG", denoted *sKTG*, is a trivalent oriented 1- or 2-tangle embedded in $\mathbb{R}^3$ with a cyclic orientation of edges meeting at each vertex, and in addition each vertex is equipped with a sign and one of the three incident edges is marked as distinguished (sometimes denoted by a thicker line). Our pictorial convention will be that a vertex drawn in a "$\curlywedge$" shape with all strands oriented up and the top strand distinguished is always positive and a vertex drawn in a "Y" shape with strands oriented up and the bottom strand distinguished is always negative (see Figure 18).

The algebraic structure *sKTG* has one kind of objects for each skeleton (a skeleton is a uni-trivalent graph with signed vertices but no embedding), as well as several operations: orientation switch, edge unzip, tangle insertion, disjoint union of 1-tangles, and stick-on. Orientation switch of either of the non-distinguished strands changes the sign of the vertex, switching the orientation of the distinguished strand does not. Unzip of an edge is only allowed if the edge is distinguished at both of its ends and the vertices at either end are of opposite signs. The stick-on operation can be done in either one of the two ways shown on the right (i.e., the stuck-on edge can be attached at a vertex of either sign, but it can not become the distinguished edge of that vertex).



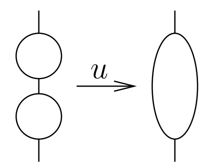
To consider expansions of *sKTG*, and ultimately the compatibility of these with Z^w, we first note that *sKTG* is finitely generated (and therefore any expansion Z^u is determined by its values on finitely many generators). The proof of this is not hard but somewhat lengthy, so we postpone it to Section 5.2.

Proposition 4.13. *The algebraic structure sKTG is finitely generated by the following list of elements:*



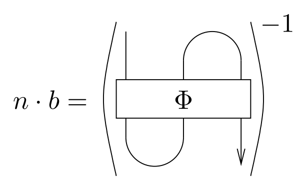
Note that we ignore strand orientations for simplicity in the statement of this proposition; this is not a problem as orientation switches are allowed in *sKTG* without restriction.

4.6.1. *Homomorphic expansions for sKTG.* Suppose that $Z^u : sKTG \to \mathcal{A}^u$ is a homomorphic expansion. We hope to determine the value of Z^u on each of the generators.



The value of Z^u on the single strand is an element of $\mathcal{A}^u(\uparrow)$ whose square is itself, hence it is 1. The value of the bubble is an element $x \in \mathcal{A}^u(\uparrow_2)$, as all chords can be pushed to the "bubble" part using the VI relation. Two bubbles can be composed and unzipped to produce a single bubble (see on the right), hence we have $x^2 = x$, which implies $x = 1$ in $\mathcal{A}^u(\uparrow_2)$.

Recall that a Drinfel'd associator is a group-like element $\Phi \in \mathcal{A}^u(\uparrow_3)$ along with a group-like element $R^u \in \mathcal{A}^u(\uparrow_2)$ satisfying the so-called pentagon and positive and negative hexagon equations, as well as a non-degeneracy and mirror skew-symmetry property. For a detailed explanation see Section 4 of [BND]; associators were first defined in [Dr2]. We claim that the Z^u-value Φ of the right associator, along with the value R^u of the right twist forms a Drinfel'd associator pair. The proof of this statement is the same as the proof of Theorem 4.2 of [BND], with minor modifications (making heavy use of the assumption that Z^u is homomorphic). It is easy to check by composition and unzips that the value of the left associator and the left twist are Φ^{-1} and $(R^u)^{-1}$, respectively. Note that if Φ is required to be a *horizontal chord* associator (i.e., all the chords of Φ are horizontal on three strands) then R^u is forced to be $e^{c/2}$ where c denotes a single chord. Note that the reverse is not true: there exist non-horizontal chord associators Φ that satisfy the hexagon equations with $R^u = e^{c/2}$.



Let b and n denote the Z^u-values of the balloon and the noose, respectively. Note that using the VI relation all chord endings can be pushed to the "looped" strands, so b and n live in $\mathcal{A}^u(\uparrow)$, as seen in Figure 16. The argument in that figure shows that $n \cdot b$ is the inverse in $\mathcal{A}^u(\uparrow)$ of "an associator on a squiggly strand", as shown on the right. In Figure 16 we start with the *sKTG* on the top left and either apply Z^u followed by unzipping the edges marked by stars, or first unzip the same edges and then apply Z^u. Since Z^u is homomorphic, the two results in the bottom right corner must agree. (Note that two of the four unzips we perform are "illegal", as the strand directions can't match. However, it is easy to get around this issue by inserting small bubbles at the top of the balloon and the bottom of the noose, and switching the appropriate edge orientations before and after the unzips. The Z^u-value of a bubble is 1, hence this will not effect the computation and so we ignore the issue for simplicity.)

In addition, it follows from Theorem 4.2 of [BND] via deleting two edges that the inverse of an "associator on a squiggly strand" is ν, the invariant of the unknot. To summarize, we have proven the following:

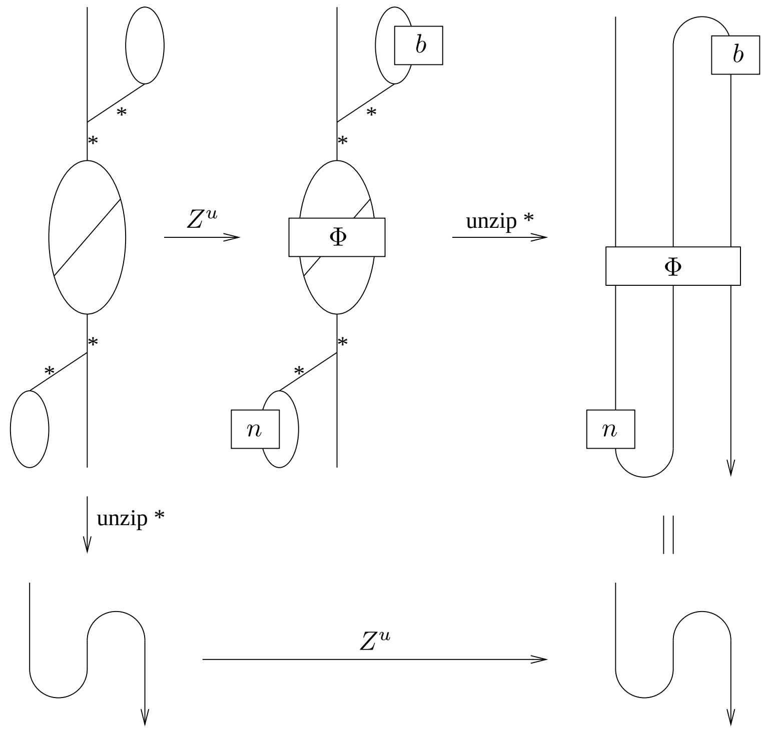
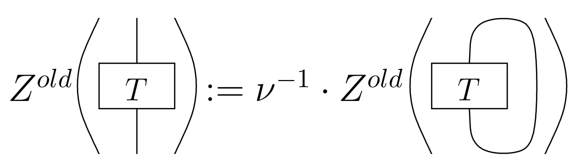


Figure 16. Unzipping a noose and a balloon to a squiggle.

Lemma 4.14. *If Z^u is a homomorphic expansion then the Z^u values of the strand and the bubble are 1, the values of the right associator and right twist form an associator pair (Φ, R^u), and the values of the left twist and left associator are inverses of these. With n and b denoting the value of the noose and the balloon, respectively, and ν being the invariant of the unknot, we have $n \cdot b = \nu$ in $\mathcal{A}^u(\uparrow)$.*

The natural question to ask is whether any triple (Φ, R^u, n) gives rise to a homomorphic expansion. We don't know whether this is true, but we do know that any pair (Φ, R^u) gives rise to a "nearly homomorphic" expansion of KTGs [MO, CL, Da], and we can construct a homomorphic expansion for *sKTG* from any of these (as shown below). However, all of these expansions take the same specific value on the noose and the balloon (also see below). We don't know whether there really is a one parameter family of homomorphic expansions Z^u for each choice of (Φ, R^u) or if we are simply missing a relation.



We now construct explicit homomorphic expansions $Z^u\colon$ *sKTG* $\to \mathcal{A}^u$ from any Z^{old} (where Z^{old} stands for an extension to KTGs of the Kontsevich integral) as follows. First of all we need to interpret Z^{old} as an invariant of 2-tangles. This can be done by connecting the top and bottom ends by a non-interacting long strand followed by a normalization, as shown on the right. By "multiplying by ν^{-1}" we mean that after computing Z^{old} we insert ν^{-1} on the long strand (recall that ν is the "invariant of the unknot"). We interpret Z^{old} of a 1-tangle as follows: stick the 1-tangle onto a single strand to obtain a 2-tangle, then proceed as above. The result will only have chords on the 1-tangle (using that the extensions of the

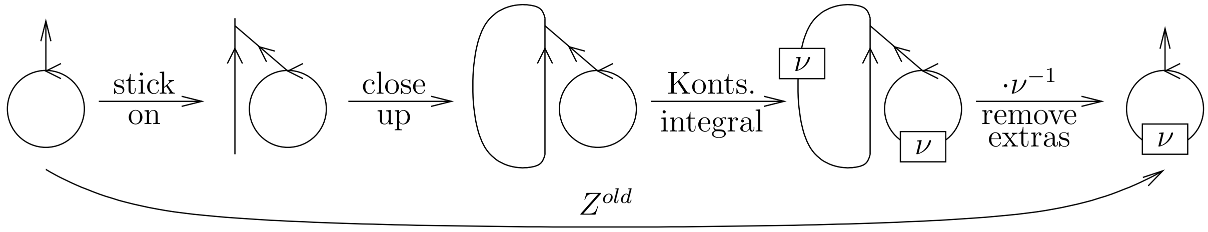


Figure 17. Computing the Z^{old} value of the noose. The third step uses that the Kontsevich integral of KTGs is homomorphic with respect to the "connected sum" operation and that the value of the unknot is ν (see [BND] for an explanation of both of these facts).

Kontsevich Integral are homomorphic with respect to "connected sums"), so we define the result to be the value of Z^{old} on the 1-tangle. As an example, we compute the value of Z^{old} for the noose in Figure 17 (note that the computation for the balloon is the same).

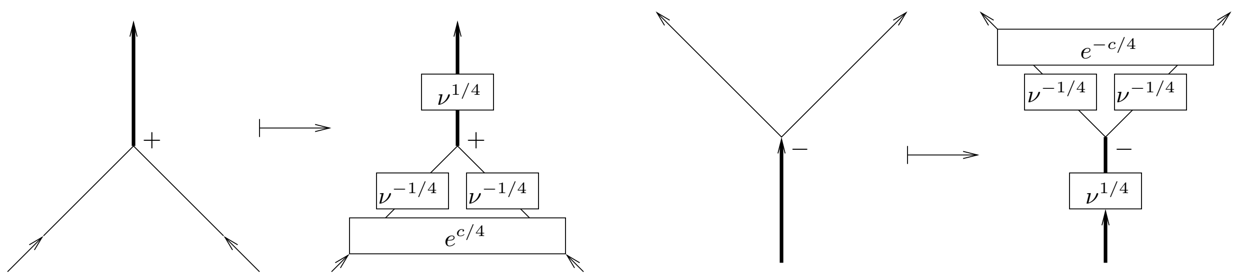
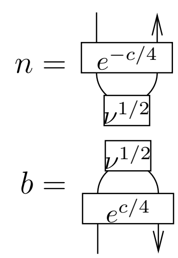


Figure 18. Normalizations for Z^u at the vertices.



Now to construct a homomorphic Z^u from Z^{old} we add normalizations near the vertices, as in Figure 18, where c denotes a single chord. Checking that Z^u is a homomorphic expansion is a simple calculation using the almost homomorphicity of Z^{old}, which we leave to the reader. The reader can also verify that Z^u of the strand and the bubble is 1 as it should be. Z^u of the right twist is $e^{c/2}$ and Z^u of the right associator is a Drinfel'd associator Φ (note that Φ depends on which Z^{old} was used). From the calculation of Figure 17 it follows that the Z^u value of the balloon and the noose (for any Z^{old}) are as shown on the right, and indeed $n \cdot b = \nu$.

4.7. **The relationship between** *sKTG* **and** *wTF***.** We move on to the question of compatibility between the homomorphic expansions Z^u and Z^w (from now on we are going to refer to the homomorphic expansion of *wTF* — called Z in the previous section — as Z^w to avoid confusion).

There is a map $a\colon sKTG \to wTF$, given by interpreting *sKTG* diagrams as *wTF* diagrams. In particular, positive vertices (of edge orientations as shown in Figure 18) are interpreted as the positive *wTF* vertex and negative vertices as the negative vertex . (The map a can also be interpreted topologically as Satoh's tubing map.) The induced map $\alpha\colon \mathcal{A}^u \to \mathcal{A}^{sw}$ is as defined in Section 3.3, that is, α maps each chord to the sum of its two possible orientations. Hence we can ask whether the two expansions are compatible (or can be chosen to be compatible), which takes us to the main result of this section:

Theorem 4.15. *Let Z^u be a homomorphic expansion for sKTG with the properties that Φ is a horizontal chord associator and $n = e^{-c/4}\nu^{1/2}$ in the sense of Section 4.6.1.*[16] *Then there exists a homomorphic expansion Z^w for wTF compatible with Z^u in the sense that the square on the right commutes.*

$$\begin{tikzcd} \mathit{sKTG} \arrow[r, "a"] \arrow[d, "Z^u"] & \mathit{wTF} \arrow[d, "Z^w"] \\ \mathcal{A}^u \arrow[r, "\alpha"] & \mathcal{A}^{sw} \end{tikzcd} \tag{18}$$

Furthermore, such Z^w are in one to one correspondence[17] *with "symmetric solutions of the KV problem" satisfying the KV equations (14), the "twist equation" (20) and the associator equation (22).*

Before moving on to the proof let us state and prove the following Lemma, to be used repeatedly in the proof of the theorem.

Lemma 4.16. *If a and b are group-like elements in $\mathcal{A}^{sw}(\uparrow_n)$, then $a = b$ if and only if $\pi(a) = \pi(b)$ and $aa^* = bb^*$. Here π is the projection induced by $\pi\colon \mathcal{P}^w(\uparrow_n) \to \mathfrak{tder}_n \oplus \mathfrak{a}_n$ (see Section 3.2), and $*$ refers to the adjoint map of Definition 3.23. In the notation of this section $*$ is applying the adjoint A on all strands.*

Proof. Write $a = e^w e^{uD}$ and $b = e^{w'}e^{uD'}$, where $w \in \mathfrak{tr}_n$, $D \in \mathfrak{tder}_n \oplus \mathfrak{a}_n$ and $u\colon \mathfrak{tder}_n \oplus \mathfrak{a}_n \to \mathcal{P}_n$ is the "upper" map of Section 3.2. Assume that $\pi(a) = \pi(b)$ and $aa^* = bb^*$. Since $\pi(a) = e^D$ and $\pi(b) = e^{D'}$, we conclude that $D = D'$. Now we compute $aa^* = e^w e^{uD} e^{-lD} e^w = e^w e^{j(D)} e^w$, where $j\colon \mathfrak{tder}_n \to \mathfrak{tr}_n$ is the map defined in Section 5.1 of [AT] and discussed in 3.24 of this paper. Now note that both w and $j(D)$ are elements of $\mathfrak{tr}_n$, hence they commute, so $aa^* = e^{2w+j(D)}$. Thus, $aa^* = bb^*$ means that $e^{2w+j(D)} = e^{2w'+j(D)}$, which implies that $w = w'$ and $a = b$. □

Proof of Theorem 4.15. In addition to being a homomorphic expansion for wTF, Z^w has to satisfy an the added condition of being compatible with Z^u. Since sKTG is finitely generated, this translates to one additional equation for each generator of sKTG, some of which are automatically satisfied. To deal with the others, we use the machinery established in the previous sections to translate these equations to conditions on F, and they turn out to be the properties studied in [AT] which link solutions of the KV problem with Drinfel'd associators.

To start, note that for the single strand and the bubble the commutativity of the square (18) is satisfied with any Z^w: both the Z^u and Z^w values are 1 (note that the Z^w value of the bubble is 1 due to the unitarity (12) of Z^w). Each of the other generators will require more study.

Commutativity of (18) for the twists. Recall that the Z^u-value of the right twist (for a Z^u with horizontal chord Φ) is $R^u = e^{c/2}$; and note that its Z^w-value is $V^{-1}RV^{21}$, where $R = e^{a_{12}}$ is the Z^w-value of the crossing (and a_{12} is a single arrow pointing from strand 1 to strand 2). Hence the commutativity of (18) for the right twist is equivalent to the "Twist Equation" $\alpha(R^u) = V^{-1}RV^{21}$. By definition of α, $\alpha(R^u) = e^{\frac{1}{2}(a_{12}+a_{21})}$, where a_{12} and a_{21} are

[16] It will become apparent that in the proof we only use slightly weaker but less aesthetic conditions on Z^u.

[17] An even nicer theorem would be a classification of homomorphic expansions for the combined algebraic structure $\left(\mathit{sKTG} \xrightarrow{a} \mathit{wTF}\right)$ in terms of solutions of the KV problem. The two obstacles to this are clarifying whether there is a free choice of n for Z^u, and — probably much harder — how much of the horizontal chord condition is necessary for a compatible Z^w to exist.

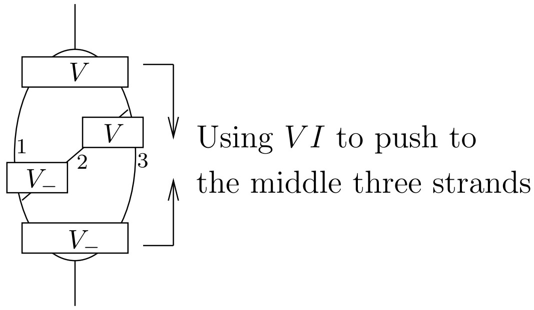


Figure 19. The Z^w-value of the right associator.

single arrows pointing from strand 1 to 2 and 2 to 1, respectively. Hence we have

$$e^{\frac{1}{2}(a_{12}+a_{21})} = V^{-1}RV^{21}. \tag{19}$$

To translate this to the language of [AT], we use Lemma 4.16, which implies that it is enough for V to satisfy the Twist Equation "on tree level" (i.e., after applying π), and for which the adjoint condition of the Lemma holds.

We first prove that the adjoint condition holds for any homomorphic expansion of $w\!T\!F$. Multiplying the left hand side of the Twist Equation by its adjoint, we get

$$e^{\frac{1}{2}(a_{12}+a_{21})}\big(e^{\frac{1}{2}(a_{12}+a_{21})}\big)^* = e^{\frac{1}{2}(a_{12}+a_{21})}e^{-\frac{1}{2}(a_{12}+a_{21})} = 1.$$

As for the right hand side, we have to compute $V^{-1}RV^{21}(V^{21})^*R^*(V^{-1})^*$. Since V is unitary (Equation (12)), $VV^* = V\cdot A_1A_2(V) = 1$. Now $R = e^{a_{12}}$, so $R^* = e^{-a_{12}} = R^{-1}$, hence the expression on the right hand side also simplifies to 1, as needed.

As for the "tree level" of the Twist Equation, recall that in Section 4.3 we used Alekseev and Torossian's solution $F \in \mathrm{TAut}_2$ to the Kashiwara–Vergne equations [AT] to find solutions V to equations (11),(12) and (13). We produced V from F by setting $F = e^{D^{21}}$ with $D \in \mathfrak{tder}_2^s$, $b := \frac{-j(F)}{2} \in \mathfrak{tr}_2$ and $V := e^b e^{uD}$, so F is "the tree part" of V, up to renumbering strands. Hence, the tree level Twist Equation translates to a new equation for F. Substituting $V = e^b e^{uD}$ into the Twist Equation we obtain $e^{\frac{1}{2}(a_{12}+a_{21})} = e^{-uD}e^{-b}e^{a_{12}}e^{b^{21}}e^{uD^{21}}$, and applying π, we get

$$e^{\frac{1}{2}(a_{12}+a_{21})} = (F^{21})^{-1}e^{a_{12}}F. \tag{20}$$

In [AT] the solutions F of the KV equations which also satisfy this equation are called "symmetric solutions of the Kashiwara-Vergne problem" discussed in Sections 8.2 and 8.3. (Note that in [AT] R denotes $e^{a_{21}}$).

Commutativity of (18) *for the associators.* Recall that the Z^u value of the right associator is a Drinfel'd associator $\Phi \in \mathcal{A}^u(\uparrow_3)$; for the Z^w value see Figure 19. Hence the new condition on V is the following:

$$\alpha(\Phi) = V_-^{(12)3}V_-^{12}V^{23}V^{1(23)} \qquad \text{in} \qquad \mathcal{A}^{sw}(\uparrow_3) \tag{21}$$

Again we treat the "tree and wheel parts" separately using Lemma 4.16. As Φ is by definition group-like, let us denote $\Phi =: e^{\phi}$. We first verify that the "wheel part" or adjoint condition of the Lemma holds. Starting with the right hand side of Equation (21), the unitarity $VV^* = 1$ of V implies that

$$V_-^{(12)3}V_-^{12}V^{23}V^{1(23)}(V^{1(23)})^*(V^{23})^*(V_-^{12})^*(V_-^{(12)3})^* = 1.$$

For the left hand side of (21) we need to show that $e^{\alpha(\phi)}(e^{\alpha(\phi)})^* = 1$ as well, and this is true for any *horizontal chord* associator. Indeed, restricted to the α-images of horizontal chords $*$ is multiplication by -1, and as it is an anti-Lie morphism, this fact extends to the Lie algebra generated by α-images of horizontal chords. Hence $e^{\alpha(\phi)}(e^{\alpha(\phi)})^* = e^{\alpha(\phi)}e^{\alpha(\phi)^*} = e^{\alpha(\phi)}e^{-\alpha(\phi)} = 1$.

On to the tree part, applying π to Equation (21) and keeping in mind that $V_- = V^{-1}$ by the unitarity of V, we obtain

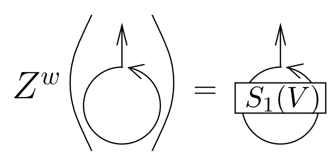
$$e^{\pi\alpha(\phi)} = (F^{3(12)})^{-1}(F^{21})^{-1}F^{32}F^{(23)1} = e^{-D^{(12)3}}e^{-D^{12}}e^{D^{23}}e^{D^{1(23)}} \quad \text{in } \mathrm{SAut}_3 := \exp(\mathfrak{sder}_3) \subset \mathrm{TAut}_3 \,. \tag{22}$$

This is Equation (26) of [AT], up to re-numbering strands 1 and 2 as 2 and 1[18]. The following fact from [AT] (their Theorem 7.5, Propositions 9.2 and 9.3 combined) implies that there is a solution F to the KV equations (14) which also satisfies (20) and (22).

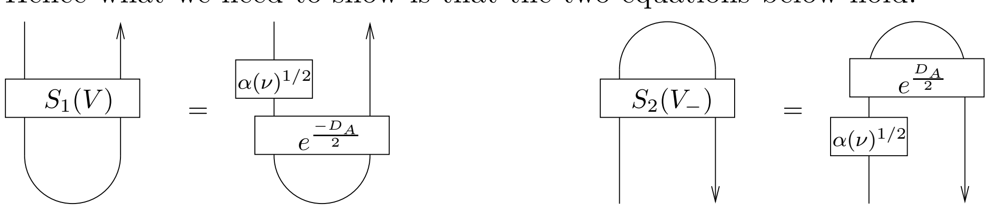
Fact 4.17. *If $\Phi' = e^{\phi'}$ is an associator in SAut_3 so that $j(\Phi') = 0$[19] then Equation (22) has a solution $F = e^{D^{21}}$ which is also a solution to the KV equations, and all such solutions are symmetric (i.e. verify the Twist Equation (20)).* □

To use this Fact, we need to show that $\Phi' := \pi\alpha(\Phi)$ is an associator in SAut_3 and that $j(\Phi') = j(\pi\alpha(\Phi)) = 0$. The latter is the unitarity of Φ which is already proven. The former follows from the fact that Φ is an associator and the fact (Theorem 3.25) that the image of $\pi\alpha$ is contained in $\mathfrak{sder}$ (ignoring degree 1 terms, which are not present in an associator anyway).

In summary, the condition of the Fact are satisfied and so there exists a solution F which in turn induces a Z^w which is compatible with Z^u for the strand, the bubble, the twists and the associators. That is, all generators of *sKTG* except possibly the balloon and the noose. As the last step of the proof of Theorem 4.15 we show that any such Z^w also automatically make (18) commutative for the balloon and the noose.



Commutativity of (18) for the balloon and the noose. Since we know the Z^u-values B and n of the balloon and the noose, we start by computing Z^w of the noose. Z^w assigns a V value to the vertex with the first strand orientation switched as shown in the figure on the right. The balloon is the same, except with a negative vertex and the second strand reversed. Hence what we need to show is that the two equations below hold:



[18] Note that in [AT] "Φ' is an associator" means that Φ' satisfies the pentagon equation, mirror skew-symmetry, and positive and negative hexagon equations in the space SAut_3. These equations are stated in [AT] as equations (25), (29), (30), and (31), and the hexagon equations are stated with strands 1 and 2 re-named to 2 and 1 as compared to [Dr2] and [BND]. This is consistent with $F = e^{D^{21}}$.

[19] The condition $j(\phi') = 0$ is equivalent to the condition $\Phi \in KRV_3^0$ in [AT]. The relevant definitions in [AT] can be found in Remark 4.2 and at the bottom of page 434 (before Section 5.2).

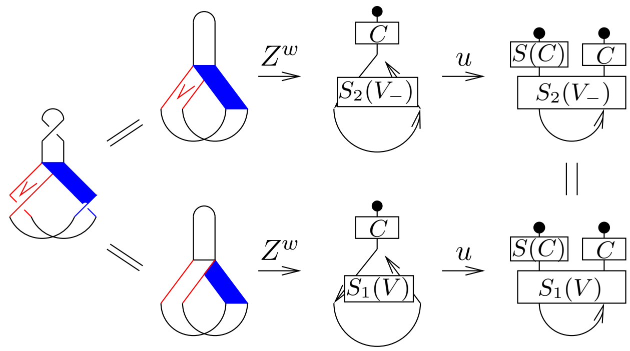


Figure 20. The proof of Equation (24). Note that the unzips are "illegal", as the strand directions don't match. This can be fixed by inserting a small bubble at the bottom of the noose and doing a number of orientation switches. As this doesn't change the result or the main argument, we suppress the issue for simplicity. Equation (24) is obtained from this result by multiplying by $S(C)^{-1}$ on the bottom and by C^{-1} on the top.

Let us denote the left hand side of the first equation above by n^w and b^w (the Z^w value of the noose and the balloon, respectively). We will start by proving that the product of these two equations holds, namely that $n^w b^w = \alpha(\nu)$. (We used that any local (small) arrow diagram on a single strand is central in $\mathcal{A}^{sw}(\uparrow_n)$, hence the cancellations.) This product equation is satisfied due to an argument identical to that of Figure 16, but carried out in *wTF*, and using that by the compatibility with associators, Z^w of an associator is $\alpha(\Phi)$.

What remains is to show that the noose and balloon equations hold individually. In light of the results so far, it is sufficient to show that

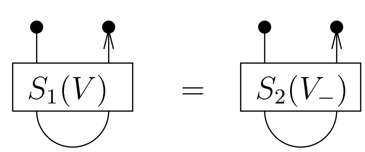
$$n^w = b^w \cdot e^{-D_A}, \tag{23}$$

where D_A stands for a single arrow on one strand (whose direction doesn't matter due to the *RI* relation. As stated in [WKO1, Theorem 3.15], $\mathcal{A}^{sw}(\uparrow_1)$ is the polynomial algebra freely generated by the arrow D_A and wheels of degrees 2 and higher. Since V is group-like, n^w (resp. b^w) is an exponential e^{A_1} (resp. e^{A_2}) with $A_1, A_2 \in \mathcal{A}^{sw}(\uparrow_1)$. We want to show that $e^{A_1} = e^{A_2} \cdot e^{-D_A}$, equivalently that $A_1 = A_2 - D_A$.

In degree 1, this can be done by explicit verification. Let $A_1^{\geq 2}$ and $A_2^{\geq 2}$ denote the degree 2 and higher parts of A_1 and A_2, respectively. We claim that capping the strand at both its top and its bottom takes e^{A_1} to $e^{A_1^{\geq 2}}$, and similarly e^{A_2} to $e^{A_2^{\geq 2}}$. (In other words, capping kills arrows but leaves wheels un-changed.) This can be proven similarly to the proof of Lemma 4.6, but using

$$F' := \sum_{k_1, k_2 = 0}^{\infty} \frac{(-1)^{k_1+k_2}}{k_1! k_2!} D_A^{k_1+k_2} S_L^{k_1} S_R^{k_2}$$

in place of F in the proof. What we need to prove, then, is the following equality, and the proof is shown in Figure 20.



. (24)

This concluses the proof of Theorem 4.15. $\square$

Recall from Section 3.3 that there is no commutative square linking $Z^u\colon u\mathcal{T} \to \mathcal{A}^u$ and $Z^w\colon w\mathcal{T} \to \mathcal{A}^{sw}$, for the simple reason that the Kontsevich integral for tangles Z^u is not canonical, but depends on a choice of parenthesizations for the "bottom" and the "top" strands of a tangle T. Yet given such choices, a tangle T can be "closed up with trees" as within the proof of Proposition 4.13 (see Section 5) into an *sKTG* which we will denote G. For G a commutativity statement does hold as we have just proven. The Z^u and Z^w invariants of T and of G differ only by a number of vertex-normalizations and vertex-values on skeleton-trees at the bottom or at the top of G, and using VI, these values can slide so they are placed on the original skeleton of T. This is summarized as the following proposition:

Proposition 4.18. *Let n and n' be natural numbers. Given choices c and and c' of parenthesizations of n and n' strands respectively, there exists invertible elements $C \in \mathcal{A}^{sw}(\uparrow_n)$ and $C' \in \mathcal{A}^{sw}(\uparrow_{n'})$ so that for any u-tangle T with n "bottom" ends and n' "top" ends we have*

$$\alpha Z^u_{c,c'}(T) = C^{-1} Z^w(aT) C',$$

where $Z^u_{c,c'}$ denotes the usual Kontsevich integral of T with bottom and top parenthesizations c and c'.

For u-braids the above proposition may be stated with $c = c'$ and then C and C' are the same.

5. Odds and Ends

5.1. **Motivation for circuit algebras: electronic circuits.** Electronic circuits are made of "components" that can be wired together in many ways. On a logical level, we only care to know which pin of which component is connected with which other pin of the same or other component. On a logical level, we don't really need to know how the wires between those pins are embedded in space (see Figures 21 and 22). "Printed Circuit Boards" (PCBs) are operators that make smaller components ("chips") into bigger ones ("circuits") — logically speaking, a PCB is simply a set of "wiring instructions", telling us which pins on which components are made to connect (and again, we never care precisely how the wires are routed provided they reach their intended destinations, and ever since the invention of multi-layered PCBs, all conceivable topologies for wiring are actually realizable). PCBs can be composed (think "plugging a graphics card onto a motherboard"); the result of a composition of PCBs, logically speaking, is simply a larger PCB which takes a larger number of components as inputs and outputs a larger circuit. Finally, it doesn't matter if several PCB are connected together and then the chips are placed on them, or if the chips are placed first and the PCBs are connected later; the resulting overall circuit remains the same.

5.2. **Proof of Proposition 4.13.** We are going to ignore strand orientations throughout this proof for simplicity. This is not an issue as orientation switches are allowed in *sKTG* without restriction. We are also going to omit vertex signs from the pictures given the pictorial convention stated in Section 4.6.

We need to prove that any *sKTG* (call it G) can be built from the generators listed in the statement of the proposition, using *sKTG* operations. To show this, consider a Morse drawing of G, that is, a planar projection of G with a height function so that all singularities along the strands are Morse and so that every "feature" of the projection (local minima and maxima, crossings and vertices) occurs at a different height.

Figure 21. The J-K flip flop, a very basic memory cell, is an electronic circuit that can be realized using 9 components — two triple-input "and" gates, two standard "nor" gates, and 5 "junctions" in which 3 wires connect (many engineers would not consider the junctions to be real components, but we do). Note that the "crossing" in the middle of the figure is merely a projection artifact and does not indicate an electrical connection, and that electronically speaking, we need not specify how this crossing may be implemented in $\mathbb{R}^3$. The J-K flip flop has 5 external connections (labelled J, K, CP, Q, and Q') and hence in the circuit algebra of computer parts, it lives in C_5. In the directed circuit algebra of computer parts it would be in $C_{3,2}$ as it has 3 incoming wires (J, CP, and K) and two outgoing wires (Q and Q').

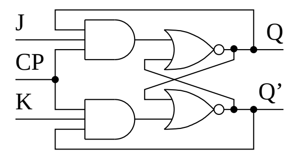
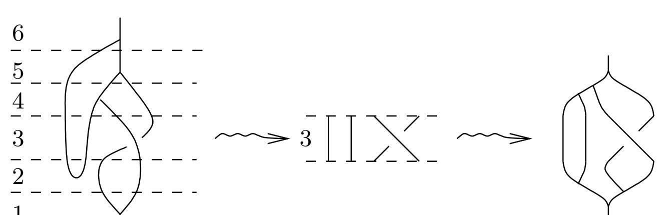


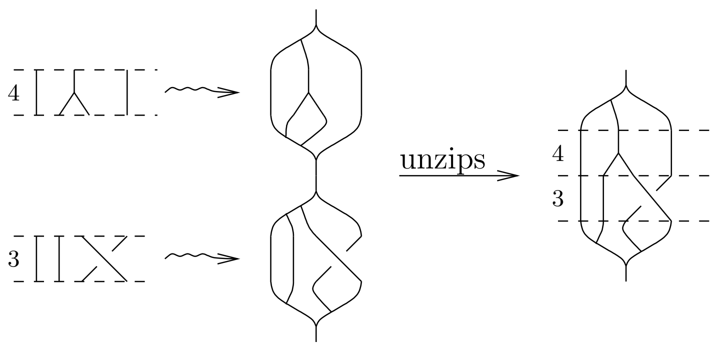
Figure 22. The circuit algebra product of 4 big black components and 1 small black component carried out using a green wiring diagram, is an even bigger component that has many golden connections (at bottom). When plugged into a yet bigger circuit, the CPU board of a laptop, our circuit functions as 4,294,967,296 binary memory cells.

The idea in short is to decompose G into levels of this Morse drawing where at each level only one "feature" occurs. The levels themselves are not *sKTG*'s, but we show that the composition of the levels can be achieved by composing their "closed-up" *sKTG* versions followed by some unzips. Each feature gives rise to a generator by "closing up" extra ends at its top and bottom. We then show that we can construct each level using the generators and the tangle insert operation.

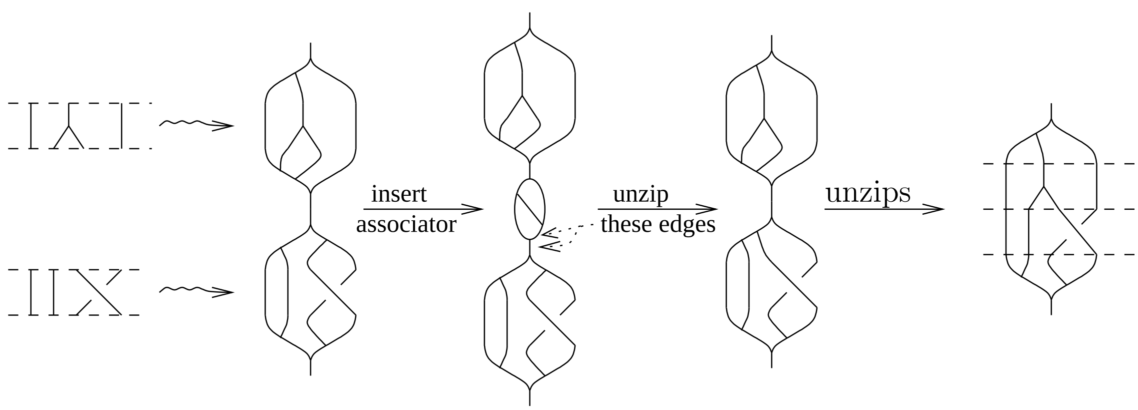
So let us decompose G into a composition of trivalent tangles ("levels"), each of which has one "feature" and (possibly) some straight vertical strands. Note that by isotopy we can make sure that every level has strands ending at both its bottom and top, except for the first or the last level in the case of 1-tangles. An example of level decomposition is shown in the figure below. Note that the levels are generally not elements of *sKTG* (have too many ends). However, we can turn each of them into a $(1,1)$-tangle (or a 1-tangle in case of the aforementioned top first or last levels) by "closing up" their tops and bottoms by arbitrary trees. In the example below we show this for one level of the Morse-drawn *sKTG* containing a crossing and two vertical strands.



Now we can compose the *sKTG*'s obtained from closing up each level. Each tree that we used to close up the tops and bottoms of levels determines a "parenthesization" of the strand endings. If these parenthesizations match on the top of each level with the bottom of the next, then we can recreate tangle composition of the levels by composing their closed versions followed by a number of unzips performed on the connecting trees. This is illustrated in the example below, for two consecutive levels of the *sKTG* of the previous example.

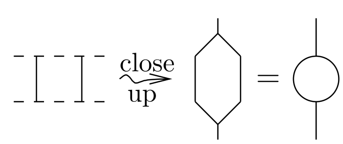


If the trees used to close up consecutive levels correspond to different parenthesizations, then we can use insertion of the left and right associators (the 5th and 6th pictures of the list of generators in the statement of the theorem) to change one parenthesization to match the other. This is illustrated in the figure below.

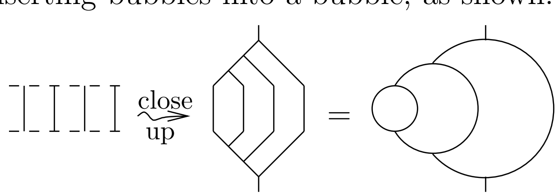


So far we have shown that G can be assembled from closed versions of the levels in its Morse drawing. The closed versions of the levels of G are simpler *sKTG*'s, and it remains to show that these can be obtained from the generators using *sKTG* operations.

Let us examine what each level might look like. First of all, in the absence of any "features" a level might be a single strand, in which case it is the first generator itself. Two parallel strands when closed up become the "bubble", as shown on the right.

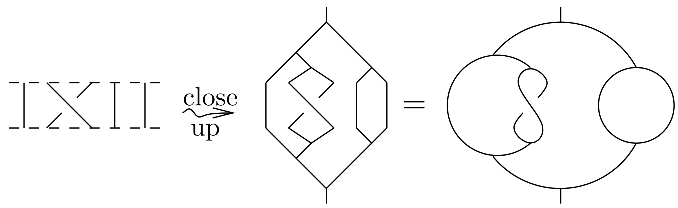


Now suppose that a level consists of n parallel strands, and that the trees used to close it up on the top and bottom are horizontal mirror images of each other, as shown below (if not, then this can be achieved by associator insertions and unzips). We want to show that this *sKTG* can be obtained from the generators using *sKTG* operations. Indeed, this can be achieved by repeatedly inserting bubbles into a bubble, as shown:



A level consisting of a single crossing becomes a left or right twist when closed up (depending on the sign of the crossing). Similarly, a single vertex becomes a bubble. A single minimum or maximum becomes a noose or a balloon, respectively.

It remains to see that the *sKTG*'s obtained when closing up simple features accompanied by more through strands can be built from the generators. A minimum accompanied by an extra strand gives rise to the *sKTG* obtained by sticking a noose onto a vertical strand (similarly, a balloon for a maximum). In the case of all the other simple features and for minima and maxima accompanied by more strands, we inserting the already generated elements into nested bubbles (bubbles inserted into bubbles), as in the example shown below. This completes the proof.



□

6. Glossary of notation

Greek letters, then Latin, then symbols:

δ	Satoh's tube map	3.4
Δ	co-product	3.2
ι	inclusion $\mathfrak{tr}_n \to \mathcal{P}^w(\uparrow_n)$	3.2
ν	the invariant of the unknot	4.6
π	the projection $\mathcal{P}^w(\uparrow_n) \to \mathfrak{a}_n \oplus \mathfrak{tder}_n$	3.2
ϕ	log of an associator	4.6
Φ	an associator	4.6
ψ_β	"operations"	2.1
$\mathfrak{a}_n$	n-dimensional Abelian Lie algebra	3.2
$\mathcal{A}$	a candidate associated graded structure	2.3
$\mathcal{A}^{sv}$	$\mathcal{D}^v$ mod 6T, RI	3.1
$\mathcal{A}^{sw}$	$\mathcal{D}^w$ mod $\overrightarrow{4T}$, TC, RI	3.1
$\mathcal{A}^{sw}$	grad *wTF*^{o}	4.2
$\mathcal{A}^{sw}$	grad *wTF*	4.5.2
$\mathcal{A}^{(s)w}$	$\mathcal{A}^w$ and/or $\mathcal{A}^{sw}$	4.2
$\mathcal{A}^u$	chord diagrams mod rels for KTGs	4.6
$\mathcal{A}^v$	$\mathcal{D}^v$ mod 6T	3.1
$\mathcal{A}^w$	$\mathcal{D}^w$ mod $\overrightarrow{4T}$, TC	3.1
$\mathcal{A}^w$	grad *wTF*^{o} without RI	4.2
$\mathcal{A}^-(\uparrow_n)$	$\mathcal{A}^-$ for pure n-tangles	3.2
A_e	1D orientation reversal	4.1.3
Ass	associative words	3.2
Ass$^+$	non-empty associative words	3.2
$\mathcal{B}^w_n$	n-coloured unitrivalent arrow diagrams	3.2
C	the invariant of a cap	4.3
CP	the Cap-Pull relation	4.1.2, 4.2
CW	Cap-Wen relations	4.5.1
c	a chord in $\mathcal{A}^u$	4.6
$\mathfrak{der}$	Lie-algebra derivations	3.2
$\mathcal{D}^v$, $\mathcal{D}^w$	arrow diagrams for v/w-tangles	3.1
div	the "divergence"	3.2
F	a map $\mathcal{A}^w \to \mathcal{A}^w$	4.2
F	the main [AT] unknown	4.4
FR	Flip Relations	4.5.1, 4.5.2
fil	a filtered structure	2.3
$\mathcal{I}$	augmentation ideal	2.2
J	a map $\mathrm{TAut}_n \to \exp(\mathfrak{tr}_n)$	3.2
j	a map $\mathrm{TAut}_n \to \mathfrak{tr}_n$	3.2
KTG	Knotted Trivalent Graphs	4.6
$\mathfrak{lie}_n$	free Lie algebra	3.2
l	a map $\mathfrak{tder}_n \to \mathcal{P}^w(\uparrow_n)$	3.2
$\mathcal{O}$	an "algebraic structure"	2.1
$\mathcal{P}^w_n$	primitives of $\mathcal{B}^w_n$	3.2
$\mathcal{P}^-(\uparrow_n)$	primitives of $\mathcal{A}^-(\uparrow_n)$	3.2
grad	associated graded structure	2.2
R	the invariant of a crossing	4.3
R4	a Reidemeister move for foams/graphs	4.1.2

$\mathfrak{sder}$	special derivations	3.3
$\mathcal{S}$	the circuit algebra of skeletons	2.4
SAut_n	the group $\exp(\mathfrak{sder}_n)$	4.6
S_k	complete orientation reversal	3.5
S_e	complete orientation reversal	4.1.3
$sKTG$	signed long KTGs	4.6
TV	Twisted Vertex relations	4.5.1
$\mathfrak{tder}$	tangential derivations	3.2
$\mathfrak{tr}_n$	cyclic words	3.2
$\mathfrak{tr}_n^s$	cyclic words mod degree 1	3.2
TAut_n	the group $\exp(\mathfrak{tder}_n)$	3.2
u	a map $\mathfrak{tder}_n \to \mathcal{P}^w(\uparrow_n)$	3.2
u_e	strand unzips	4.1.3
$u\mathcal{T}$	u-tangles	3.3
V, V^+	the invariant of a (positive) vertex	4.3
V^-	the invariant of a negative vertex	4.3
II	Vertex Invariance	4.2
$v\mathcal{T}$	v-tangles	3.1
$v\mathcal{T}(\uparrow_n)$	pure n-component v-tangles	3.2
W	$Z(w)$	4.5.3
W^2	Wen squared	4.5.1
w	the wen	4.5
$w\mathcal{T}$	w-tangles	3.1
$w\mathcal{T}(\uparrow_n)$	pure n-component w-tangles	3.2
$w\mathcal{TF}$	w-tangled foams with wens	4.5
$w\mathcal{TF}^o$	orientable w-tangled foams	4.1
Z	expansions	throughout
$Z_{\mathcal{A}}$	an $\mathcal{A}$-expansion	2.3
4T	$4T$ relations	4.6
$\uparrow$	a "long" strand	throughout
$\uparrow$	the quandle operation	2.1
$*$	the adjoint on $\mathcal{A}^w(\uparrow_n)$	3.2

References

[AT] A. Alekseev and E. Meinrenken, *The Kashiwara-Vergne conjecture and Drinfel'd's associators,* Inventiones Mathematicae, **164** (2006) 615–634, arXiv:0506499.

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfel'd's associators,* Annals of Mathematics, **175** (2012) 415–463, arXiv:0802.4300.

[AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfel'd's associators, braid groups and an explicit solution of the Kashiwara-Vergne equations,* Publications Mathématiques de L'IHÉS, **112-1** (2010) 143–189, arXiv:0903.4067.

[BN1] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology **34** (1995) 423–472.

[BN2] D. Bar-Natan, *Non-associative tangles,* in *Geometric topology* (proceedings of the Georgia international topology conference), (W. H. Kazez, ed.), 139–183, Amer. Math. Soc. and International Press, Providence, 1997.

[BN3] D. Bar-Natan, *On Associators and the Grothendieck-Teichmuller Group I,* Selecta Mathematica, New Series **4** (1998) 183–212.

[BN4] D. Bar-Natan, *Algebraic Knot Theory — A Call for Action,* web document, 2006, `http://www.math.toronto.edu/~drorbn/papers/AKT-CFA.html`.

[BND] D. Bar-Natan and Z. Dancso, *Homomorphic expansions for knotted trivalent graphs*, Journal of Knot Theory and its Ramifications Vol. **22**, No. 1 (2013) arXiv:1103.1896

[BGRT] D. Bar-Natan, S. Garoufalidis, L. Rozansky and D. P. Thurston, *Wheels, wheeling, and the Kontsevich integral of the unknot,* Israel Journal of Mathematics **119** (2000) 217–237, arXiv:q-alg/9703025.

[BHLR] D. Bar-Natan, I. Halacheva, L. Leung, and F. Roukema, *Some Dimensions of Spaces of Finite Type Invariants of Virtual Knots,* submitted.

[BLT] D. Bar-Natan, T. Q. T. Le, and D. P. Thurston, *Two applications of elementary knot theory to Lie algebras and Vassiliev invariants,* Geometry and Topology **7-1** (2003) 1–31, arXiv:math.QA/0204311.

[BP] B. Berceanu and S. Papadima, *Universal Representations of Braid and Braid-Permutation Groups,* J. of Knot Theory and its Ramifications **18-7** (2009) 973–983, arXiv:0708.0634.

[BH] T. Brendle and A. Hatcher, *Configuration Spaces of Rings and Wickets,* arXiv:0805.4354.

[CL] D. Cheptea and T. Q. T. Le: *A TQFT associated to the LMO invariant of three-dimensional manifolds,* Commun. Math. Physics **272** (2007) 601–634

[CS] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams,* Mathematical Surveys and Monographs **55**, American Mathematical Society, Providence 1998.

[D] D. M. Dahm, *A generalization of braid theory*, PhD Thesis, Princeton university, 1962.

[Da] Z. Dancso, *On a Kontsevich Integral for Knotted Trivalent Graphs,* in *Algebraic and Geometric Topology* **10** (2010) 1317–1365, arXiv:0811.4615.

[Dr1] V. G. Drinfel'd, *Quantum Groups,* in *Proceedings of the International Congress of Mathematicians,* 798–820, Berkeley, 1986.

[Dr2] V. G. Drinfel'd, *Quasi-Hopf Algebras,* Leningrad Math. J. **1** (1990) 1419–1457.

[Dr3] V. G. Drinfel'd, *On Quasitriangular Quasi-Hopf Algebras and a Group Closely Connected with* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$*,* Leningrad Math. J. **2** (1991) 829–860.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I,* Selecta Mathematica, New Series **2** (1996) 1–41, arXiv:q-alg/9506005.

[FRR] R. Fenn, R. Rimanyi, and C. Rourke, *The Braid-Permutation Group,* Topology **36** (1997) 123–135.

[Gol] D. L. Goldsmith, *The Theory of Motion Groups,* Mich. Math. J. **28-1** (1981) 3–17.

[Jon] V. Jones, *Planar algebras, I,* New Zealand Journal of Mathematics, to appear, arXiv:math.QA/9909027.

[Joy] D. Joyce, *A Classifying Invariant of Knots, the Knot Quandle,* Journal of Pure and Appl. Algebra **23** (1982) 37–65.

[KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff Formula and Invariant Hyperfunctions,* Invent. Math. **47** (1978) 249–272.

[Ka] L. H. Kauffman, *Virtual Knot Theory,* European J. Comb. **20** (1999) 663–690, arXiv:math.GT/9811028.

[Kup] G. Kuperberg, *What is a Virtual Link?,* Algebr. Geom. Topol. **3** (2003) 587–591, arXiv:math.GT/0208039.

[LM] T. Q. T. Le and J. Murakami, *The universal Vassiliev-Kontsevich invariant for framed oriented links,* Compositio Math. **102** (1996) 41–64, arXiv:hep-th/9401016.

[Lei] Tom Leinster, *Higher Operads, Higher Categories,* London Mathematical Society Lecture Note Series **298**, Cambridge University Press, ISBN 0-521-53215-9, arXiv:math.CT/0305049.

[Lev] J. Levine, *Addendum and Correction to: "Homology Cylinders: an Enlargement of the Mapping Class Group,* Alg. Geom. Top. **2** (2002), 1197–1204, arXiv:math.GT/0207290

[Lod] J-L. Loday, *Une version non commutative des algebres de Lie: des algebres de Leibniz,* Enseign. math. (2) **39** (3-4): 269–293.

[Mc] J. McCool, *On Basis-Conjugating Automorphisms of Free Groups,* Can. J. Math. **38-6** (1986) 1525–1529.

[MM] J. Milnor and J. Moore, *On the structure of Hopf algebras,* Annals of Math. **81** (1965) 211–264.

[MO] J. Murakami, and T. Ohtsuki, *Topological quantum field theory for the universal quantum invariant,* Communications in Mathematical Physics **188** 3 (1997) 501–520.

[Sa] S. Satoh, *Virtual Knot Presentations of Ribbon Torus Knots,* J. of Knot Theory and its Ramifications **9-4** (2000) 531–542.

[WKO0] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne,* earlier web version of the first two papers of this series in one. Paper, videos (wClips) and related files at `http://www.math.toronto.edu/~drorbn/papers/WKO/`. The arXiv:1309.7155 edition may be older.

[WKO1] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: Braids, Knots and the Alexander Polynomial,* `http://www.math.toronto.edu/drorbn/LOP.html#WKO1`, arXiv:1405.1956.

[WKO3] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects III: the Double Tree Construction,* in preparation.

Department of Mathematics, University of Toronto, Toronto Ontario M5S 2E4, Canada
E-mail address: `drorbn@math.toronto.edu, zsuzsi@math.toronto.edu`
URL: `http://www.math.toronto.edu/~drorbn, http://www.math.toronto.edu/zsuzsi`