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Finite type invariants of w -knotted objects II: tangles, foams and the Kashiwara–Vergne problem

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Abstract This is the second in a series of papers dedicated to studying w -knots, and more generally, w -knotted objects (w -braids, w -tangles, etc.). These are classes of knotted objects that are wider but weaker than their “usual” counterparts. To get (say) w -knots from usual knots (or u -knots), one has to allow non-planar “virtual” knot diagrams, hence enlarging the base set of knots. But then one imposes a new relation beyond the ordinary collection of Reidemeister moves, called the “overcrossings commute” relation, making w -knotted objects a bit weaker once again. Satoh (J. Knot Theory Ramif. 9-4:531–542, 2000) studied several classes of w -knotted objects (under the name “weakly-virtual”) and has shown them to be closely related to certain classes of knotted surfaces in \mathbb{R}^4 . In this article we study finite type invariants of w -tangles and w -trivalent graphs (also referred to as w -tangled foams). Much as the spaces \mathcal{A} of chord diagrams for ordinary knotted objects are related to metrized Lie algebras, the spaces \mathcal{A}^w of “arrow diagrams” for w -knotted objects are related to not-necessarily-metrized Lie algebras. Many questions concerning w -knotted objects turn out to be equivalent to questions about Lie algebras. Most notably we find that

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 20 w-foams.

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50 1 Introduction

51 This is the second in a series of papers on w-knotted objects. In the first paper [38], we
 52 took a classical approach to studying finite type invariants of w-braids and w-knots and
 53 proved that the universal finite type invariant for w-knots is essentially the Alexander
 54 polynomial. In this paper we will study finite type invariants of w-tangles and w-
 55 tangled foams from a more algebraic point of view, and prove that “homomorphic”
 56 universal finite type invariants of w-tangled foams are in one-to-one correspondence
 57 with solutions to the (Alekseev–Torossian version of) the Kashiwara–Vergne problem
 58 in Lie theory. Mathematically, this paper does not depend on the results of [38] in
 59 any significant way, and the reader familiar with the theory of finite type invariants
 60 will have no difficulty reading this paper without having read [38]. However, since

61 this paper starts with an abstract re-phrasing of the well-known finite type story in
 62 terms of general algebraic structures, readers who need an introduction to finite type
 63 invariants may find it more pleasant to read [38] first (especially Sects. 1, 2 and 3.1–
 64 3.5).

65 1.1 Motivation and hopes

66 This article and its siblings [38,39] are efforts towards a larger goal. Namely, we
 67 believe many of the difficult algebraic equations in mathematics, especially those that
 68 are written in graded spaces, more especially those that are related in one way or
 69 another to quantum groups [18], and to the work of Etingof and Kazhdan [21], can be
 70 understood, and indeed would appear more natural, in terms of finite type invariants
 71 of various topological objects.

72 This work was inspired by Alekseev and Torossian’s results [2] on Drinfel’d associ-
 73 ators and the Kashiwara–Vergne conjecture, both of which fall into the aforementioned
 74 class of “difficult equations in graded spaces”. The Kashiwara–Vergne conjecture—
 75 proposed in 1978 [26] and proven in 2006 by Alekseev and Meinrenken [1]—has
 76 strong implications in Lie theory and harmonic analysis, and is a cousin of the Dufo
 77 isomorphism, which was shown to be knot-theoretic in [11]. We also know that
 78 Drinfel’d’s theory of associators [19] can be interpreted as a theory of well-behaved
 79 universal finite type invariants of parenthesized tangles¹ [5,29], or of knotted trivalent
 80 graphs [17].

81 In Sect. 4 we will re-interpret the Kashiwara–Vergne conjecture as the problem of
 82 finding a “homomorphic” universal finite type invariant of a class of w -knotted trivalent
 83 graphs (more accurately named w -tangled foams). This result fits into a bigger picture
 84 incorporating usual, virtual and w -knotted objects and their theories of finite type
 85 invariants, connected by the inclusion map from usual to virtual, and the projection
 86 from virtual to w -knotted objects. In a sense that will be made precise in Sect. 2,
 87 usual and w -knotted objects with this mapping form a unified algebraic structure, and
 88 the mysterious relationship between Drinfel’d associators and the Kashiwara–Vergne
 89 conjecture is explained as a theory of finite type invariants for this larger structure.
 90 This will be the topic of Sect. 4.6.

91 We are optimistic that this paper is a step towards re-interpreting the work of
 92 Etingof and Kazhdan [21] on quantization of Lie bi-algebras as a construction of a
 93 well-behaved universal finite type invariant of virtual knots [27] or of a similar class
 94 of virtually knotted objects. However, w -knotted objects are quite interesting in their
 95 own right, both topologically and algebraically: they are related to combinatorial group
 96 theory, to groups of movies of flying rings in \mathbb{R}^3 , and more generally, to certain classes
 97 of knotted surfaces in \mathbb{R}^4 . The references include [13,22,23,33,36].

98 In [38] we studied the universal finite type invariants of w -braids and w -knots, the
 99 latter of which turns out to be essentially the Alexander polynomial. A more thorough
 100 introduction about our “hopes and dreams” and the u - v - w big picture can also be found
 101 in [38].

¹ “ q -tangles” in [29], “non-associative tangles” in [5].

1.2 A brief overview and large-scale explanation

We are going to start by developing the algebraic ingredients of the paper in Sect. 2. The general notion of an *algebraic structure* lets us treat spaces of a topological or diagrammatic nature in a unified algebraic manner. All of braids, w-braids, w-knots, w-tangles, etc., and their associated chord- or arrow-diagrammatic counterparts form algebraic structures, and so do any number of these spaces combined, with maps between them.

We then introduce *associated graded structures* with respect to a specific filtration, the machine which in our case takes an algebraic structure of “topological nature” (say, braids with n strands) and produces the corresponding diagrammatic space (for braids, horizontal chord diagrams on n vertical strands). This is done by taking the associated graded space with respect to a given filtration, namely the powers of the augmentation ideal in the algebraic structure.

An *expansion*, sometimes called a universal finite type invariant, is a map from an algebraic structure (in this case one of topological nature) to its associated graded (a structure of combinatorial/diagrammatic nature), with a certain non-degeneracy property. A *homomorphic expansion* is one that is in addition “well behaved” with respect to the *operations* of the algebraic structure (such as composition and strand doubling for braids, for example).

The three main results of the paper are as follows:

- (1) As mentioned before, our goal is to provide a topological framework for the Kashiwara–Vergne (KV) problem. The first result in that direction is Theorem 4.9, in which we establish a bijection between certain homomorphic expansions of *w-tangled foams* (introduced in Sect. 4) and solutions of the Kashiwara–Vergne equations. More precisely, “certain” homomorphic expansions means ones that are group-like (a commonly used condition), and subject to another very minor technical condition. Section 3 leads up to this result by studying the simpler case of *w-tangles* and identifying building blocks of its associated graded structure as the spaces which appear in the [2] formulation of the KV equations.
- (2) In Theorem 4.11 we study an unoriented version of w-tangled foams, and prove that homomorphic expansions for this space (group-like and subject to the same minor condition) are in one-to-one correspondence with solutions to the KV problem with *even Duflo function*. This sets the stage for perhaps the most interesting result of the paper:
- (3) Section 4.7 marries the theory above with the theory of ordinary (not w-) knotted trivalent graphs (KTGs). For technical reasons explained in Sect. 4, we work with a signed version of KTGs (sKTG). Roughly speaking, homomorphic expansions for sKTGs are determined by a *Drinfel’d associator*. Furthermore, sKTGs map naturally into w-tangled foams.

In Theorem 4.15 we prove that any homomorphic expansion of sKTGs coming from a *horizontal chord* associator has a compatible homomorphic expansion of w-tangled foams, and furthermore, these expansions are in one-to-one correspondence with *symmetric* solutions of the KV problem. This gives a topological explanation for the relationship between Drinfel’d associators and the KV conjecture.

We note that in [39] we'll further capitalize on these insights to provide a topological proof and interpretation for Alekseev, Enriquez and Torossian's explicit solutions for the KV conjecture in terms of associators [3].

Several of the structures of a topological nature in this paper (w-tangles and w-foams) are introduced as *Reidemeister theories*. That is, the spaces are built from pictorial generators (such as crossings) which can be connected arbitrarily, and the resulting pictures are then factored out by certain relations ("Reidemeister moves"). Technically speaking, this is done using the framework of *circuit algebras* (similar to planar algebras but without the planarity requirement) which are introduced in Sect. 2.

One of the fundamental theorems of classical knot theory is Reidemeister's theorem, which states that isotopy classes of knots are in bijection with *knot diagrams* modulo Reidemeister moves. In our case, w-knotted objects have a Reidemeister description and a topological interpretation in terms of ribbon knotted tubes in \mathbb{R}^4 . However, the analogue of the Reidemeister theorem, i.e. the statement that these two interpretations coincide, is only known for w-braids [13, 16, 33].

For w-tangles and w-foams (and w-knots as well) there is a map δ from the Reidemeister presentation to the appropriate class of ribbon 2-knotted objects in \mathbb{R}^4 . In our case this means that all the generators have a local topological interpretation and the relations represent isotopies. The map δ is certainly a surjection, but it is only conjectured to be injective (in other words, it is possible that some relations are missing).

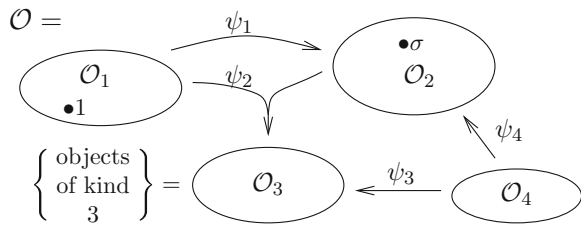
The main difficulty in proving the injectivity of δ lies in the management of the ribbon structure. A ribbon 2-knot is a knotted sphere or long tube in \mathbb{R}^4 which admits a filling with only certain types of singularities. While there are Reidemeister theorems for general 2-knots in \mathbb{R}^4 [15], the techniques don't translate well to ribbon 2-knots, mainly because it is not well understood how different ribbon structures (fillings) of the same ribbon 2-knot can be obtained from each other through Reidemeister type moves. The completion of such a theorem would be of great interest. We suspect that even if δ is not injective, the present set of generators and relations describes a set of ribbon-knotted tubes in \mathbb{R}^4 with possibly some extra combinatorial information, similarly to how, say, dropping the *R1* relation in classical knot theory results in a Reidemeister theory for framed knots with rotation numbers.

The paper is organized as follows: we start with a discussion of general algebraic structures, associated graded structures, expansions (universal finite type invariants) and "circuit algebras" in Sect. 2. In Sect. 3 we study w-tangles and identify some of the spaces [2] where the KV conjecture "lives" as the spaces of "arrow diagrams" (the w-analogue of chord diagrams) for certain w-tangles. In Sect. 4 we study w-tangled foams and we prove the main theorems discussed above. For more detailed information consult the "Section summary" paragraphs at the beginning of each of the sections. A glossary of notation is on page 56.

2 Algebraic structures, expansions, and circuit algebras

Section summary In this section we introduce the associated graded structure of an "arbitrary algebraic structure" with respect to powers of its augmentation ideal (Sects. 2.1, 2.2) and introduce the notions of "expansions" and "homomor-

Fig. 1 An algebraic structure \mathcal{O} with 4 kinds of objects and one binary, 3 unary and two 0-nary operations (the constants 1 and σ)



phic expansions” (2.3). Everything is so general that practically anything is an example, yet our main goal is to set the language for the examples of w-tangles and w-tangled foams, which appear later in this paper. Both of these examples are types of “circuit algebras”, and hence we end this section with a general discussion of circuit algebras (Sect. 2.4).

2.1 Algebraic structures

An “algebraic structure” \mathcal{O} is some collection (\mathcal{O}_α) of sets of objects of different kinds, where the subscript α denotes the “kind” of the objects in \mathcal{O}_α , along with some collection of “operations” ψ_β , where each ψ_β is an arbitrary map with domain some product $\mathcal{O}_{\alpha_1} \times \cdots \times \mathcal{O}_{\alpha_k}$ of sets of objects, and range a single set \mathcal{O}_{α_0} (so operations may be unary or binary or multinary, but they always return a value of some fixed kind). We also allow some named “constants” within some \mathcal{O}_α ’s (or equivalently, allow some 0-nary operations).² The operations may or may not be subject to axioms—an “axiom” is an identity asserting that some composition of operations is equal to some other composition of operations.

Figure 1 illustrates the general notion of an algebraic structure. Here are a few specific examples:

- We will use $\langle b \rangle$, the free group on one generator b , as a running example throughout this chapter (of course $\langle b \rangle$ is isomorphic to \mathbb{Z}). This is an algebraic structure with one kind of objects, a binary operation “multiplication”, a unary operation “inverse”, one constant “the identity”, and the expected axioms.
- Groups in general: one kind of objects, one binary “multiplication”, one unary “inverse”, one constant “the identity”, and some axioms.
- Group homomorphisms: Two kinds of objects, one for each group. 7 operations—3 for each of the two groups and the homomorphism itself, going between the two groups. Many axioms.
- A group acting on a set, a group extension, a split group extension and many other examples from group theory.

² Alternatively define “algebraic structures” using the theory of “multicategories” [30]. Using this language, an algebraic structure is simply a functor from some “structure” multicategory \mathcal{C} into the multicategory **Set** (or into **Vect**, if all \mathcal{O}_i are vector spaces and all operations are multi-linear). A “morphism” between two algebraic structures over the same multicategory \mathcal{C} is a natural transformation between the two functors representing those structures.

- 217 • A quandle is a set with an operation \uparrow , satisfying $(x \uparrow y) \uparrow z = (x \uparrow y) \uparrow (y \uparrow z)$
- 218 and some further minor axioms. This is an algebraic structure with one kind of
- 219 objects and one operation. See [37] for an analysis of quandles from the perspective
- 220 of this paper.
- 221 • Planar algebras as in [24] and circuit algebras as in Sect. 2.4.
- 222 • The algebra of knotted trivalent graphs as in [7, 17].
- 223 • Let $\zeta : B \rightarrow S$ be an arbitrary homomorphism of groups (though our notation
- 224 suggests what we have in mind— B may well be braids, and S may well be permu-
- 225 tations). We can consider an algebraic structure \mathcal{O} whose kinds are the elements
- 226 of S , for which the objects of kind $s \in S$ are the elements of $\mathcal{O}_s := \zeta^{-1}(s)$, and
- 227 with the product in B defining operations $\mathcal{O}_{s_1} \times \mathcal{O}_{s_2} \rightarrow \mathcal{O}_{s_1 s_2}$.
- 228 • W -tangles and w -foams, studied in the following two sections of this paper.
- 229 • Clearly, many more examples appear throughout mathematics.

2.2 Associated graded structures

231 Any algebraic structure \mathcal{O} has an “especially natural” associated graded structure: that

232 is, we take the associated structure with respect to a specific and natural filtration. This

233 will be a repeating construction throughout the rest of this paper series.

234 First extend \mathcal{O} to allow formal linear combinations of objects of the same kind

235 (extending the operations in a linear or multi-linear manner), then let \mathcal{I} , the “augmen-

236 tation ideal”, be the sub-structure made out of all such combinations in which the sum

237 of coefficients is 0, then let \mathcal{I}^m be the set of all outputs of algebraic expressions (that

238 is, arbitrary compositions of the operations in \mathcal{O}) that have at least m inputs in \mathcal{I} (and

239 possibly, further inputs in \mathcal{O}), and finally, set

$$240 \quad \text{grad } \mathcal{O} := \bigoplus_{m \geq 0} \mathcal{I}^m / \mathcal{I}^{m+1}. \quad (1)$$

241 Clearly, with the operations inherited from \mathcal{O} , the associated graded $\text{grad } \mathcal{O}$ is again

242 algebraic structure with the same multi-graph of spaces and operations, but with new

243 objects and with new operations that may or may not satisfy the axioms satisfied by

244 the operations of \mathcal{O} . The main new feature in $\text{grad } \mathcal{O}$ is that it is a “graded” structure;

245 we denote the degree m piece $\mathcal{I}^m / \mathcal{I}^{m+1}$ of $\text{grad } \mathcal{O}$ by $\text{grad}_m \mathcal{O}$.

246 We believe that many of the most interesting graded structures that appear in math-

247 ematics are the result of this construction (i.e., as associated graded structures with

248 respect to powers of the augmentation ideal), and that many of the interesting graded

249 equations that appear in mathematics arise when one tries to find “expansions”, or

250 “universal finite type invariants”, which are also morphisms³ $Z : \mathcal{O} \rightarrow \text{grad } \mathcal{O}$ (see

251 Sect. 2.3) or when one studies “automorphisms” of such expansions.⁴ Indeed, the paper

³ Indeed, if \mathcal{O} is finitely presented then finding such a morphism $Z : \mathcal{O} \rightarrow \text{grad } \mathcal{O}$ amounts to finding its values on the generators of \mathcal{O} , subject to the relations of \mathcal{O} . Thus it is equivalent to solving a system of equations written in some graded spaces.

⁴ The Drinfel’d graded Grothendieck–Teichmüller group GRT is an example of such an automorphism group. See [6, 20].

252 you are reading now is really the study of the associated graded structures of various
 253 algebraic structures associated with w-knotted objects. We would like to believe that
 254 much of the theory of quantum groups (at “generic” \hbar) will eventually be shown to be
 255 a study of the associated graded structures of various algebraic structures associated
 256 with v-knotted objects.

257 *Example 2.1* We compute the associated graded structure of the running example $\langle b \rangle$.
 258 Allowing formal \mathbb{Q} -linear combinations of elements we get $\mathbb{Q}\langle b \rangle = \mathbb{Q}[b, b^{-1}]$. The
 259 augmentation ideal \mathcal{I} is generated by differences $(b^n - 1)$ as a vector space (where
 260 $1 = b^0$), and generated by $(b - 1)$ as an ideal.

261 We claim that $\text{grad} \langle b \rangle \cong \mathbb{Q}[[c]]$, the algebra of power series in one variable. To
 262 show this, consider the map $\pi : \mathbb{Q}[[c]] \rightarrow \text{grad} \langle b \rangle$ by setting $\pi(c) = [b - 1] \pmod{\mathcal{I}^2}$.
 263 It is easy to show explicitly that π is surjective. For example, in degree 1, we need to
 264 show that $b - 1$ generates $\mathcal{I}/\mathcal{I}^2$. indeed, $(b^n - 1) - n(b - 1)$ has a double zero at $b = 1$,
 265 and hence $f = \frac{(b^n - 1) - n(b - 1)}{(b - 1)^2}$ is a polynomial, and $b^n - 1 = n(b - 1) + f(b - 1)^2$.
 266 So modulo $(b - 1)^2 \in \mathcal{I}^2$, $b^n - 1 = n(b - 1)$. A similar argument works to show that
 267 $(b - 1)^k$ generates $\mathcal{I}^k/\mathcal{I}^{k+1}$.

268 Note that $\langle b \rangle$ can also be thought of as the pure braid group on two strands: b would
 269 be a “full twist” and c can be represented as a single “horizontal chord”. In other knot
 270 theoretic settings, it is generally relatively easy to find a “candidate associated graded”
 271 and a map π , which can be shown to be surjective by explicit means.

272 To show that π is injective we are going to use the machinery of “expansions”
 273 which is the tool we use to accomplish similar tasks in the later sections of this paper.

274 We end this section with two more examples of computing associated graded struc-
 275 tures: the proof of Proposition 2.2 is an exercise; for the proof of Proposition 2.3 see
 276 [37].

277 **Proposition 2.2** *If G is a group, $\text{grad} G$ is a graded associative algebra with unit. Sim-*
 278 *ilarly, the associated graded structure of a group homomorphism is a homomorphism*
 279 *of graded associative algebras.*

280 **Proposition 2.3** *If Q is a unital quandle, $\text{grad}_0 Q$ is one-dimensional and $\text{grad}_{>0} Q$*
 281 *is a graded right Leibniz algebra⁵ generated by $\text{grad}_1 Q$.*

282 2.3 Expansions and homomorphic expansions

283 We start with the definition. Given an algebraic structure \mathcal{O} let $\text{fil } \mathcal{O}$ denote the filtered
 284 structure of linear combinations of objects in \mathcal{O} (respecting kinds), filtered by the
 285 powers (\mathcal{I}^m) of the augmentation ideal \mathcal{I} . Recall also that any graded space $G =$
 286 $\bigoplus_m G_m$ is automatically filtered, by $(\bigoplus_{n \geq m} G_n)_{m=0}^\infty$.

287 **Definition 2.4** An “expansion” Z for \mathcal{O} is a map $Z : \mathcal{O} \rightarrow \text{grad } \mathcal{O}$ that pre-
 288 serves the kinds of objects and whose linear extension (also called Z) to $\text{fil } \mathcal{O}$

⁵ A Leibniz algebra is a Lie algebra without anti-commutativity, as defined by Loday in [32].

289 respects the filtration of both sides, and for which $(\text{gr } Z) : (\text{gr fil } \mathcal{O} = \text{grad } \mathcal{O}) \rightarrow$
 290 $(\text{gr grad } \mathcal{O} = \text{grad } \mathcal{O})$ is the identity map of $\text{grad } \mathcal{O}$; we refer to this as the “universality
 291 property”.

292 In practical terms, this is equivalent to saying that Z is a map $\mathcal{O} \rightarrow \text{grad } \mathcal{O}$ whose
 293 restriction to \mathcal{I}^m vanishes in degrees less than m (in $\text{grad } \mathcal{O}$) and whose degree m piece
 294 is the projection $\mathcal{I}^m \rightarrow \mathcal{I}^m / \mathcal{I}^{m+1}$.

295 We come now to what is perhaps the most crucial definition in this paper.

296 **Definition 2.5** A “homomorphic expansion” is an expansion which also commutes
 297 with all the algebraic operations defined on the algebraic structure \mathcal{O} .

298 **Why bother with homomorphic expansions?** Primarily, for two reasons:

- 299 • Often $\text{grad } \mathcal{O}$ is simpler to work with than \mathcal{O} ; for one, it is graded and so it allows
 300 for finite “degree by degree” computations, whereas often times, such as in many
 301 topological examples, anything in \mathcal{O} is inherently infinite. Thus it can be beneficial
 302 to translate questions about \mathcal{O} to questions about $\text{grad } \mathcal{O}$. A simplistic example
 303 would be, “is some element $a \in \mathcal{O}$ the square (relative to some fixed operation)
 304 of an element $b \in \mathcal{O}$?”. Well, if Z is a homomorphic expansion and by a finite
 305 computation it can be shown that $Z(a)$ is not a square already in degree 7 in $\text{grad } \mathcal{O}$,
 306 then we’ve given a conclusive negative answer to the example question. Some less
 307 simplistic and more relevant examples appear in [7].
- 308 • Often $\text{grad } \mathcal{O}$ is “finitely presented”, meaning that it is generated by some finitely
 309 many elements $g_1, \dots, g_k \in \mathcal{O}$, subject to some relations $R_1 \dots R_n$ that can be
 310 written in terms of g_1, \dots, g_k and the operations of \mathcal{O} . In this case, finding a
 311 homomorphic expansion Z is essentially equivalent to guessing the values of Z
 312 on g_1, \dots, g_k , in such a manner that these values $Z(g_1), \dots, Z(g_k)$ would satisfy
 313 the $\text{grad } \mathcal{O}$ versions of the relations $R_1 \dots R_n$. So finding Z amounts to solving
 314 equations in graded spaces. It is often the case (as will be demonstrated in this paper;
 315 see also [5, 6]) that these equations are very interesting for their own algebraic sake,
 316 and that viewing such equations as arising from an attempt to solve a problem about
 317 \mathcal{O} sheds further light on their meaning.

318 In practice, often the first difficulty in searching for an expansion (or a homomorphic
 319 expansion) $Z : \mathcal{O} \rightarrow \text{grad } \mathcal{O}$ is that its would-be target space $\text{grad } \mathcal{O}$ is hard to identify.
 320 It is typically easy to make a suggestion \mathcal{A} for what $\text{grad } \mathcal{O}$ could be. It is typically
 321 easy to come up with a reasonable generating set \mathcal{D}_m for \mathcal{I}^m (keep some knot theoretic
 322 examples in mind, or \mathbb{Z} in Example 2.1). It is a bit harder but not exceedingly difficult
 323 to discover some relations \mathcal{R} satisfied by the elements of the image of \mathcal{D} in $\mathcal{I}^m / \mathcal{I}^{m+1}$
 324 $(4T, \overrightarrow{4T}, \text{ and more in knot theory, there are no relations for } \mathbb{Z})$. Thus we set $\mathcal{A} := \mathcal{D} / \mathcal{R}$;
 325 but it is often very hard to be sure that we found everything that ought to go in \mathcal{R} ; so
 326 perhaps our suggestion \mathcal{A} is still too big? Finding $4T$ for example was actually not
 327 *that* easy. Could we have missed some further relations that are hiding in \mathcal{A} ?

328 The notion of an \mathcal{A} -expansion, defined below, solves two problems at once. Once
 329 we find an \mathcal{A} -expansion we know that we’ve identified $\text{grad } \mathcal{O}$ correctly, and we
 330 automatically get what we really wanted, a $(\text{grad } \mathcal{O})$ -valued expansion.

331 **Definition 2.6** A “candidate associated graded structure” for an algebraic structure
 332 \mathcal{O} is a graded structure \mathcal{A} with the same operations as \mathcal{O} along with a homomorphic
 333 surjective graded map $\pi : \mathcal{A} \rightarrow \text{grad } \mathcal{O}$. An “ \mathcal{A} -expansion” is a kind and filtration
 334 respecting map $Z_{\mathcal{A}} : \mathcal{O} \rightarrow \mathcal{A}$ for which $(\text{gr } Z_{\mathcal{A}}) \circ \pi : \mathcal{A} \rightarrow \mathcal{A}$ is the identity. One
 335 can similarly define “homomorphic \mathcal{A} -expansions”.

$$\begin{array}{ccc}
 & & \mathcal{A} \\
 & \nearrow Z_{\mathcal{A}} & \uparrow \text{gr } Z_{\mathcal{A}} \\
 \mathcal{O} & & \text{grad } \mathcal{O} \\
 & \xrightarrow{Z} & \\
 & & \downarrow \pi
 \end{array}$$

337 **Proposition 2.7** If \mathcal{A} is a candidate associated graded of \mathcal{O} and $Z_{\mathcal{A}} : \mathcal{O} \rightarrow \mathcal{A}$ is
 338 a homomorphic \mathcal{A} -expansion, then $\pi : \mathcal{A} \rightarrow \text{grad } \mathcal{O}$ is an isomorphism and $Z :=$
 339 $\pi \circ Z_{\mathcal{A}}$ is a homomorphic expansion. (Often in this case, \mathcal{A} is identified with $\text{grad } \mathcal{O}$
 340 and $Z_{\mathcal{A}}$ is identified with Z).

341 *Proof* π is surjective by birth. Since $(\text{gr } Z_{\mathcal{A}}) \circ \pi$ is the identity, π is also injective
 342 and hence it is an isomorphism. The rest is immediate. \square

343 *Example 2.8* Back to $\langle b \rangle$, in Example 2.1 we found a candidate associated graded
 344 structure $\mathcal{A} = \mathbb{Q}[[c]]$ and a map $\pi : c \mapsto [b - 1]$. According to Proposition 2.7,
 345 it is enough to find a homomorphic \mathcal{A} -expansion, that is, an algebra homomor-
 346 phism $Z_{\mathcal{A}} : \mathbb{Q}\langle b \rangle \rightarrow \mathbb{Q}[[c]]$ such that $\text{gr } Z_{\mathcal{A}} \circ \pi$ is the identity of $\mathbb{Q}[[c]]$. It is
 347 a straightforward calculation to check that any algebra map defined by $Z_{\mathcal{A}}(b) =$
 348 $1 + c + \{\text{higher order terms}\}$ satisfies this property. If one seeks a “group-like” homo-
 349 morphic expansion then $Z_{\mathcal{A}}(b) = e^c$ is the only solution. In either case, exhibiting
 350 $Z_{\mathcal{A}}$ proves that π is injective and hence \mathcal{A} is the associated graded structure of $\langle b \rangle$.

351 2.4 Circuit algebras

352 “Circuit algebras” are so common and everyday, and they make such a useful language
 353 (definitely for the purposes of this paper, but also elsewhere), we find it hard to believe
 354 they haven’t made it into the standard mathematical vocabulary.⁶ People familiar with
 355 planar algebras [24] may note that circuit algebras are just the same as planar algebras,
 356 except with the planarity requirement dropped from the “connection diagrams” (and
 357 all colourings are dropped as well).

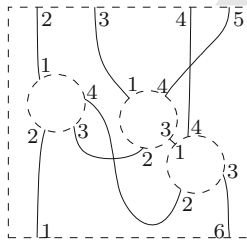
358 In our context, the main utility of circuit algebras is that they allow for a much
 359 simpler presentation of v (irtual)- and w -tangles. There are planar algebra presentations
 360 of v - and w -tangles, generated by the usual crossings and the “virtual crossing”,
 361 modulo the usual as well as the “virtual” and “mixed” Reidemeister moves. Switching
 362 from planar algebras to circuit algebras however renders the extra generators and
 363 relations unnecessary: the “virtual crossing” becomes merely a circuit algebra artifact,
 364 and the new Reidemeister moves are implied by the circuit algebra structure (see
 365 Warning 3.3, Definition 3.4, and Remark 3.5).

⁶ Or have they, and we have been looking the wrong way?

Author Proof

366 The everyday intuition for circuit algebras comes from electronic circuits, whose
 367 components can be wired together in many, not necessarily planar, ways, and it is
 368 not important to know how these wires are embedded in space. For details and more
 369 motivation see Sect. 5.1. We start formalizing this image by defining “wiring dia-
 370 grams”, the abstract analogs of printed circuit boards. Let \mathbb{N} denote the set of natural
 371 numbers including 0, and for $n \in \mathbb{N}$ let \underline{n} denote some fixed set with n elements, say
 372 $\{1, 2, \dots, n\}$.

373 **Definition 2.9** Let $k, n, n_1, \dots, n_k \in \mathbb{N}$ be natural numbers. A “wiring diagram”
 374 D with inputs $\underline{n}_1, \dots, \underline{n}_k$ and outputs \underline{n} is an unoriented compact 1-manifold whose
 375 boundary is $\underline{n} \sqcup \underline{n}_1 \sqcup \dots \sqcup \underline{n}_k$, regarded up to homeomorphism (on the right is an
 376 example with $k = 3, n = 6$, and $n_1 = n_2 = n_3 = 4$). In strictly combinatorial
 377 terms, it is a pairing⁷ of the elements of the set $\underline{n} \sqcup \underline{n}_1 \sqcup \dots \sqcup \underline{n}_k$ along with a
 378 single further natural number that counts closed circles. If $D_1; \dots; D_m$ are wiring
 379 diagrams with inputs $\underline{n}_{11}, \dots, \underline{n}_{1k_1}; \dots; \underline{n}_{m1}, \dots, \underline{n}_{mk_m}$ and outputs $\underline{n}_1; \dots; \underline{n}_m$ and
 380 D is a wiring diagram with inputs $\underline{n}_1; \dots; \underline{n}_m$ and outputs \underline{n} , there is an obvious
 381 “composition” $D(D_1, \dots, D_m)$ (obtained by gluing the corresponding 1-manifolds,
 382 and also describable in completely combinatorial terms) which is a wiring diagram
 383 with inputs $(\underline{n}_{ij})_{1 \leq i \leq k_j, 1 \leq j \leq m}$ and outputs \underline{n} (note that closed circles may be created
 384 in $D(D_1, \dots, D_m)$ even if none existed in D and in $D_1; \dots; D_m$).



385
386

387 A circuit algebra is an algebraic structure (in the sense of Sect. 2.2) whose operations
 388 are parametrized by wiring diagrams. Here’s a formal definition:

389 **Definition 2.10** A circuit algebra consists of the following data:

- 390 • For every natural number $n \geq 0$ a set (or a \mathbb{Z} -module) C_n “of circuits with n legs”.
- 391 • For any wiring diagram D with inputs $\underline{n}_1, \dots, \underline{n}_k$ and outputs \underline{n} , an operation
 392 (denoted by the same letter) $D: C_{n_1} \times \dots \times C_{n_k} \rightarrow C_n$ (or linear $D: C_{n_1} \otimes$
 393 $\dots \otimes C_{n_k} \rightarrow C_n$ if we work with \mathbb{Z} -modules).

394 We insist that the obvious “identity” wiring diagrams with \underline{n} inputs and \underline{n} outputs act
 395 as the identity of C_n , and that the actions of wiring diagrams be compatible in the
 396 obvious sense with the composition operation on wiring diagrams.

⁷ We mean “pairing” in the sense of combinatorics, not in the sense of linear algebra. That is, an involution without fixed point.

397 A silly but useful example of a circuit algebra is the circuit algebra \mathcal{S} of empty
 398 circuits, or in our context, of “skeletons”. The circuits with n legs for \mathcal{S} are wiring
 399 diagrams with n outputs and no inputs; namely, they are 1-manifolds with boundary
 400 \underline{n} (so n must be even).

401 More generally one may pick some collection of “basic components” (analogous
 402 to logic gates and junctions for electronic circuits as in Fig. 21) and speak of the “free
 403 circuit algebra” generated by these components; even more generally we can speak of
 404 circuit algebras given in terms of “generators and relations”. (In the case of electronics,
 405 our relations may include the likes of De Morgan’s law $\neg(p \vee q) = (\neg p) \wedge (\neg q)$ and
 406 the laws governing the placement of resistors in parallel or in series.) We feel there
 407 is no need to present the details here, yet many examples of circuit algebras given in
 408 terms of generators and relations appear in this paper, starting with the next section.
 409 We will use the notation $C = CA\langle G \mid R \rangle$ to denote the circuit algebra generated by
 410 a collection of elements G subject to some collection R of relations.

411 People familiar with electric circuits know that connectors sometimes come in
 412 “male” and “female” versions, and that you can’t plug a USB cable into a headphone
 413 jack. Thus one may define “directed circuit algebras” in which the wiring diagrams
 414 are oriented, the circuit sets C_n get replaced by $C_{p,q}$ for “circuits with p incoming
 415 wires and q outgoing wires” and only orientation preserving connections are ever
 416 allowed.⁸ Likewise there is a “coloured” version of everything, in which the wires
 417 may be coloured by the elements of some given set X (which may include among its
 418 members the elements “USB” and “audio”) and in which connections are allowed only
 419 if the colour coding is respected. We will leave the formal definitions of directed and
 420 coloured circuit algebras, as well as the definitions of directed and coloured analogues
 421 of the skeletons algebra \mathcal{S} and generators and relations for directed and coloured
 422 algebras, as an exercise.

423 Note that there is an obvious notion of “a morphism between two circuit algebras”
 424 and that circuit algebras (directed or not, coloured or not) form a category. We feel that a
 425 precise definition is not needed. A lovely example is the “implementation morphism”
 426 of logic circuits in the style of Fig. 21 in Sect. 5 into more basic circuits made of
 427 transistors and resistors.

428 Perhaps the prime mathematical example of a circuit algebra is tensor algebra. If t_1
 429 is an element (a “circuit”) in some tensor product of vector spaces and their duals, and
 430 t_2 is the same except in a possibly different tensor product of vector spaces and their
 431 duals, then once an appropriate pairing D (a “wiring diagram”) of the relevant vector
 432 spaces is chosen, t_1 and t_2 can be contracted (“wired together”) to make a new tensor
 433 $D(t_1, t_2)$. The pairing D must pair a vector space with its own dual, and so this circuit
 434 algebra is coloured by the set of vector spaces involved, and directed, by declaring
 435 (say) that some vector spaces are of one gender and their duals are of the other. We
 436 have in fact encountered this circuit algebra in [38, Sect. 3.5].

437 Let G be a group. A G -graded algebra A is a collection $\{A_g : g \in G\}$ of vector
 438 spaces, along with products $A_g \otimes A_h \rightarrow A_{gh}$ that induce an overall structure of an

⁸ By convention we label the boundary points of such circuits $1, \dots, p+q$, with the first p labels reserved for the incoming wires and the last q for the outgoing. The inputs of wiring diagrams must be labeled in the opposite way for the numberings to match.

439 algebra on $A := \bigoplus_{g \in G} A_g$. In a similar vein, we define the notion of an \mathcal{S} -graded
440 circuit algebra:

441 **Definition 2.11** An \mathcal{S} -graded circuit algebra, or a “circuit algebra with skeletons”, is
442 an algebraic structure C with spaces C_β , one for each element β of the circuit algebra
443 of skeletons \mathcal{S} , along with composition operations $D_{\beta_1, \dots, \beta_k} : C_{\beta_1} \times \dots \times C_{\beta_k} \rightarrow C_\beta$,
444 defined whenever D is a wiring diagram and $\beta = D(\beta_1, \dots, \beta_k)$, so that with the
445 obvious induced structure, $\coprod_{\beta} C_\beta$ is a circuit algebra. A similar definition can be
446 made if/when the skeletons are taken to be directed or coloured.

447 Loosely speaking, a circuit algebra with skeletons is a circuit algebra in which
448 every element T has a well-defined skeleton $\zeta(T) \in \mathcal{S}$. Yet note that as an algebraic
449 structure a circuit algebra with skeletons has more “spaces” than an ordinary circuit
450 algebra, for its spaces are enumerated by skeleton and not merely by integers. The
451 prime examples for circuit algebras with skeletons appear in the next section.

452 3 w-Tangles

453 **Section summary** In Sect. 3.1 we introduce v-tangles and w-tangles, the obvious
454 v- and w- counterparts of the standard knot-theoretic notion of “tangles”, and
455 briefly discuss their finite type invariants and their associated spaces of “arrow
456 diagrams”, $\mathcal{A}^v(\uparrow_n)$ and $\mathcal{A}^w(\uparrow_n)$. We then construct a homomorphic expansion
457 Z , or a “well-behaved” universal finite type invariant for w-tangles. The only
458 algebraic tool we need to use is $\exp(a) := \sum a^n/n!$ (Sec. 3.1 is in fact a routine
459 extension of parts of [38, Sect. 3.1]). In Sec. 3.2 we show that $\mathcal{A}^w(\uparrow_n) \cong$
460 $\mathcal{U}(\mathfrak{a}_n \oplus \mathfrak{td}\mathfrak{r}_n \times \mathfrak{tr}_n)$, where \mathfrak{a}_n is an Abelian algebra of rank n and where $\mathfrak{td}\mathfrak{r}_n$
461 and \mathfrak{tr}_n , two of the primary spaces used by Alekseev and Torossian [2], have
462 simple descriptions in terms of cyclic words and free Lie algebras. We also show
463 that some functionals studied in [2], div and j , have a natural interpretation in our
464 language. In 3.3 we discuss a subclass of w-tangles called “special” w-tangles,
465 and relate them by similar means to Alekseev and Torossian’s $\mathfrak{sd}\mathfrak{r}_n$ and to “tree
466 level” ordinary Vassiliev theory. Some conventions are described in Sec. 3.4 and
467 the uniqueness of Z is studied in Sec. 3.5.

468 3.1 v-tangles and w-tangles

469 With the task of defining circuit algebras completed in Sect. 2.4, the definition of
470 v-tangles and w-tangles is simple.

471 **Definition 3.1** The (\mathcal{S} -graded) circuit algebra vD of v-tangle diagrams is the \mathcal{S} -graded
472 directed circuit algebra freely generated by two generators in $C_{2,2}$ called the positive
473 crossing, $\begin{array}{c} 4 \nearrow 3 \\ 1 \searrow 2 \end{array}$, and the negative crossing, $\begin{array}{c} 4 \nwarrow 3 \\ 1 \swarrow 2 \end{array}$. In as much as possible we suppress
474 the leg-numbering below; with this in mind, $vD := vD := \text{CA} \langle \nearrow, \nwarrow \rangle$. The skeleton

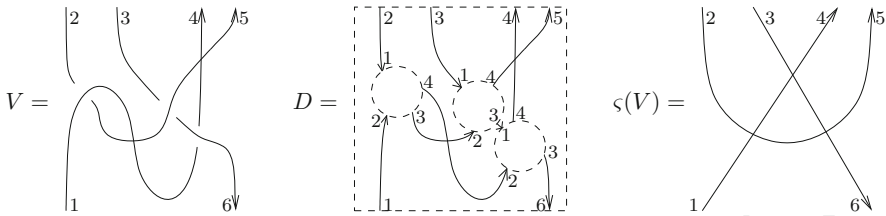


Fig. 2 $V \in vD_{3,3}$ is a v -tangle diagram. V is the result of applying the circuit algebra operation $D : C_{2,2} \times C_{2,2} \times C_{2,2} \rightarrow C_{3,3}$, given by the wiring diagram shown, acting on *two negative crossings and one positive crossing*. In other words $V = D(\nearrow, \searrow, \nearrow)$. The skeleton of V is given by $\zeta(V) = D(\bowtie, \bowtie, \bowtie)$, which is equal in \mathcal{S} to the diagram shown here. Note that we usually suppress the circuit algebra numbering of boundary points. Note also that the apparent “virtual crossings” of V are not virtual crossings but merely part of the circuit algebra structure, see Warning 3.3. The same is true for the crossings appearing in the skeleton $\zeta(V)$

475 of both crossings is the element $\begin{matrix} 4 & \nearrow & 3 \\ 1 & \times & 2 \end{matrix}$ (the pairing of 1&3 and 2&4) in $\mathcal{S}_{2,2}$. That is,
 476 $\zeta(\nearrow) = \zeta(\searrow) = \bowtie$.

477 **Example 3.2** An example of a v -tangle diagram V is shown the left side of Fig. 2. V
 478 is a circuit algebra composition of two negative crossings and one positive crossing by
 479 the wiring diagram D , as shown. The right side of the same figure shows the skeleton
 480 $\zeta(V)$ of V : to produce the skeleton, replace each crossing by the element \bowtie in \mathcal{S}
 481 and apply the same wiring diagram. The elements of \mathcal{S} are oriented 1-manifolds with
 482 numbered boundary points, and hence the result is equal to the one shown in the figure.

483 **Warning 3.3** People familiar with the planar presentation of virtual tangles may be
 484 accustomed to the notion of there being another type of crossing: the “virtual crossing”.
 485 The main point of introducing circuit algebras (as opposed to working with planar
 486 algebras) is to eliminate the need for virtual crossings: they become part of the CA
 487 structure. This greatly simplifies the presentation of both v - and w -tangles: there is
 488 one less generator, as seen above, and far fewer relations, as we explain in Remark 3.5.

489 **Definition 3.4** The (\mathcal{S} -graded) circuit algebra vT of v -tangles is the \mathcal{S} -graded directed
 490 circuit algebra of v -tangle diagrams vD , modulo the R1^f, R2 and R3 moves as depicted
 491 in Fig. 3. These relations make sense as circuit algebra relations between the two
 492 generators, and preserve skeleta. To obtain the circuit algebra wT of w -tangles we
 493 also mod out by the OC relation of Fig. 3 (note that each side in that relation involves
 494 only two generators, with the apparent third “virtual” crossing being merely a circuit
 495 algebra artifact). In fewer words, $vT := CA\langle \nearrow, \searrow \mid \rho = \alpha, \bowtie = |, \bowtie = \bowtie \rangle$,
 496 and $wT := vT / \bowtie = \bowtie$.

497 **Remark 3.5** One may also define v -tangles and w -tangles using the language of planar
 498 algebras, except then another generator is required (the “virtual crossing”) and also a
 499 number of further relations shown in Fig. 3 (VR1–VR3, M), and some of the operations

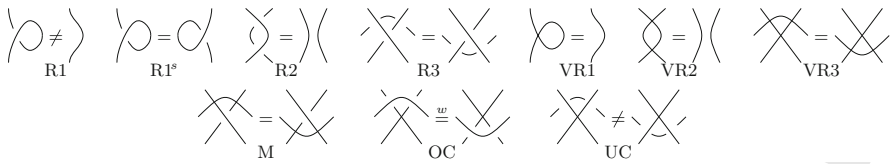


Fig. 3 The relations (“Reidemeister moves”) $R1^s$, $R2$ and $R3$ define v -tangles, adding OC to these defines w -tangles. $VR1$, $VR2$, $VR3$ and M are not necessary as the circuit algebra presentation eliminates the need for “virtual crossings” as generators. $R1$ is not imposed for framing reasons, and not imposing UC breaks the symmetry between over and under crossings in wT

(non-planar wirings) become less elegant to define. In our context “virtual crossings” are automatically present (but unimportant) as part of the circuit algebra structure, and the “virtual Reidemeister moves” $VR1$ – $VR3$ and M are also automatically true. In fact, the “rerouting move” known in the planar presentation, which says that a purely virtual strand of a v -tangle diagram can be re-routed in any other purely virtual way, is precisely the statement that virtual crossings are unimportant, and the language of circuit algebras makes this fact manifest.

Remark 3.6 For $S \in \mathcal{S}$ a given skeleton, that is, an oriented 1-manifold with numbered ends, let us denote by $vT(S)$ and $wT(S)$, respectively, the v - and w -tangles with skeleton S . That is, $vT(S)$ and $wT(S)$ are the pre-images of S under the skeleton map ζ . Note that in our case the skeleton map is “forgetting topology”, in other words, forgetting the under/over information of crossings, resulting in empty circuits. With this notation, $wT(\uparrow)$, the set of w -tangles whose skeleton is a single line, is exactly the set of (long) w -knots discussed in [38, Section 3]. Note also that $wT(\uparrow_n)$, the set of w -tangles whose skeleton is n lines, includes w -braids with n strands [38, Section 2] but it is more general. Neither w -knots nor w -braids are circuit algebras.

Remark 3.7 Since we do not mod out by the $R1$ relation, only by its weak (or “spun”) version $R1^s$, it is more appropriate to call our class of v/w -tangles *framed v/w -tangles*. (Recall that framed u -tangles are characterized as the planar algebra generated by the positive and negative crossings modulo the $R1^s$, $R2$ and $R3$ relations.) However, since we are for the most part interested in studying the framed theories (cf. Comment 4.4), we will reserve the unqualified name for the framed case, and will explicitly write “unframed v/w -tangles” if we wish to mod out by $R1$. For a more detailed explanation of framings and $R1$ moves, see [38, Remark 3.5].

Our next task is to study the associated graded structures $\text{grad } vT$ and $\text{grad } wT$ of vT and wT . These are “arrow diagram spaces on tangle skeletons”: directed analogues of the chord diagram spaces of ordinary finite type invariant theory, and even more similar to the arrow diagram spaces for braids and knots discussed in [38]. Our convention for figures will be to show skeletons as thick lines with thin arrows (directed chords). Again, the language of circuit algebras makes defining these spaces exceedingly simple.

Definition 3.8 The (\mathcal{S} -graded) circuit algebra $\mathcal{D}^v = \mathcal{D}^w$ of arrow diagrams is the graded and \mathcal{S} -graded directed circuit algebra generated by a single degree 1 generator

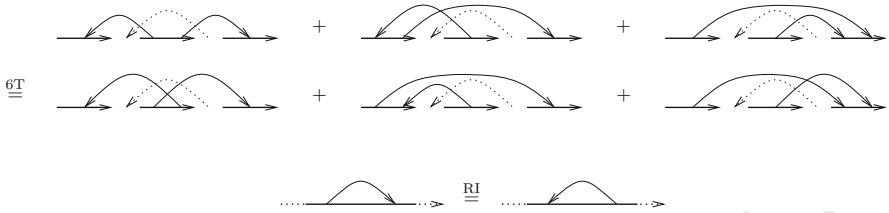


Fig. 4 Relations for v-arrow diagrams on tangle skeletons. Skeleta parts that are not connected can lie on separate skeleton components; and the *dotted arrow* that remains in the same position means “all other arrows remain the same throughout”

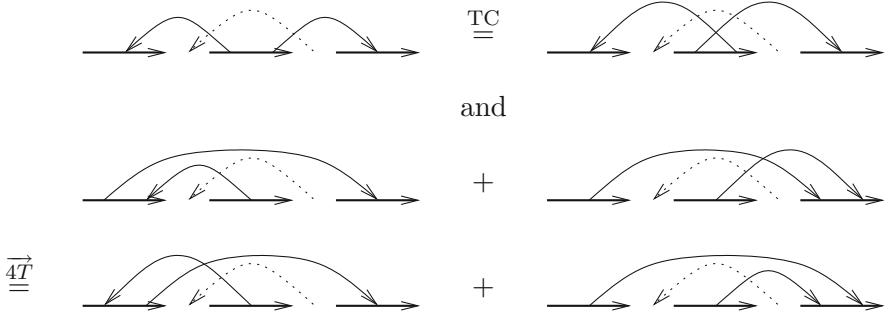
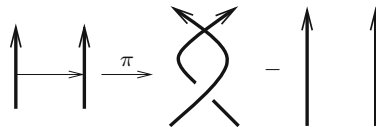


Fig. 5 Relations for w-arrow diagrams on tangle skeletons

533 a in $C_{2,2}$ called “the arrow” as shown on the right, with the obvious meaning for
 534 its skeleton. There are morphisms $\pi : \mathcal{D}^v \rightarrow vT$ and $\pi : \mathcal{D}^w \rightarrow wT$ defined by
 535 mapping the arrow to an overcrossing minus a no-crossing. (On the right some virtual
 536 crossings were added to make the skeleta match). Let \mathcal{A}^v be $\mathcal{D}^v/6T$, let $\mathcal{A}^w :=$
 537 $\mathcal{A}^v/TC = \mathcal{D}^w/(4T, TC)$, and let $\mathcal{A}^{sv} := \mathcal{A}^v/RI$ and $\mathcal{A}^{sw} := \mathcal{A}^w/RI$ as usual,
 538 with $RI, 6T, \overrightarrow{4T}$, and TC being the relations shown in Figs. 4 and 5. Note that the pair
 539 of relations $(\overrightarrow{4T}, TC)$ is equivalent to the pair $(6T, TC)$, as discussed in [38, Section
 540 2.3.1].



541
542

543 **Proposition 3.9** *The maps π above induce surjections $\pi : \mathcal{A}^{sv} \rightarrow \text{grad } vT$ and*
 544 *$\pi : \mathcal{A}^{sw} \rightarrow \text{grad } wT$. Hence in the language of Definition 2.6, \mathcal{A}^{sv} and \mathcal{A}^{sw} are*
 545 *candidate associated graded structures of vT and wT .*

546 *Proof* Proving that π is well-defined amounts to checking directly that the RI and $6T$
 547 or $RI, \overrightarrow{4T}$ and TC relations are in the kernel of π . (Just like in the finite type theory of

virtual knots and braids.) Thanks to the circuit algebra structure, it is enough to verify the surjectivity of π in degree 1. We leave this as an exercise for the reader. \square

We do not know if \mathcal{A}^{sv} is indeed the associated graded of vT (also see [10]). Yet in the w case, the picture is simple:

Theorem 3.10 *The assignment $\nearrow \mapsto e^a$ (with e^a denoting the exponential of a single arrow from the over strand to the under strand, interpreted via its power series) extends to a well defined $Z: wT \rightarrow \mathcal{A}^{sw}$. The resulting map Z is a homomorphic \mathcal{A}^{sw} -expansion, and in particular, $\mathcal{A}^{sw} \cong \text{grad } wT$ and Z is a homomorphic expansion.*

Proof The proof is essentially the same as the proof of [38, Theorem 2.15], and follows [2, 12]. One needs to check that Z satisfies the Reidemeister moves and the OC relation. $R1^s$ follows easily from RI , $R2$ is obvious, TC implies OC . For $R3$, let $\mathcal{A}^{sw}(\uparrow_n)$ denote the space of “arrow diagrams on n vertical strands”. We need to verify that $R := e^a \in \mathcal{A}^{sw}(\uparrow_2)$ satisfies the Yang–Baxter equation

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}, \quad \text{in } \mathcal{A}^{sw}(\uparrow_3),$$

where $R^{ij} = e^{a_{ij}}$ means “place R on strands i and j ”. By $4T$ and TC relations, both sides of the equation can be reduced to $e^{a_{12}+a_{13}+a_{23}}$, proving the Reidemeister invariance of Z .

Z is by definition a circuit algebra homomorphism. Hence to show that Z is an \mathcal{A}^{sw} -expansion we only need to check the universality property in degree one, where it is very easy. The rest follows from Proposition 2.7. \square

Remark 3.11 Note that the restriction of Z to w -knots and w -braids (in the sense of Remark 3.6) recovers the expansions constructed in [38]. Note also that the filtration and associated graded structure for w -braids fits into the general algebraic framework of Sect. 2 by applying the machinery to the skeleton-graded group of w -braids instead the circuit algebra of w -tangles. (The skeleton of a w -braid is the permutation it represents.) However, as w -knots do not form a finitely presented algebraic structure in the sense of Sect. 2, the “finite type” filtration used in [38] does not arise as powers of any augmentation ideal. This captures the reason why w -knots are “the wrong objects to study”, as we have mentioned at the beginning of Section 3 of [38].

In a similar spirit to [38, Definition 3.12], one may define a “ w -Jacobi diagram” on an arbitrary skeleton:

Definition 3.12 A “ w -Jacobi diagram on a tangle skeleton”⁹ is a graph made of the following ingredients:

- An oriented “skeleton” consisting of long lines and circles (i.e., an oriented one-manifold). In figures we draw the skeleton lines thicker.
- Other directed edges, usually called “arrows”.
- Trivalent “skeleton vertices” in which an arrow starts or ends on the skeleton line.

⁹ We usually short this to “ w -Jacobi diagram”, or sometimes “arrow diagram” or just “diagram”.



Fig. 6 The \overrightarrow{STU} relations for arrow diagrams, with their “central edges” marked e for easier memorization



Fig. 7 The \overrightarrow{AS} and \overrightarrow{IHX} relations

- Trivalent “internal vertices” in which two arrows end and one arrow begins. The internal vertices are cyclically oriented; in figures the assumed orientation is always counterclockwise unless marked otherwise. Furthermore, all trivalent vertices must be connected to the skeleton via arrows (but not necessarily following the direction of the arrows).

Note that we allow multiple and loop arrow edges, as long as trivalence and the two-in-one-out rule is respected.

Formal linear combinations of (w-Jacobi) arrow diagrams form a circuit algebra. We denote by \mathcal{A}^{wt} the quotient of the circuit algebra of arrow diagrams modulo the $\overrightarrow{STU}_1, \overrightarrow{STU}_2$ relations of Fig. 6, and the TC relation. We denote \mathcal{A}^{wt} modulo the RI relation by \mathcal{A}^{swt} . We then have the following “bracket-rise” theorem:

Theorem 3.13 *The obvious inclusion of arrow diagrams (with no internal vertices) into w-Jacobi diagrams descends to a map $\bar{\iota} : \mathcal{A}^w \rightarrow \mathcal{A}^{wt}$, which is a circuit algebra isomorphism. Furthermore, the \overrightarrow{AS} and \overrightarrow{IHX} relations of Fig. 7 hold in \mathcal{A}^{wt} . Consequently, it is also true that $\mathcal{A}^{sw} \cong \mathcal{A}^{swt}$.*

Proof In the proof of [38, Theorem 3.13] we showed this for long w-knots (i.e., tangles whose skeleton is a single long line). That proof applies here verbatim, noting that it does not make use of the connectivity of the skeleton.

In short, to check that $\bar{\iota}$ is well-defined, we need to show that the \overrightarrow{STU} relations imply the $4\overrightarrow{T}$ relation. This is shown in Fig. 8. To show that $\bar{\iota}$ is an isomorphism, we construct an inverse $\mathcal{A}^{wt} \rightarrow \mathcal{A}^w$, which “eliminates all internal vertices” using a sequence of \overrightarrow{STU} relations. Checking that this is well-defined requires some case analysis; the fact that it is an inverse to $\bar{\iota}$ is obvious. Verifying that the \overrightarrow{AS} and \overrightarrow{IHX} relations hold in \mathcal{A}^{wt} is an easy exercise. \square

Given the above theorem, we no longer keep the distinction between \mathcal{A}^w and \mathcal{A}^{wt} and between \mathcal{A}^{sw} and \mathcal{A}^{swt} .

We recall from [38] that a “ k -wheel”, sometimes denoted w_k , is an arrow diagram consisting of an oriented cycle of arrows with k incoming “spokes”, the tails of which rest on the skeleton. An example is shown in Fig. 9. In this language, the RI relation can be rephrased using the \overrightarrow{STU} relation to say that all one-wheels are 0, or $w_1 = 0$.

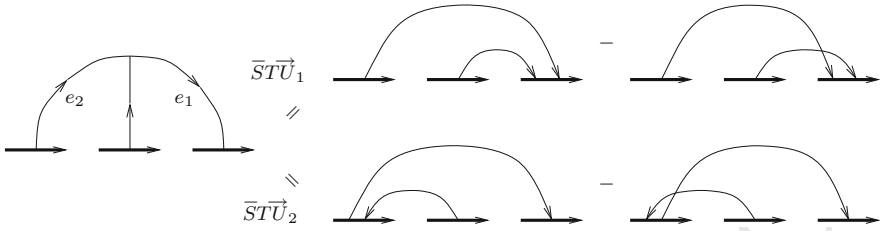


Fig. 8 Applying \overrightarrow{STU}_1 and \overrightarrow{STU}_2 to the diagram on the left, we get the two sides of $4\vec{T}$

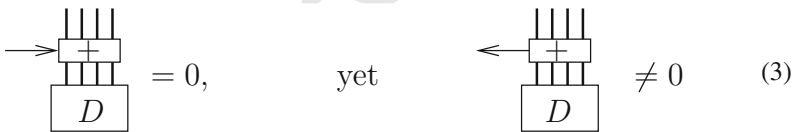
Fig. 9 A 4-wheel and the RI relation re-phrased



615 *Remark 3.14* Note that if T is an arbitrary w tangle, then the equality on the left side
 616 of the figure below always holds, while the one on the right generally doesn't:



617 (2)
 618 The arrow diagram version of this statement is that if D is an arbitrary arrow diagram
 619 in \mathcal{A}^w , then the left side equality in the figure below always holds (we will sometimes
 620 refer to this as the “head-invariance” of arrow diagrams), while the right side equality
 621 (“tail-invariance”) generally fails.



623 We leave it to the reader to ascertain that Eq. (2) implies Eq. (3). There is also a direct
 624 proof of Eq. (3) which we also leave to the reader, though see an analogous statement
 625 and proof in [5, Lemma 3.4]. Finally note that a restricted version of tail-invariance
 626 does hold—see Sect. 3.3.

627 **3.2 $\mathcal{A}^w(\uparrow_n)$ and the Alekseev–Torossian spaces**

628 **Definition 3.15** Let $\mathcal{A}^v(\uparrow_n)$ be the part of \mathcal{A}^v in which the skeleton is the disjoint
 629 union of n directed lines, with similar definitions for $\mathcal{A}^w(\uparrow_n)$, $\mathcal{A}^{sv}(\uparrow_n)$, and $\mathcal{A}^{sw}(\uparrow_n)$.

Theorem 3.16 (Diagrammatic PBW Theorem) *Let \mathcal{B}_n^w denote the space of uni-trivalent diagrams¹⁰ with symmetrized ends coloured with colours in some n -element set (say $\{x_1, \dots, x_n\}$), modulo the \overrightarrow{AS} and \overrightarrow{IHX} relations of Fig. 7. Then there is an isomorphism $\mathcal{A}^w(\uparrow_n) \cong \mathcal{B}_n^w$.*

Proof sketch Readers familiar with the diagrammatic PBW theorem [4, Theorem 8] will note that the proof carries through almost verbatim. There is a map $\chi : \mathcal{B}_n^w \rightarrow \mathcal{A}^w(\uparrow_n)$, which sends each uni-trivalent diagram to the average of all ways of attaching their univalent ends to the skeleton of n lines, so that ends of colour x_i are attached to the strand numbered i . I.e., a diagram with k_i uni-valent vertices of colour x_i is sent to a sum of $\prod_i k_i!$ terms, divided by $\prod_i k_i!$.

The goal is to show that χ is an isomorphism by constructing an inverse for it. The image of χ are *symmetric* sums of diagrams, that is, sums of diagrams that are invariant under permuting arrow endings on the same skeleton component. One can show that in fact any arrow diagram D in $\mathcal{A}^w(\uparrow_n)$ is equivalent via \overrightarrow{STU} and TC relations to a symmetric sum. The obvious candidate is its “symmetrization” $Sym(D)$: the average of all ways of permuting the arrow endings on each skeleton component of D . It is not true that each diagram is equivalent to its symmetrization (hence, the “simply delete the skeleton” map is not an inverse for χ), but it is true that $D - Sym(D)$ has fewer skeleton vertices (lower degree) than D , hence we can construct χ^{-1} inductively. The fact that this inductive procedure is well-defined requires a proof; that proof is essentially the same as the proof of the corresponding fact in [4, Theorem 8].

Both $\mathcal{A}^w(\uparrow_n)$ and \mathcal{B}_n^w have a natural bi-algebra structure. In $\mathcal{A}^w(\uparrow_n)$ multiplication is given by stacking. For a diagram $D \in \mathcal{A}^w(\uparrow_n)$, the co-product $\Delta(D)$ is given by the sum of all ways of dividing D between a “left co-factor” and a “right cofactor” so that the connected components of $D - S$ are kept intact, where S is the skeleton of D . In \mathcal{B}_n^w multiplication is given by disjoint union, and Δ is the sum of all ways of dividing the connected components of a diagram between two co-factors (here there is no skeleton). Note that the isomorphism χ above is a co-algebra isomorphism, but not an algebra homomorphism.

The primitives \mathcal{P}_n^w of \mathcal{B}_n^w are the connected diagrams (and hence the primitives of $\mathcal{A}^w(\uparrow_n)$ are the diagrams that remain connected even when the skeleton is removed). Given the “two in one out” rule for internal vertices, the diagrams in \mathcal{P}_n^w can only be trees (diagrams with no cycles) or wheels (a single oriented cycle with a number of “spokes”, or leaves, attached to it). “Wheels of trees” can be reduced to simple wheels by repeatedly using \overrightarrow{IHX} , as in Fig. 10.

Thus as a vector space \mathcal{P}_n^w is easy to identify. It is a direct sum $\mathcal{P}_n^w = \langle \text{trees} \rangle \oplus \langle \text{wheels} \rangle$. The wheels part is simply the graded vector space generated by all cyclic words in the letters x_1, \dots, x_n . Alekseev and Torossian [2] denote the space of cyclic words by tt_n , and so shall we. The trees in \mathcal{P}_n^w have leafs coloured x_1, \dots, x_n . Modulo \overrightarrow{AS} and \overrightarrow{IHX} , they correspond to elements of the free Lie algebra lie_n on the generators x_1, \dots, x_n . But the root of each such tree also carries a label in $\{x_1, \dots, x_n\}$, hence

¹⁰ Oriented graphs with vertex degrees either 1 or 3, where trivalent vertices must have two edges incoming and one edge outgoing and are cyclically oriented.

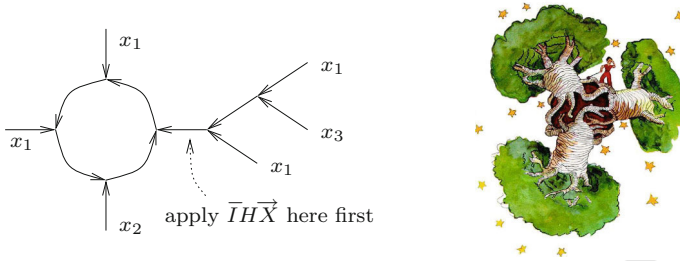


Fig. 10 A wheel of trees can be reduced to a combination of wheels, and a wheel of trees with a Little Prince

671 there are n types of such trees as separated by their roots, and so \mathcal{P}_n^w is isomorphic to
 672 the direct sum $\mathfrak{tr}_n \oplus \bigoplus_{i=1}^n \mathfrak{lie}_n$.

673 Note that with \mathcal{B}_n^{sw} and \mathcal{P}_n^{sw} defined in the analogous manner (i.e., factoring out
 674 by one-wheels, as in the RI relation), we can also conclude that $\mathcal{P}_n^{sw} \cong \mathfrak{tr}_n / (\deg 1) \oplus$
 675 $\bigoplus_{i=1}^n \mathfrak{lie}_n$.

676 By the Milnor–Moore theorem [34], $\mathcal{A}^w(\uparrow_n)$ is isomorphic to the universal envelop-
 677 ing algebra $\mathcal{U}(\mathcal{P}_n^w)$, with \mathcal{P}_n^w identified as the subspace $\mathcal{P}^w(\uparrow_n)$ of primitives of
 678 $\mathcal{A}^w(\uparrow_n)$ using the PBW symmetrization map $\chi: \mathcal{B}_n^w \rightarrow \mathcal{A}^w(\uparrow_n)$. Thus in order to
 679 understand $\mathcal{A}^w(\uparrow_n)$ as an associative algebra, it is enough to understand the Lie algebra
 680 structure induced on \mathcal{P}_n^w via the commutator bracket of $\mathcal{A}^w(\uparrow_n)$.

681 Our goal is to identify $\mathcal{P}^w(\uparrow_n)$ as the Lie algebra $\mathfrak{tr}_n \rtimes (\mathfrak{a}_n \oplus \mathfrak{tder}_n)$, which in
 682 itself is a combination of the Lie algebras \mathfrak{a}_n , \mathfrak{tder}_n and \mathfrak{tr}_n studied by Alekseev and
 683 Torossian [2]. Here are the relevant definitions:

684 **Definition 3.17** Let \mathfrak{a}_n denote the vector space with basis x_1, \dots, x_n , also regarded
 685 as an Abelian Lie algebra of dimension n . As before, let $\mathfrak{lie}_n = \mathfrak{lie}(\mathfrak{a}_n)$ denote the
 686 free Lie algebra on n generators, now identified as the basis elements of \mathfrak{a}_n . Let
 687 $\mathfrak{der}_n = \mathfrak{der}(\mathfrak{lie}_n)$ be the (graded) Lie algebra of derivations acting on \mathfrak{lie}_n , and let

688
$$\mathfrak{tder}_n = \{D \in \mathfrak{der}_n : \forall i \exists a_i \text{ s.t. } D(x_i) = [x_i, a_i]\}$$

689 denote the subalgebra of “tangential derivations”. A tangential derivation D is deter-
 690 mined by the a_i ’s for which $D(x_i) = [x_i, a_i]$, and determines them up to the
 691 ambiguity $a_i \mapsto a_i + \alpha_i x_i$, where the α_i ’s are scalars. Thus as vector spaces,
 692 $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_{i=1}^n \mathfrak{lie}_n$.

693 **Definition 3.18** Let $\text{Ass}_n = \mathcal{U}(\mathfrak{lie}_n)$ be the free associative algebra “of words”, and let
 694 Ass_n^+ be the degree > 0 part of Ass_n . As before, we let $\mathfrak{tr}_n = \text{Ass}_n^+ / (x_{i_1} x_{i_2} \cdots x_{i_m} =$
 695 $x_{i_2} \cdots x_{i_m} x_{i_1})$ denote “cyclic words” or “(coloured) wheels”. Ass_n , Ass_n^+ , and \mathfrak{tr}_n are
 696 \mathfrak{tder}_n -modules and there is an obvious equivariant “trace” $\text{tr}: \text{Ass}_n^+ \rightarrow \mathfrak{tr}_n$.

697 **Proposition 3.19** *There is a split short exact sequence of Lie algebras*

698
$$0 \longrightarrow \mathfrak{tr}_n \xrightarrow{\iota} \mathcal{P}^w(\uparrow_n) \xrightarrow{\pi} \mathfrak{a}_n \oplus \mathfrak{tder}_n \longrightarrow 0.$$

699 *Proof* The inclusion ι is defined the natural way: $\mathfrak{t}\mathfrak{n}$ is spanned by coloured “floating”
 700 wheels, and such a wheel is mapped into $\mathcal{P}^w(\uparrow_n)$ by attaching its ends to their assigned
 701 strands in arbitrary order. Note that this is well-defined: wheels have only tails, and
 702 tails commute.

703 As vector spaces, the statement is already proven: $\mathcal{P}^w(\uparrow_n)$ is generated by trees
 704 and wheels (with the all arrow endings fixed on n strands). When factoring out by the
 705 wheels, only trees remain. Trees have one head and many tails. All the tails commute
 706 with each other, and commuting a tail with a head on a strand costs a wheel (by \overrightarrow{STU}),
 707 thus in the quotient the head also commutes with the tails. Therefore, the quotient
 708 is the space of floating (coloured) trees, which we have previously identified with
 709 $\bigoplus_{i=1}^n \mathfrak{lie}_n \cong \mathfrak{a}_n \oplus \mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n$.

710 It remains to show that the maps ι and π are Lie algebra maps as well. For ι this is
 711 easy: the Lie algebra $\mathfrak{t}\mathfrak{n}$ is commutative, and is mapped to the commutative (due to
 712 TC) subalgebra of $\mathcal{P}^w(\uparrow_n)$ generated by wheels.

713 To show that π is a map of Lie algebras we give two proofs, first a “hands-on” one,
 714 then a “conceptual” one.

715 **Hands-on argument** \mathfrak{a}_n is the image of single arrows on one strand. These commute
 716 with everything in $\mathcal{P}^w(\uparrow_n)$, and so does \mathfrak{a}_n in the direct sum $\mathfrak{a}_n \oplus \mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n$.

717 It remains to show that the bracket of $\mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n$ works the same way as commuting
 718 trees in $\mathcal{P}^w(\uparrow_n)$. Let D and D' be elements of $\mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n$ represented by (a_1, \dots, a_n) and
 719 (a'_1, \dots, a'_n) , meaning that $D(x_i) = [x_i, a_i]$ and $D'(x_i) = [x_i, a'_i]$ for $i = 1, \dots, n$.
 720 Let us compute the commutator of these elements:

$$\begin{aligned} 721 \quad [D, D'](x_i) &= (DD' - D'D)(x_i) = D[x_i, a'_i] - D'[x_i, a_i] \\ 722 &= [[x_i, a_i], a'_i] + [x_i, Da'_i] - [[x_i, a'_i], a_i] - [x_i, D'a_i] \\ 723 &= [x_i, Da'_i - D'a_i + [a_i, a'_i]]. \end{aligned}$$

724 Now let T and T' be two trees in $\mathcal{P}^w(\uparrow_n)/\mathfrak{t}\mathfrak{n}$, their heads on strands i and j ,
 725 respectively (i may or may not equal j). Let us denote by a_i (resp. a'_j) the element
 726 in \mathfrak{lie}_n given by forming the appropriate commutator of the colours of the tails of T 's
 727 (resp. T'). In $\mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n$, let $D = \pi(T)$ and $D' = \pi(T')$. D and D' are determined by
 728 $(0, \dots, a_i, \dots, 0)$, and $(0, \dots, a'_j, \dots, 0)$, respectively. (In each case, the i -th or the
 729 j -th is the only non-zero component.) The commutator of these elements is given
 730 by $[D, D'](x_i) = [Da'_i - D'a_i + [a_i, a'_i], x_i]$, and $[D, D'](x_j) = [Da'_j - D'a_j +$
 731 $[a_j, a'_j], x_j]$. Note that unless $i = j$, $a_j = a'_i = 0$.

732 In $\mathcal{P}^w(\uparrow_n)/\mathfrak{t}\mathfrak{n}$, all tails commute, as well as a head of a tree with its own tails.
 733 Therefore, commuting two trees only incurs a cost when commuting a head of one
 734 tree over the tails of the other on the same strand, and the two heads over each other,
 735 if they are on the same strand.

736 If $i \neq j$, then commuting the head of T over the tails of T' by \overrightarrow{STU} costs a sum
 737 of trees given by Da'_j , with heads on strand j , while moving the head of T' over the
 738 tails of T costs exactly $-D'a_i$, with heads on strand i , as needed.

739 If $i = j$, then everything happens on strand i , and the cost is $(Da'_i - D'a_i + [a_i, a'_i])$,
 740 where the last term happens when the two heads cross each other.

741 **Conceptual argument** There is an action of $\mathcal{P}^w(\uparrow_n)$ on lie_n , as follows: introduce and
 742 extra strand on the right. An element L of lie_n corresponds to a tree with its head on
 743 the extra strand. Its commutator with an element of $\mathcal{P}^w(\uparrow_n)$ (considered as an element
 744 of $\mathcal{P}^w(\uparrow_{n+1})$ by the obvious inclusion) is again a tree with head on strand $(n + 1)$,
 745 defined to be the result of the action.

746 Since L has only tails on the first n strands, elements of tr_n , which also only have
 747 tails, act trivially. So do single (local) arrows on one strand (\mathfrak{a}_n). It remains to show
 748 that trees act as tdet_n , and it is enough to check this on the generators of lie_n (as the
 749 Leibniz rule is obviously satisfied). The generators of lie_n are arrows pointing from
 750 one of the first n strands, say strand i , to strand $(n + 1)$. A tree with head on strand
 751 i acts on this element, according \overrightarrow{STU} , by forming the commutator, which is exactly
 752 the action of tdet_n . □

753 To identify $\mathcal{P}^w(\uparrow_n)$ as the semidirect product $\text{tr}_n \rtimes (\mathfrak{a}_n \oplus \text{tdet}_n)$, it remains to show
 754 that the short exact sequence of the Proposition splits. This is indeed the case, although
 755 not canonically. Two —of the many— splitting maps $u, l : \text{tdet}_n \oplus \mathfrak{a}_n \rightarrow \mathcal{P}^w(\uparrow_n)$ are
 756 described as follows: $\text{tdet}_n \oplus \mathfrak{a}_n$ is identified with $\bigoplus_{i=1}^n \text{lie}_n$, which in turn is identified
 757 with floating (coloured) trees. A map to $\mathcal{P}^w(\uparrow_n)$ can be given by specifying how to
 758 place the legs on their specified strands. A tree may have many tails but has only one
 759 head, and due to TC , only the positioning of the head matters. Let u (for *upper*) be
 760 the map placing the head of each tree above all its tails on the same strand, while l (for
 761 *lower*) places the head below all the tails. It is obvious that these are both Lie algebra
 762 maps and that $\pi \circ u$ and $\pi \circ l$ are both the identity of $\text{tdet}_n \oplus \mathfrak{a}_n$. This makes $\mathcal{P}^w(\uparrow_n)$
 763 a semidirect product.

764 *Remark 3.20* Let tr_n^s denote tr_n mod out by its degree one part (one-wheels). Since
 765 the RI relation is in the kernel of π , there is a similar split exact sequence

$$766 \quad 0 \rightarrow \text{tr}_n^s \xrightarrow{\bar{\iota}} \mathcal{P}^{sw} \xrightarrow{\bar{\pi}} \mathfrak{a}_n \oplus \text{tdet}_n.$$

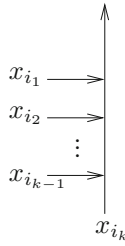
767 **Definition 3.21** For any $D \in \text{tdet}_n$, $(l - u)D$ is in the kernel of π , therefore is in the
 768 image of ι , so $\iota^{-1}(l - u)D$ makes sense. We call this element $\text{div}D$.

769 **Definition 3.22** In [2] div is defined as follows: $\text{div}(a_1, \dots, a_n) := \sum_{k=1}^n \text{tr}((\partial_k a_k) x_k)$,
 770 where ∂_k picks out the words of a sum which end in x_k and deletes their last letter x_k ,
 771 and deletes all other words (the ones which do not end in x_k).

772 **Proposition 3.23** *The div of Definition 3.21 and the div of [2] are the same.*

773 *Proof* It is enough to verify the claim for the linear generators of tdet_n , namely, ele-
 774 ments of the form $(0, \dots, a_j, \dots, 0)$, where $a_j \in \text{lie}_n$ or equivalently, single (floating,
 775 coloured) trees, where the colour of the head is j . By the Jacobi identity, each a_j can
 776 be written in a form $a_j = [x_{i_1}, [x_{i_2}, [\dots, x_{i_k}]] \dots]$. Equivalently, by \overrightarrow{THX} , each tree
 777 has a standard “comb” form, as shown on the picture on the right.

Author Proof



778

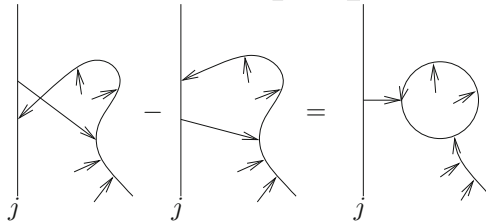
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780 For an associative word $Y = y_1 y_2 \dots y_l \in \text{Ass}_n^+$, we introduce the notation $[Y] :=$
 781 $[y_1, [y_2, [\dots, y_l] \dots]]$. The div of [2] picks out the words that end in x_j , forgets the
 782 rest, and considers these as cyclic words. Therefore, by interpreting the Lie brackets
 783 as commutators, one can easily check that for a_j written as above,

$$784 \quad \text{div}((0, \dots, a_j, \dots, 0)) = \sum_{\alpha: i_\alpha = x_j} -x_{i_1} \dots x_{i_{\alpha-1}} [x_{i_{\alpha+1}} \dots x_{i_k}] x_j. \quad (4)$$

785 □

786 In Definition 3.21, div of a tree is the difference between attaching its head on the
 787 appropriate strand (here, strand j) below all of its tails and above. As shown in the
 788 figure on the right, moving the head across each of the tails on strand j requires an
 789 \overrightarrow{STU} relation, which “costs” a wheel (of trees, which is equivalent to a sum of honest
 790 wheels). Namely, the head gets connected to the tail in question. So div of the tree
 791 represented by a_j is given by



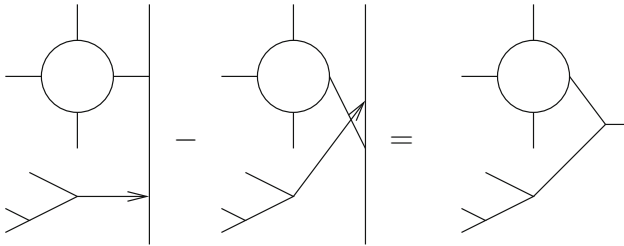
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$$793 \quad \sum_{\alpha: x_{i_\alpha} = j} \text{“connect the head to the } \alpha \text{ leaf”}.$$

794 This in turn gets mapped to the formula above via the correspondence between
 795 wheels and cyclic words. □

796 *Remark 3.24* There is an action of \mathfrak{tder}_n on \mathfrak{tr}_n as follows. Represent a cyclic word
 797 $w \in \mathfrak{tr}_n$ as a wheel in $\mathcal{P}^w(\uparrow_n)$ via the map ι . Given an element $D \in \mathfrak{tder}_n$, $u(D)$,
 798 as defined above, is a tree in $\mathcal{P}^w(\uparrow_n)$ whose head is above all of its tails. We define
 799 $D \cdot w := \iota^{-1}(u(D)\iota(w) - \iota(w)u(D))$. Note that $u(D)\iota(w) - \iota(w)u(D)$ is in the image
 800 of ι , i.e., a linear combination of wheels, for the following reason. The wheel $\iota(w)$

801 has only tails. As we commute the tree $u(D)$ across the wheel, the head of the tree is
 802 commuted across tails of the wheel on the same strand. Each time this happens the
 803 cost, by the \overrightarrow{STU} relation, is a wheel with the tree attached to it, as shown on the
 804 right, which in turn (by $\overrightarrow{IH\bar{X}}$ relations, as Fig. 10 shows) is a sum of wheels. Once
 805 the head of the tree has been moved to the top, the tails of the tree commute up for
 806 free by TC . Note that the alternative definition, $D \cdot w := \iota^{-1}(l(D)\iota(w) - \iota(w)l(D))$
 807 is in fact equal to the definition above.



809 **Definition 3.25** In [2], the group $TAut_n$ is defined as $\exp(\mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n)$. Note that $\mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n$
 810 is positively graded, hence it integrates to a group. Note also that $TAut_n$ is the group
 811 of “basis-conjugating” automorphisms of \mathfrak{lie}_n , i.e., for $g \in TAut_n$, and any $x_i, i =$
 812 $1, \dots, n$ generator of \mathfrak{lie}_n , there exists an element $g_i \in \exp(\mathfrak{lie}_n)$ such that $g(x_i) =$
 813 $g_i^{-1}x_i g_i$.

814 The action of $\mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n$ on $\mathfrak{t}\mathfrak{r}_n$ lifts to an action of $TAut_n$ on $\mathfrak{t}\mathfrak{r}_n$, by interpreting
 815 exponentials formally, in other words e^D acts as $\sum_{n=0}^{\infty} \frac{D^n}{n!}$. The lifted action is by
 816 conjugation: for $w \in \mathfrak{t}\mathfrak{r}_n$ and $e^D \in TAut_n, e^D \cdot w = \iota^{-1}(e^{uD}\iota(w)e^{-uD})$.

817 Recall that in Section 5.1 of [2] Alekseev and Torossian construct a map
 818 $j: TAut_n \rightarrow \mathfrak{t}\mathfrak{r}_n$ which is characterized by two properties: the cocycle property

$$j(gh) = j(g) + g \cdot j(h), \tag{5}$$

820 where in the second term multiplication by g denotes the action described above; and the
 821 condition

$$\frac{d}{ds} j(\exp(sD))|_{s=0} = \text{div}(D). \tag{6}$$

823 Now let us interpret j in our context.

824 **Definition 3.26** The adjoint map $*$: $\mathcal{A}^w(\uparrow_n) \rightarrow \mathcal{A}^w(\uparrow_n)$ acts by “flipping over
 825 diagrams and negating arrow heads on the skeleton”. In other words, for an arrow
 826 diagram D ,

$$D^* := (-1)^{\#\{\text{tails on skeleton}\}} S(D),$$

828 where S denotes the map which switches the orientation of the skeleton strands (i.e.
 829 flips the diagram over), and multiplies by $(-1)^{\#\text{skeleton vertices}}$.

830 **Proposition 3.27** For $D \in \mathfrak{tdet}_n$, define a map $J : TAut_n \rightarrow \exp(\mathfrak{tr}_n)$ by $J(e^D) :=$
 831 $e^{uD}(e^{uD})^*$. Then

$$832 \quad \exp(j(e^D)) = J(e^D).$$

833 *Proof* Note that $(e^{uD})^* = e^{-lD}$, due to “Tails Commute” and the fact that a tree has
 834 only one head.

835 Let us check that $\log J$ satisfies properties (5) and (6). Namely, with $g = e^{D_1}$ and
 836 $h = e^{D_2}$, and using that \mathfrak{tr}_n is commutative, we need to show that

$$837 \quad J(e^{D_1}e^{D_2}) = J(e^{D_1})(e^{uD_1} \cdot J(e^{D_2})), \quad (7)$$

838 where \cdot denotes the action of \mathfrak{tdet}_n on \mathfrak{tr}_n ; and that

$$839 \quad \frac{d}{ds} J(e^{sD})|_{s=0} = \operatorname{div} D. \quad (8)$$

840 Indeed, with $\operatorname{BCH}(D_1, D_2) = \log e^{D_1}e^{D_2}$ being the standard Baker–Campbell–
 841 Hausdorff formula,

$$\begin{aligned} 842 \quad J(e^{D_1}e^{D_2}) &= J(e^{\operatorname{BCH}(D_1, D_2)}) = e^{u(\operatorname{BCH}(D_1, D_2))} e^{-l(\operatorname{BCH}(D_1, D_2))} \\ 843 \quad &= e^{\operatorname{BCH}(uD_1, uD_2)} e^{-\operatorname{BCH}(lD_1, lD_2)} \\ 844 \quad &= e^{uD_1} e^{uD_2} e^{-lD_2} e^{-lD_1} = e^{uD_1} (e^{uD_2} e^{-lD_2}) e^{-uD_1} e^{lD_1} \\ 845 \quad &= (e^{uD_1} \cdot J(D_2)) J(D_1), \end{aligned}$$

846 as needed.

847 As for condition (6), a direct computation of the derivative yields

$$848 \quad \frac{d}{ds} J(e^{sD})|_{s=0} = uD - lD = \operatorname{div} D,$$

849 as desired. □

850 3.3 The relationship with u-tangles

851 Let uT be the planar algebra of classical, or “usual” tangles. There is a map $a : uT \rightarrow$
 852 wT of u -tangles into w -tangles: algebraically, it is defined in the obvious way on the
 853 planar algebra generators of uT . (It can also be interpreted topologically as Satoh’s
 854 tubing map, see [38, Section 3.1.1], where a u -tangle is a tangle drawn on a sphere.
 855 However, it is only conjectured that the circuit algebra presented here is a Reidemeister
 856 theory for “tangled ribbon tubes in \mathbb{R}^4 ”.) The map a induces a corresponding map
 857 $\alpha : \mathcal{A}^u \rightarrow \mathcal{A}^{sw}$, which maps an ordinary Jacobi diagram (i.e., unoriented chords with
 858 internal trivalent vertices modulo the usual AS , IHX and STU relations) to the sum
 859 of all possible orientations of its chords (many of which are zero in \mathcal{A}^{sw} due to the
 860 “two in one out” rule).

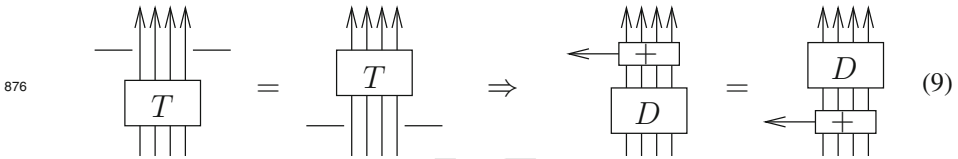
861 It is tempting to ask whether the square on the left commutes. Unfortunately, this
 862 question hardly makes sense, as there is no canonical choice for the dotted line in it.
 863 Similarly to the braid case of [38, Section 2.5.5], the definition of the homomorphic
 864 expansion (Kontsevich integral) for u -tangles typically depends on various choices of
 865 “parenthesizations”. Choosing parenthesizations, this square becomes commutative
 866 up to some fixed corrections. The details are in Proposition 4.18.

$$867 \begin{array}{ccc} uT & \xrightarrow{Z^u} & \mathcal{A}^u \\ \downarrow a & & \downarrow \alpha \\ wT & \xrightarrow{Z^w} & \mathcal{A}^{sw} \end{array}$$

868 Yet already at this point we can recover something from the existence of the map
 869 $a : uT \rightarrow wT$, namely an interpretation of the Alekseev and Torossian [2] space of
 870 special derivations,

$$871 \mathfrak{sd\epsilon r}_n := \left\{ D \in \mathfrak{t\delta\epsilon r}_n : D \left(\sum_{i=1}^n x_i \right) = 0 \right\}.$$

872 Recall from Remark 3.14 that in general it is not possible to slide a strand under an
 873 arbitrary w -tangle. However, it is possible to slide strands freely under tangles *in the*
 874 *image of a* , and thus by reasoning similar to the reasoning in Remark 3.14, diagrams
 875 D in the image of α respect “tail-invariance”:



877 Let $\mathcal{P}^u(\uparrow_n)$ denote the primitives of $\mathcal{A}^u(\uparrow_n)$, that is, Jacobi diagrams that remain
 878 connected when the skeleton is removed. Remember that $\mathcal{P}^w(\uparrow_n)$ stands for the primi-
 879 tives of $\mathcal{A}^w(\uparrow_n)$. Equation (9) readily implies that the image of the composition

$$880 \mathcal{P}^u(\uparrow_n) \xrightarrow{\alpha} \mathcal{P}^w(\uparrow_n) \xrightarrow{\pi} \mathfrak{a}_n \oplus \mathfrak{t\delta\epsilon r}_n$$

881 is contained in $\mathfrak{a}_n \oplus \mathfrak{sd\epsilon r}_n$. Even better is true.

882 **Theorem 3.28** *The image of $\pi \alpha$ is precisely $\mathfrak{a}_n \oplus \mathfrak{sd\epsilon r}_n$.*

883 This theorem was first proven by Drinfel’d (Lemma after Proposition 6.1 in [20]),
 884 but the proof we give here is due to Levine [31].

885 *Proof* Let lie_n^d denote the degree d piece of lie_n . Let V_n be the vector space with basis
 886 x_1, x_2, \dots, x_n . Note that

887
$$V_n \otimes \text{lie}_n^d \cong \bigoplus_{i=1}^n \text{lie}_n^d \cong (\text{tder}_n \oplus \mathfrak{a}_n)^d,$$

888 where tder_n is graded by the number of tails of a tree, and \mathfrak{a}_n is contained in degree 1.
 889 The bracket defines a map $\beta: V_n \otimes \text{lie}_n^d \rightarrow \text{lie}_n^{d+1}$: for $a_i \in \text{lie}_n^d$ where $i = 1, \dots, n$,
 890 the “tree” $D = (a_1, a_2, \dots, a_n) \in (\text{tder}_n \oplus \mathfrak{a}_n)^d$ is mapped to

891
$$\beta(D) = \sum_{i=1}^n [x_i, a_i] = D \left(\sum_{i=1}^n x_i \right),$$

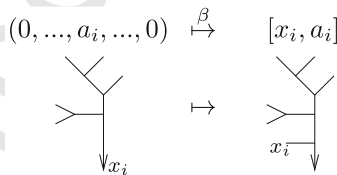
892 where the first equality is by the definition of tensor product and the bracket, and the
 893 second is by the definition of the action of tder_n on lie_n .

894 Since \mathfrak{a}_n is contained in degree 1, by definition $\text{sder}_n^d = (\ker \beta)^d$ for $d \geq 2$. In
 895 degree 1, \mathfrak{a}_n is obviously in the kernel, hence $(\ker \beta)^1 = \mathfrak{a}_n \oplus \text{sder}_n^1$. So overall,
 896 $\ker \beta = \mathfrak{a}_n \oplus \text{sder}_n$.

897 We want to study the image of the map $\mathcal{P}^u(\uparrow^n) \xrightarrow{\pi\alpha} \mathfrak{a}_n \oplus \text{tder}_n$. Under α , all
 898 connected Jacobi diagrams that are not trees or wheels go to zero, and under π so do
 899 all wheels. Furthermore, π maps trees that live on n strands to “floating” trees with
 900 univalent vertices coloured by the strand they used to end on. So for determining the
 901 image, we may replace $\mathcal{P}^u(\uparrow^n)$ by the space \mathcal{T}_n of connected *unoriented* “floating
 902 trees” (uni-trivalent graphs), the ends (univalent vertices) of which are coloured by
 903 the $\{x_i\}_{i=1, \dots, n}$. We denote the degree d piece of \mathcal{T}_n , i.e., the space of trees with $d + 1$
 904 ends, by \mathcal{T}_n^d . Abusing notation, we shall denote the map induced by $\pi\alpha$ on \mathcal{T}_n by
 905 $\alpha: \mathcal{T}_n \rightarrow \mathfrak{a}_n \oplus \text{tder}_n$. Since choosing a “head” determines the entire orientation of
 906 a tree by the two-in-one-out rule, α maps a tree in \mathcal{T}_n^d to the sum of $d + 1$ ways of
 907 choosing one of the ends to be the “head”.

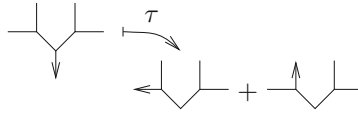
908 We want to show that $\ker \beta = \text{im } \alpha$. This is equivalent to saying that $\bar{\beta}$ is injective,
 909 where $\bar{\beta}: V_n \otimes \text{lie}_n / \text{im } \alpha \rightarrow \text{lie}_n$ is map induced by β on the quotient by $\text{im } \alpha$. \square

910 The degree d piece of $V_n \otimes \text{lie}_n$, in the pictorial description, is generated by floating
 911 trees with d tails and one head, all coloured by $x_i, i = 1, \dots, n$. This is mapped to
 912 lie_n^{d+1} , which is isomorphic to the space of floating trees with $d + 1$ tails and one head,
 913 where only the tails are coloured by the x_i . The map β acts as shown on the picture
 914 on the right.



915
 916 We show that $\bar{\beta}$ is injective by exhibiting a map $\tau: \text{lie}_n^{d+1} \rightarrow V_n \otimes \text{lie}_n / \text{im } \alpha$ so
 917 that $\tau\bar{\beta} = I$. τ is defined as follows: given a tree with one head and $d + 1$ tails τ acts

918 by deleting the head and the arc connecting it to the rest of the tree and summing over
 919 all ways of choosing a new head from one of the tails on the left half of the tree relative
 920 to the original placement of the head (see the picture on the right). As long as we show
 921 that τ is well-defined, it follows from the definition and the pictorial description of β
 922 that $\tau\bar{\beta} = I$.



924 For well-definedness we need to check that the images of \vec{AS} and $\vec{IH\bar{X}}$
 925 under τ are in the image of α . This we do in the picture below. In both cases it is
 926 enough to check the case when the “head” of the relation is the head of the tree itself,
 927 as otherwise an \vec{AS} or $\vec{IH\bar{X}}$ relation in the domain is mapped to an \vec{AS} or $\vec{IH\bar{X}}$
 928 relation, thus zero, in the image.

929
$$\vec{AS}: \left(\text{tree} \right) + \left(\text{tree} \right) \xrightarrow{\tau} \left(\text{wavy} \right) + \left(\text{wavy} \right) \in \text{im } \alpha$$

930

931
$$\vec{IH\bar{X}}: \left(\text{tree} \right) - \left(\text{tree} \right) + \left(\text{tree} \right) \xrightarrow{\tau} \left(\text{tree} \right) - \left(\text{tree} \right) - \left(\text{tree} \right) + \left(\text{tree} \right) + \left(\text{tree} \right) =$$

$$= - \left(\text{tree} \right) - \left(\text{tree} \right) - \left(\text{tree} \right) \in \text{im } \alpha$$

932 In the $\vec{IH\bar{X}}$ picture, in higher degrees A, B and C may denote an entire tree. In this
 933 case, the arrow at A (for example) means the sum of all head choices from the tree A .

934 *Comment 3.29* In view of the relation between the right half of Eq. (9) and the special
 935 derivations \mathfrak{sdet} , it makes sense to call w-tangles that satisfy the condition in the left
 936 half of Eq. (9) “special”. The a images of u-tangles are thus special. We do not know
 937 if the global version of Theorem 3.28 holds true. Namely, we do not know whether
 938 every special w-tangle is the a -image of a u-tangle.

939 **3.4 The local topology of w-tangles**

940 So far throughout this section we have presented w -tangles as a Reidemeister theory: a
 941 circuit algebra given by generators and relations. There is a topological intuition behind
 942 this definition: we can interpret the strings of a w -tangle diagram as oriented tubes in
 943 \mathbb{R}^4 , as shown in Fig. 11. Each tube has a 3-dimensional “filling”, and each crossings
 944 represents a ribbon intersection between the tubes where the one corresponding to
 945 the under-strand intersects the filling of the over-strand. (For an explanation of ribbon



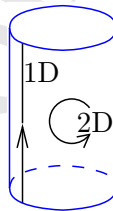
Fig. 11 A virtual crossing corresponds to non-intersecting tubes, while a crossing means that the tube corresponding to the under strand “goes through” the tube corresponding to the over strand

946 intersections see [38, Section 2.2.2].) In Fig. 11 we use the drawing conventions of
 947 [15]: we draw surfaces as if projected from \mathbb{R}^4 to \mathbb{R}^3 , and cut them open when they
 948 are “hidden” by something with a higher 4-th coordinate.

949 Note that w-braids can also be thought of in terms of flying rings, with “time” being
 950 the fourth dimension; this is equivalent to the tube interpretation in the obvious way.
 951 In this language a crossing represents a ring (the under strand), flying through another
 952 (the over strand). This is described in detail in [38, Section 2.2.1].

953 The assignment of tangled ribbon tubes in \mathbb{R}^4 to w-tangles is well-defined (the
 954 Reidemeister and OC relations are satisfied), and after Satoh [36] we call it the tubing
 955 map and denote it by δ : $\{\text{w-tangles}\} \rightarrow \{\text{Ribbon tubes in } \mathbb{R}^4\}$. It is natural to expect
 956 that δ is an isomorphism, and indeed it is a surjection. However, the injectivity of
 957 δ remains unproven even for long w-knots. Nonetheless, ribbon tubes in \mathbb{R}^4 will
 958 serve as the topological motivation and local topological interpretation behind the
 959 circuit algebras presented in this paper. In [38, Section 3.1.1] we present a topological
 960 construction for δ . We will mention that construction occasionally in this paper, but
 961 only for motivational purposes.

962 We observe that the ribbon tubes in the image of δ are endowed with two orientations,
 963 we will call these the 1- and 2-dimensional orientations. The one dimensional
 964 orientation is the direction of the tube as a “strand” of the tangle. In other words,
 965 each tube has a “core”¹¹: a distinguished line along the tube, which is oriented as a
 966 1-dimensional manifold. Furthermore, the tube as a 2-dimensional surface is oriented
 967 as given by δ . An example is shown on the right.

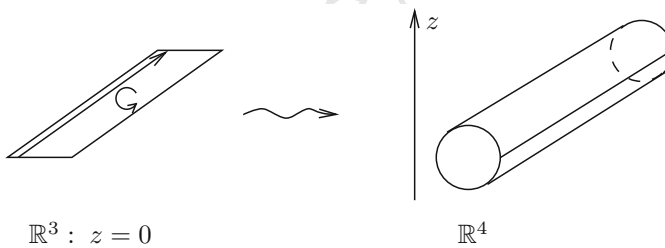


968
 969 Next we wish to understand the topological meaning of crossing signs. Recall that
 970 a tube in \mathbb{R}^4 has a “filling”: a solid (3-dimensional) cylinder embedded in \mathbb{R}^4 , with
 971 boundary the tube, and the 2D orientation of the tube induces an orientation of its
 972 filling as a 3-dimensional manifold. At a (non-virtual) crossing the core of one tube

¹¹ The core of Lord Voldemort’s wand was made of a phoenix feather.

973 intersects the filling of another transversely. Due to the complementary dimensions,
 974 the intersection is a single point, and the 1D orientation of the core along with the 3D
 975 orientation of the filling it passes through determines an orientation of the ambient
 976 space. We say that the crossing is positive if this agrees with the standard orientation of
 977 \mathbb{R}^4 , and negative otherwise. Hence, there are four types of crossings, given by whether
 978 the core of tube A intersects the filling of B or vice versa, and two possible signs in each
 979 case. In the flying ring interpretation, the 1D orientation of the tube is the direction of
 980 the flow of time. The 2D and 1D orientations of the tube together induce an orientation
 981 of the flying ring which is a cross-section of the tube at each moment. Hence, saying
 982 “below” and “above” the ring makes sense, and there are four types of crossings: ring
 983 A flies through ring B from below or from above; and ring B flies through ring A from
 984 below or from above (cf. [38, Exercise 2.7]). A crossing is positive if the inner ring
 985 comes from below, and negative otherwise.

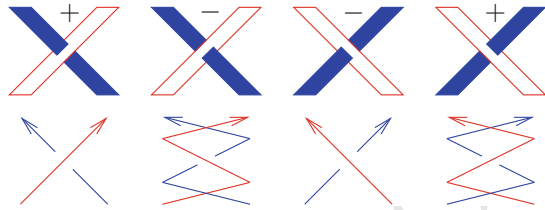
986 We take the opportunity here to introduce another notation, to be called the “band
 987 notation”, which is more suggestive of the 4D topology than the strand notation we
 988 have been using so far. We represent a tube in \mathbb{R}^4 by a picture of an oriented band
 989 in \mathbb{R}^3 . By “oriented band” we mean that it has two orientations: a 1D direction (for
 990 example an orientation of one of the edges), and a 2D orientation as a surface. To
 991 interpret the 3D picture of a band as an tube in \mathbb{R}^4 , we add an extra coordinate. Let
 992 us refer to the \mathbb{R}^3 coordinates as x , y and t , and to the extra coordinate as z . Think of
 993 \mathbb{R}^3 as being embedded in \mathbb{R}^4 as the hyperplane $z = 0$, and think of the band as being
 994 made of a thin double membrane. Push the membrane up and down in the z direction
 995 at each point as far as the distance of that point from the boundary of the band, as
 996 shown on the right. Furthermore, keep the 2D orientation of the top membrane (the
 997 one being pushed up), but reverse it on the bottom. This produces an oriented tube
 998 embedded in \mathbb{R}^4 .



999

1000 In band notation, the four possible crossings appear as in Fig. 12, where underneath
 1001 each crossing we indicate the corresponding strand picture. The signs for each type of
 1002 crossing are also shown. Note that the sign of a crossing depends on the 2D orientation
 1003 of the over-strand, as well as the 1D direction of the under-strand. Hence, switching
 1004 only the direction (1D orientation) of a strand changes the sign of the crossing if and
 1005 only if the strand involved is the under strand. However, fully changing the orientation
 1006 (both 1D and 2D) always switches the sign of the crossing. Note that switching the
 1007 strand direction in the strand notation corresponds to the complete (both 1D and 2D)
 1008 orientation switch.

Fig. 12 Crossings and crossing signs in band notation



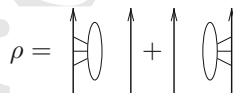
3.5 Good properties and uniqueness of the homomorphic expansion

In much the same way as in the case of braids [38, Section 2.5.1], Z has a number of good properties with respect to various tangle operations: it is group-like¹²; it commutes with adding an inert strand (note that this is a circuit algebra operation, hence it doesn't add anything beyond homomorphicity); and it commutes with deleting a strand and with strand orientation reversals. All but the last of these were explained in the context of braids and the explanations still hold. Orientation reversal $S_k : wT \rightarrow wT$ is the operation which reverses the orientation of the k -th component. Note that in the world of topology (via Satoh's tubing map) this means reversing both the 1D and the 2D orientations. The induced diagrammatic operation $S_k : \mathcal{A}^w(T) \rightarrow \mathcal{A}^w(S_k(T))$, where T denotes the skeleton of a given w -tangle, acts by multiplying each arrow diagram by (-1) raised to the power the number of arrow endings (both heads and tails) on the k -th strand, as well as reversing the strand orientation. Saying that " Z commutes with S_k " means that the appropriate square commutes.

The following theorem asserts that a well-behaved homomorphic expansion of w -tangles is unique:

Theorem 3.30 *The only homomorphic expansion satisfying the good properties described above is the Z defined in Sect. 3.1.*

Proof We first prove the following claim: Assume, by contradiction, that Z' is a different homomorphic expansion of w -tangles with the good properties described above. Let $R' = Z'(\nearrow)$ and $R = Z(\nearrow)$, and denote by ρ the lowest degree homogeneous non-vanishing term of $R' - R$. (Note that R' determines Z' , so if $Z' \neq Z$, then $R' \neq R$.) Suppose ρ is of degree k . Then we claim that $\rho = \alpha_1 w_k^1 + \alpha_2 w_k^2$ is a linear combination of w_k^1 and w_k^2 , where w_k^i denotes a k -wheel living on strand i , as shown on the right.



Before proving the claim, note that it leads to a contradiction. Let d_i denote the operation "delete strand i ". Then up to degree k , we have $d_1(R') = \alpha_2 w_k^1$ and $d_2(R') = \alpha_1 w_k^2$, but Z' is compatible with strand deletions, so $\alpha_1 = \alpha_2 = 0$. Hence Z is unique, as stated.

¹² In practice this simply means that the value of the crossing is an exponential.

Author Proof

1040 On to the proof of the claim, note that Z' being an expansion determines the degree
 1041 1 term of R' (namely, the single arrow a^{12} from strand 1 to strand 2, with coefficient
 1042 1). So we can assume that $k \geq 2$. Note also that since both R' and R are group-like,
 1043 ρ is primitive. Hence ρ is a linear combination of connected diagrams, namely trees
 1044 and wheels.

1045 Both R and R' satisfy the Reidemeister 3 relation:

$$1046 \quad R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}, \quad R'^{12}R'^{13}R'^{23} = R'^{23}R'^{13}R'^{12}$$

1047 where the superscripts denote the strands on which R is placed (compare with the proof
 1048 of Theorem 3.10). We focus our attention on the degree $k + 1$ part of the equation for
 1049 R' , and use that up to degree $k + 1$. We can write $R' = R + \rho + \mu$, where μ denotes
 1050 the degree $k + 1$ homogeneous part of $R' - R$. Thus, up to degree $k + 1$, we have

$$1051 \quad (R^{12} + \rho^{12} + \mu^{12})(R^{13} + \rho^{13} + \mu^{13})(R^{23} + \rho^{23} + \mu^{23}) \\ 1052 \quad = (R^{23} + \rho^{23} + \mu^{23})(R^{13} + \rho^{13} + \mu^{13})(R^{12} + \rho^{12} + \mu^{12}).$$

1053 The homogeneous degree $k + 1$ part of this equation is a sum of some terms which
 1054 contain ρ and some which don't. The diligent reader can check that those which don't
 1055 involve ρ cancel on both sides, either due to the fact that R satisfies the Reidemeister
 1056 3 relation, or by simple degree counting. Rearranging all the terms which do involve
 1057 ρ to the left side, we get the following equation, where a^{ij} denotes an arrow pointing
 1058 from strand i to strand j :

$$1059 \quad [a^{12}, \rho^{13}] + [\rho^{12}, a^{13}] + [a^{12}, \rho^{23}] + [\rho^{12}, a^{23}] + [a^{13}, \rho^{23}] + [\rho^{13}, a^{23}] = 0. \quad (10)$$

1060 The third and fifth terms sum to $[a^{12} + a^{13}, \rho^{23}]$, which is zero due to the “head-
 1061 invariance” of diagrams, as in Remark 3.14.

1062 We treat the tree and wheel components of ρ separately. Let us first assume that
 1063 ρ is a linear combination of trees. Recall that the space of trees on two strands is
 1064 isomorphic to $\mathfrak{lie}_2 \oplus \mathfrak{lie}_2$, the first component given by trees whose head is on the first
 1065 strand, and the second component by trees with their head on the second strand. Let
 1066 $\rho = \rho_1 + \rho_2$, where ρ_i is the projection to the i -th component for $i = 1, 2$.

1067 Note that due to TC , we have $[a^{12}, \rho_2^{13}] = [\rho_2^{12}, a^{13}] = [\rho_1^{12}, a^{23}] = 0$. So Eq.
 1068 (10) reduces to

$$1069 \quad [a^{12}, \rho_1^{13}] + [\rho_1^{12}, a^{13}] + [\rho_2^{12}, a^{23}] + [\rho_1^{13}, a^{23}] + [\rho_2^{13}, a^{23}] = 0$$

1070 The left side of this equation lives in $\bigoplus_{i=1}^3 \mathfrak{lie}_3$. Notice that only the first term lies in
 1071 the second direct sum component, while the second, third and last terms live in the
 1072 third one, and the fourth term lives in the first. This in particular means that the first
 1073 term is itself zero. By \overrightarrow{STU} , this implies

$$1074 \quad 0 = [a^{12}, \rho_1^{13}] = -[\rho_1, x_1]_2^{13},$$

1075 where $[\rho_1, x_1]_2^{13}$ means the tree defined by the element $[\rho_1, x_1] \in \text{lie}_2$, with its tails
 1076 on strands 1 and 3, and head on strand 2. Hence, $[\rho_1, x_1] = 0$, so ρ_1 is a multiple of
 1077 x_1 . The tree given by $\rho_1 = x_1$ is a degree 1 element, a possibility we have eliminated,
 1078 so $\rho_1 = 0$.

1079 Equation (10) is now reduced to

$$1080 \quad [\rho_2^{12}, a^{23}] + [\rho_2^{13}, a^{23}] = 0.$$

1081 Both terms are words in lie_3 , but notice that the first term does not involve the letter x_3 .
 1082 This means that if the second term involves x_3 at all, i.e., if ρ_2 has tails on the second
 1083 strand, then both terms have to be zero individually. Assuming this and looking at
 1084 the first term, ρ_2^{12} is a Lie word in x_1 and x_2 , which does involve x_2 by assumption.
 1085 We have $[\rho_2^{12}, a^{23}] = [x_2, \rho_2^{12}] = 0$, which implies ρ_2^{12} is a multiple of x_2 , in other
 1086 words, ρ is a single arrow on the second strand. This is ruled out by the assumption
 1087 that $k \geq 2$.

1088 On the other hand if the second term does not involve x_3 at all, then ρ_2 has no tails
 1089 on the second strand, hence it is of degree 1, but again $k \geq 2$. We have proven that the
 1090 “tree part” of ρ is zero.

1091 So ρ is a linear combination of wheels. Wheels have only tails, so the first, second
 1092 and fourth terms of (10) are zero due to the tails commute relation. What remains is
 1093 $[\rho^{13}, a^{23}] = 0$. We assert that this is true if and only if each linear component of ρ
 1094 has all of its tails on one strand.

1095 To prove this, recall each wheel of ρ^{13} represents a cyclic word in letters x_1 and
 1096 x_3 . The map $r: \rho^{13} \mapsto [\rho^{13}, a^{23}]$ is a map $\text{tt}_2 \rightarrow \text{tt}_3$, which sends each cyclic word
 1097 in letters x_1 and x_3 to the sum of all ways of substituting $[x_2, x_3]$ for one of the x_3 's in
 1098 the word. Note that if we expand the commutators, then all terms that have x_2 between
 1099 two x_3 's cancel. Hence all remaining terms will be cyclic words in x_1 and x_3 with a
 1100 single occurrence of x_2 in between an x_1 and an x_3 .

1101 We construct an almost-inverse r' to r : for a cyclic word w in tt_3 with one occurrence
 1102 of x_2 , let r' be the map that deletes x_2 from w and maps it to the resulting word in tt_2
 1103 if x_2 is followed by x_3 in w , and maps it to 0 otherwise. On the rest of tt_3 the map r'
 1104 may be defined to be 0.

1105 The composition $r'r$ takes a cyclic word in x_1 and x_3 to itself multiplied by the
 1106 number of times a letter x_3 follows a letter x_1 in it. The kernel of this map can consist
 1107 only of cyclic words that do not contain the sub-word x_3x_1 , namely, these are the
 1108 words of the form x_3^k or x_1^k . Such words are indeed in the kernel of r , so these make
 1109 up exactly the kernel of r . This is exactly what needed to be proven: all wheels in ρ
 1110 have all their tails on one strand.

1111 This concludes the proof of the claim, and the proof of the theorem. \square

1112 4 w-Tangled foams

1113 **Section summary** In this section we add “foam vertices” to w-tangles (and a few
 1114 lesser things as well) and ask the same questions we asked before; primarily, “is
 1115 there a homomorphic expansion?”. As we shall see, in the current context this

1116 question is equivalent to the Alekseev–Torossian [2] version of the Kashiwara
 1117 and Vergne [26] problem and explains the relationship between these topics and
 1118 Drinfel’d’s theory of associators.

1119 **4.1 The circuit algebra of w-tangled foams**

1120 In the same manner as we did for tangles, we will present the circuit algebra of w-
 1121 tangled foams via its Reidemeister-style diagrammatic description accompanied by a
 1122 local topological interpretation.

1123 **Definition 4.1** Let wTF^o (where o stands for “orientable”, to be explained in Sect. 4.5)
 1124 be the algebraic structure

1125
$$wTF^o = \text{CA} \left\langle \begin{array}{c} \begin{array}{|l} \nearrow \nwarrow \\ \nwarrow \nearrow \\ \bullet \\ \nearrow \nwarrow \\ \nwarrow \nearrow \end{array} \mid \begin{array}{l} \text{w-relations as in} \\ \text{Section 4.1.2} \end{array} \mid \begin{array}{l} \text{w-operations as} \\ \text{in Section 4.1.3} \end{array} \end{array} \right\rangle$$

1126 Hence wTF^o is the circuit algebra generated by the generators listed above and
 1127 described below, modulo the relations described in Sect. 4.1.2, and augmented with
 1128 several “auxiliary operations”, which are a part of the algebraic structure of wTF^o but
 1129 are not a part of its structure as a circuit algebra, as described in Sect. 4.1.3. To be more
 1130 precise, wTF^o is skeleton-graded where the circuit algebra of skeleta \mathcal{S}^o is a version
 1131 of the \mathcal{S} introduced in Sect. 2.4, but with vertices and caps included (as opposed to
 1132 only empty circuits).

1133
 1134 To be completely precise, we have to admit that wTF^o as a circuit algebra has
 1135 more generators than shown above. The last two generators are “foam vertices”, as
 1136 will be explained shortly, and exist in all possible orientations of the three strands.
 1137 Some examples are shown on the right. However, in Sect. 4.1.3 we will describe the
 1138 operation “orientation switch” which allows switching the orientation of any given
 1139 strand. In the algebraic structure which includes this extra operation in addition to the
 1140 circuit algebra structure, the generators of the definition above are enough.



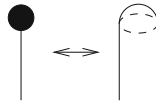
1142 **4.1.1 The generators of wTF^o**

1143 There is topological meaning to each of the generators of wTF^o : they each stand
 1144 for a certain local feature of framed knotted ribbon tubes in \mathbb{R}^4 . As in Sect. 3.4, the
 1145 tubes are oriented as 2-dimensional surfaces, and also have a distinguished core with
 1146 a 1-dimensional orientation (direction).

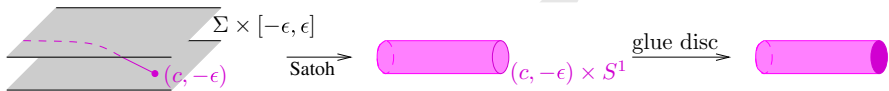
1147 The crossings are as explained in Sect. 3.4: the under-strand denotes the ring flying
 1148 through, or the “thin” tube. Recall that there really are four kinds of crossings, but
 1149 in the circuit algebra the two not shown are obtained from the two that are shown by
 1150 adding virtual crossings (see Figs. 11, 12).

1151 A bulleted end denotes a cap on the tube, or a flying ring that shrinks to a point, as in
 1152 the figure on the right. For further motivation, in terms of the topological construction
 1153 of Satoh’s tubing map [38, Section 3.1.1], the cap means that “the string is attached
 1154 to the bottom of the thickened surface”, as shown in the figure below. We Recall that
 1155 the tubing map is the composition

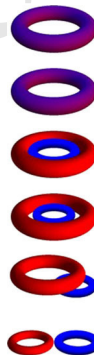
$$\gamma \times S^1 \hookrightarrow \Sigma \times [-\epsilon, \epsilon] \hookrightarrow \mathbb{R}^4.$$



1158 Here γ is a trivalent tangle “drawn on the virtual surface Σ ”, with caps ending on
 1159 $\Sigma \times \{-\epsilon\}$. The first embedding above is the product of this “drawing” with an S^1 ,
 1160 while the second arises from the unit normal bundle of Σ in \mathbb{R}^4 . For each cap $(c, -\epsilon)$
 1161 the tube resulting from Satoh’s map has a boundary component $\partial_c = (c, -\epsilon) \times S^1$.
 1162 Follow the tubing map by gluing a disc to this boundary component to obtain the
 1163 capped tube mentioned above.
 1164



1166 The last two generators denote singular “foam vertices”. As the notation suggests,
 1167 a vertex can be thought of as “half of a crossing”. To make this precise using the flying
 1168 rings interpretation, the first singular vertex represents the movie shown on the left:
 1169 the ring corresponding to the right strand approaches the ring represented by the left
 1170 strand from below, flies inside it, and then the two rings fuse (as opposed to a crossing
 1171 where the ring coming from the right would continue to fly out to above and to the
 1172 left of the other one). The second vertex is the movie where a ring splits radially into
 1173 a smaller and a larger ring, and the small one flies out to the right and below the big
 1174 one.
 1175



Author Proof

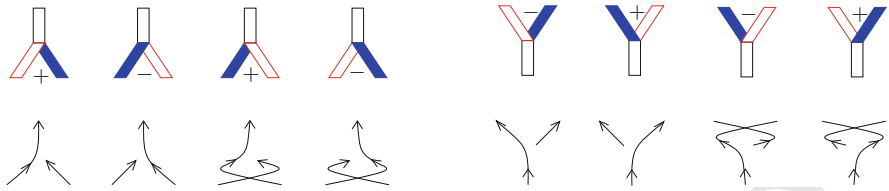


Fig. 13 Vertex types in wTF^o

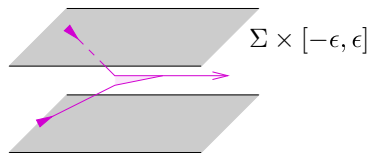
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The vertices can also be interpreted topologically via a natural extension of Satoh’s tubing map. For the first vertex, imagine the broken right strand approaching the continuous left strand directly from below in a thickened surface, as shown.



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The reader might object that there really are four types of vertices (as there are four types of crossings), and each of these can be viewed as a “fuse” or a “split” depending on the strand directions, as shown in Fig. 13. However, looking at the fuse vertices for example, observe that the last two of these can be obtained from the first two by composing with virtual crossings, which always exist in a circuit algebra.

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The sign of a vertex can be defined the same way as the sign of a crossing (see Sect. 3.4). We will sometimes refer to the first generator vertex as “the positive vertex” and to the second one as “the negative vertex”. We use the band notation for vertices the same way we do for crossings: the fully coloured band stands for the thin (inner) ring.

4.1.2 The relations of wTF^o

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In addition to the usual $R1^s$, $R2$, $R3$, and OC moves of Fig. 3, we more relations are added to describe the behaviour of the additional features.

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Comment 4.2 As before, the relations have local topological explanations, and we conjecture that they provide a Reidemeister theory for “w-tangled foams”, that is, knotted ribbon tubes with foam vertices in \mathbb{R}^4 . In this section we list the relations along with the topological reasoning behind them. However, for any rigorous purposes below, wTF^o is studied as a circuit algebra given by the declared generators and relations, with topology serving only as intuition.

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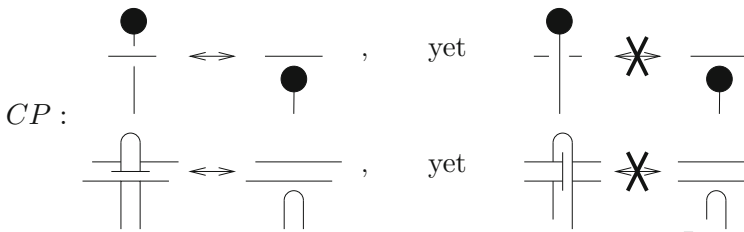
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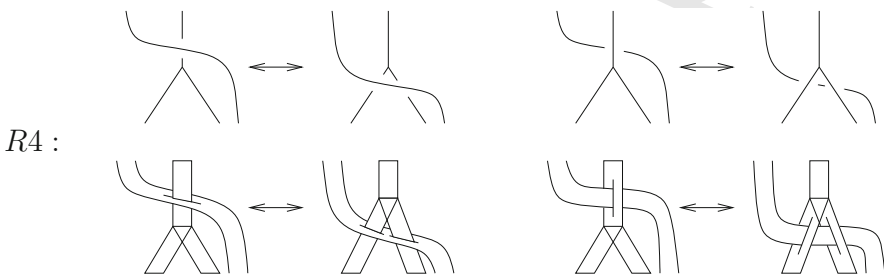
1204

1205

Recall that topologically, a cap represents a capped tube or equivalently, flying ring shrinking to a point. Hence, a cap on the thin (or under) strand can be “pulled out” from a crossing, but the same is not true for a cap on the thick (or over) strand, as shown below. This is the case for any orientation of the strands. We denote this relation by CP , for Cap Pull-out.



1207 The Reidemeister 4 relations assert that a strand can be moved under or over a
 1208 crossing, as shown in the picture below. The ambiguously drawn vertices in the picture
 1209 denote a vertex of any kind (as described in Sect. 4.1.1), and the strands can be oriented
 1210 arbitrarily. The local topological (tube or flying ring) interpretations can be read from
 1211 the pictures below. These relations will be denoted R4.



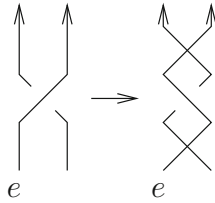
1213 4.1.3 The auxiliary operations of wTF^o

1214 The circuit algebra wTF^o is equipped with several extra operations.

1215 The first of these is the familiar orientation switch. We will, as mentioned in
 1216 Sect. 3.4, distinguish between switching both the 2D and 1D orientations, or just
 1217 the strand (1D) direction.

1218 Topologically *orientation switch*, denoted S_e , is the switch of both orientations of
 1219 the strand e . Diagrammatically (and this is the definition) S_e is the operation which
 1220 reverses the orientation of a strand in a wTF^o diagram. The reader can check that when
 1221 applying Satoh's tubing map, this amounts to reversing both the direction and the 2D
 1222 orientation of the tube arising from the strand.

1223 The operation which, in topology world, reverses a tube's direction but not its 2D
 1224 orientation is called "*adjoint*", and denoted by A_e . This is slightly more intricate to
 1225 define rigorously in terms of diagrams. In addition to reversing the direction of the
 1226 strand e of the wTF^o diagram, A_e also locally changes each crossing of e over another
 1227 strand by adding two virtual crossings, as shown on the right. We recommend for the
 1228 reader to convince themselves that this indeed represents a direction switch in topology
 1229 after reading Sect. 4.5.



1230
1231

1232 *Remark 4.3* As an example, let us observe how the negative generator vertex can be
1233 obtained from the positive generator vertex by adjoint operations and composition
1234 with virtual crossings, as shown in Fig. 14. Note that also all other vertices can be
1235 obtained from the positive vertex via orientation switch and adjoint operations and
1236 composition by virtual crossings.

1237 As a small exercise, it is worthwhile to convince ourselves of the effect of orientation
1238 switch operations on the *band picture*. For example, replace $A_1A_2A_3$ by $S_1S_2S_3$ in
1239 Fig. 14. In the strand diagram, this will only reverse the direction of the strands. The
1240 reader can check that in the band picture not only the arrows will reverse but also the
1241 blue band will switch to be on top of the red band.

1242 *Comment 4.4* Framings were discussed in Sect. 3.4, but have not played a significant
1243 role so far, except to explain the lack of a Reidemeister 1 relation. We now need to
1244 discuss framings in order to provide a topological explanation for the unzip (tube
1245 doubling) operation.

1246 In the local topological interpretation of wTF^o , strands represent ribbon-knotted
1247 tubes with foam vertices, which are also equipped with a framing, arising from the
1248 blackboard framing of the strand diagrams via Satoh’s tubing map. Strand doubling
1249 is the operation of doubling a tube by “pushing it off itself slightly” in the framing
1250 direction, as shown in Fig. 15.

1251 Recall that ribbon knotted tubes have a “filling”, with only “ribbon” self-
1252 intersections [38, Section 2.2.2]. When we double a tube, we want this ribbon property
1253 to be preserved. This is equivalent to saying that the ring obtained by pushing off any
1254 given girth of the tube in the framing direction is not linked with the original tube,
1255 which is indeed the case.

1256 Framings arising from the blackboard framing of strand diagrams via Satoh’s tubing
1257 map always match at the vertices, with the normal vectors pointing either directly

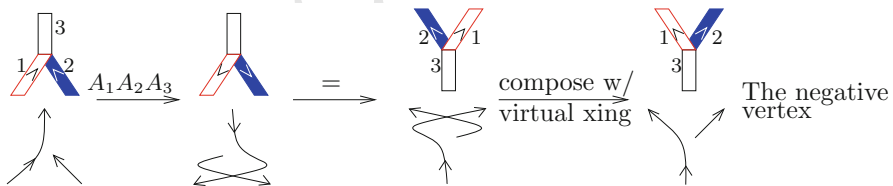
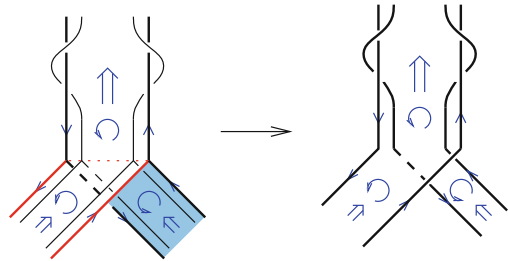
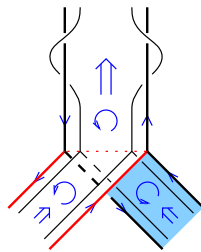


Fig. 14 Switching strand orientations at a vertex. The adjoint operation only switches the tube direction, hence in the *band picture* only the *arrows* change. To express this vertex in terms of the negative generating vertex in strand notation, we use a virtual crossing (see Fig. 13)

Fig. 15 Unzipping a tube, in band notation with orientations and framing marked



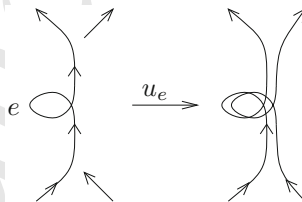
1258 towards or away from the center of the singular ring. Note that the orientations of the
 1259 three tubes may or may not match. An example of a vertex with the orientations and
 1260 framings shown is on the right. Note that the framings on the two sides of each band
 1261 are mirror images of each other, as they should be.



1262

1263 Unzip, or tube doubling is perhaps the most interesting of the auxiliary wTF^o
 1264 operations. As mentioned above, topologically this means pushing the tube off itself
 1265 slightly in the framing direction. At each of the vertices at the two ends of the doubled
 1266 tube there are two tubes to be attached to the doubled tube. At each end, the normal
 1267 vectors pointed either directly towards or away from the center, so there is an “inside”
 1268 and an “outside” ending ring. The two tubes to be attached also come as an “inside”
 1269 and an “outside” one, which defines which one to attach to which. An example is
 1270 shown in Fig. 15. Unzip can only be done if the 1D and 2D orientations match at both
 1271 ends.

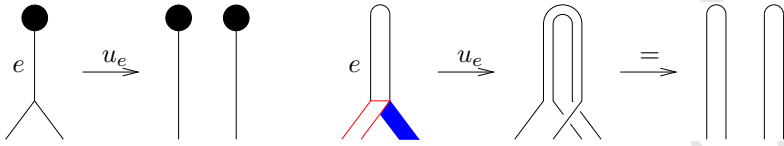
1272 To define unzip rigorously, we must talk only of strand diagrams. The natural
 1273 definition is to let u_e double the strand e using the blackboard framing, and then attach
 1274 the ends of the doubled strand to the connecting ones, as shown on the right. We restrict
 1275 unzip to strands whose two ending vertices are of different signs. This is a somewhat
 1276 artificial condition which we impose to get equations equivalent to the [2] equations.



1277

1278 A related operation, *disk unzip*, is unzip done on a capped strand, pushing the tube
 1279 off in the direction of the framing (in diagrammatic world, in the direction of the

blackboard framing), as before. An example in the line and band notations (with the framing suppressed) is shown below.



Finally, we allow deletion d_e of “long linear” strands, meaning strands that do not end in a vertex on either side. To summarize,

$$wTF^o = CA \left\langle \begin{array}{c} \nearrow, \nwarrow, \bullet, \nearrow, \nwarrow \\ \downarrow, \uparrow, \uparrow, \downarrow \end{array} \middle| \begin{array}{l} R1^s, R2, R3, R4, \\ OC, CP \end{array} \right| \left. \begin{array}{c} S_e, A_e, u_e, d_e \end{array} \right\rangle$$

The goal, as before, is to construct a homomorphic expansion for wTF^o . However, first we need to understand its target space, the associated graded structure $\text{grad } wTF^o$.

4.2 The associated graded structure

Mirroring the previous section, we describe the associated graded \mathcal{A}^{sw} of wTF^o and its “full version” \mathcal{A}^w as circuit algebras on certain generators modulo a number of relations. From now on we will write $\mathcal{A}^{(s)w}$ to mean “ \mathcal{A}^w and/or \mathcal{A}^{sw} ”.

$$\mathcal{A}^{(s)w} = CA \left\langle \begin{array}{c} \uparrow, \downarrow, \bullet, \nearrow, \nwarrow \\ \downarrow, \uparrow, \uparrow, \downarrow \end{array} \middle| \begin{array}{l} \text{relations as in} \\ \text{Section 4.2.1} \end{array} \right| \left. \begin{array}{c} \text{operations as in} \\ \text{Section 4.2.2} \end{array} \right\rangle$$

In other words, $\mathcal{A}^{(s)w}$ are the circuit algebras of arrow diagrams on trivalent (or foam) skeletons with caps; that is, skeleta are elements of \mathcal{S}^o as in Definition 4.1. Note that all but the first of the generators are skeleton features (of degree 0), and that the single arrow is the only generator of degree 1. As for the generating vertices, the same remark applies as in Definition 4.1, that is, there are more vertices with all possible strand orientations needed to generate $\mathcal{A}^{(s)w}$ as circuit algebras.

4.2.1 The relations of $\mathcal{A}^{(s)w}$

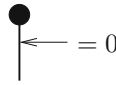
In addition to the usual $\overrightarrow{4T}$ and TC relations (see Fig. 5), as well as RI in the case of $\mathcal{A}^{sw} = \mathcal{A}^w / RI$, diagrams in $\mathcal{A}^{(s)w}$ satisfy the following additional relations:

Vertex invariance, denoted by VI, are relations arising the same way as $\overrightarrow{4T}$ does, but with the participation of a vertex as opposed to a crossing:

The equation shows three sets of diagrams. The first set shows three trivalent vertices with different strand orientations, each preceded by a \pm sign, followed by an equals sign and a zero. The second set shows three trivalent vertices with different strand orientations, each preceded by a \pm sign, followed by the word “and”. The third set shows three trivalent vertices with different strand orientations, each preceded by a \pm sign, followed by an equals sign and a zero.

1305 The other end of the arrow is in the same place throughout the relation, somewhere
 1306 outside the picture shown. The signs are positive whenever the strand on which the
 1307 arrow ends is directed towards the vertex, and negative when directed away. The
 1308 ambiguously drawn vertex means any kind of vertex, but the same one throughout.

1309 The CP relation (a cap can be pulled out from under a strand but not from over,
 1310 Sect. 4.1.2) implies that arrow heads near a cap are zero, as shown on the right. Denote
 1311 this relation also by CP. (Also note that a tail near a cap is not set to zero.)



1312
 1313 As in the previous sections, and in particular in Definition 3.12, we define a “w-
 1314 Jacobi diagram” (or just “arrow diagram”) on a foam skeleton by allowing trivalent
 1315 chord vertices. Denote the circuit algebra of formal linear combinations of arrow
 1316 diagrams by $\mathcal{A}^{(s)wt}$. We have the following bracket-rise theorem:

1317 **Theorem 4.5** *The obvious inclusion of diagrams induces a circuit algebra isomor-*
 1318 *phism $\mathcal{A}^{(s)w} \cong \mathcal{A}^{(s)wt}$. Furthermore, the \overrightarrow{AS} and \overrightarrow{THX} relations of Fig. 7 hold in*
 1319 *$\mathcal{A}^{(s)wt}$.*

1320 *Proof* Same as the proof of Theorem 3.13. □

1321 As in Sect. 3.1, the primitive elements of $\mathcal{A}^{(s)w}$ are connected diagrams (that is,
 1322 connected even with the skeleton removed), namely trees and wheels. Before moving
 1323 on to the auxiliary operations of $\mathcal{A}^{(s)w}$, let us make two useful observations:

1324 **Lemma 4.6** $\mathcal{A}^w(\uparrow)$, the part of \mathcal{A}^w with skeleton \uparrow , is isomorphic as a vector space to
 1325 the completed polynomial algebra freely generated by wheels w_k with $k \geq 1$. Likewise
 1326 $\mathcal{A}^{sw}(\uparrow)$, except here $k \geq 2$.

1327 *Proof* Any arrow diagram with an arrow head at its top is zero by the Cap Pull-out
 1328 (CP) relation. If D is an arrow diagram that has a head somewhere on the skeleton but
 1329 not at the top, then one can use repeated \overrightarrow{STU} relations to commute the head to the
 1330 top at the cost of diagrams with one fewer skeleton head.

1331 Iterating this procedure, we can get rid of all arrow heads, and hence write D as
 1332 a linear combination of diagrams having no heads on the skeleton. All connected
 1333 components of such diagrams are wheels.

1334 To prove that there are no relations between wheels in $\mathcal{A}^{(s)w}(\uparrow)$, let $S_L : \mathcal{A}^{(s)w}$
 1335 $(\uparrow_1) \rightarrow \mathcal{A}^{(s)w}(\uparrow_1)$ (resp. S_R) be the map that sends an arrow diagram to the sum of
 1336 all ways of dropping one left (resp. right) arrow (on a vertical strand, left means down
 1337 and right means up). Define

1338
$$F := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} D_R^k (S_L + S_R)^k,$$

Author Proof

1339 where D_R is a short right arrow. We leave it as an exercise for the reader to check that
 1340 F is a bi-algebra homomorphism that kills diagrams with an arrow head at the top
 1341 (i.e., CP is in the kernel of F), and F is injective on wheels. This concludes the proof.
 1342 □

1343 **Lemma 4.7** $\mathcal{A}^{(s)w}(Y) = \mathcal{A}^{(s)w}(\uparrow_2)$, where $\mathcal{A}^{(s)w}(Y)$ stands for the space of arrow
 1344 diagrams whose skeleton is a Y -graph with any orientation of the strands, and as
 1345 before $\mathcal{A}^{(s)w}(\uparrow_2)$ is the space of arrow diagrams on two strands.

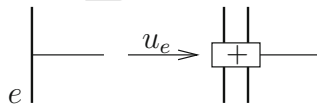
1346 *Proof* We can use the vertex invariance (VI) relation to push all arrow heads and tails
 1347 from the “trunk” of the vertex to the other two strands. □

1348 *4.2.2 The auxiliary operations of $\mathcal{A}^{(s)w}$*

1349 Recall from Sect. 3.4 that the orientation switch S_e (i.e. changing both the 1D and 2D
 1350 orientations of a strand) always changes the sign of a crossing involving the strand e .
 1351 Hence, letting S denote any foam (trivalent) skeleton, the induced arrow diagrammatic
 1352 operation is a map $S_e: \mathcal{A}^{(s)w}(S) \rightarrow \mathcal{A}^{(s)w}(S_e(S))$ which acts by multiplying each
 1353 arrow diagram by (-1) raised to the number of arrow endings on e (counting both
 1354 heads and tails).

1355 The adjoint operation A_e (i.e. switching only the strand direction), on the other
 1356 hand, only changes the sign of a crossing when the strand being switched is the under-
 1357 (or through) strand. (See Sect. 3.4 for pictures and explanation.) Therefore, the arrow
 1358 diagrammatic A_e acts by switching the direction of e and multiplying each arrow
 1359 diagram by (-1) raised to the number of arrow heads on e . Note that in $\mathcal{A}^{(s)w}(\uparrow_n)$
 1360 taking the adjoint on every strand gives the adjoint map of Definition 3.26.

1361 The arrow diagram operations induced by unzip and disc unzip (both to be denoted
 1362 u_e , and interpreted appropriately according to whether the strand e is capped) are
 1363 maps $u_e: \mathcal{A}^{(s)w}(S) \rightarrow \mathcal{A}^{(s)w}(u_e(S))$, where each arrow ending (head or tail) on e
 1364 is mapped to a sum of two arrows, one ending on each of the new strands, as shown on
 1365 the right. In other words, if in an arrow diagram D there are k arrow ends on e , then
 1366 $u_e(D)$ is a sum of 2^k arrow diagrams.



1368 The operation induced by deleting the long linear strand e is the map $d_e: \mathcal{A}^{(s)w}(S) \rightarrow$
 1369 $\mathcal{A}^{(s)w}(d_e(S))$ which kills arrow diagrams with any arrow ending (head or tail) on e ,
 1370 and leaves all else unchanged, except with e removed.

1371 **Definition 4.8** In summary,

$$\mathcal{A}^{(s)w} = \text{CA} \left\langle \left[\begin{array}{c} \uparrow, \downarrow, \bullet, \curvearrowright, \curvearrowleft, \curvearrowright \\ \overrightarrow{4T}, \text{TC}, \text{VI}, \text{CP}, \\ (\text{RI for } \mathcal{A}^{sw}) \end{array} \right] \mid S_e, A_e, u_e, d_e \right\rangle$$

1372

1373 4.3 The homomorphic expansion

1374 The following is one of the main theorems of this paper:

1375 **Theorem 4.9** [Proof in Sect. 4.4] *There exists a group-like¹³ homomorphic expansion*
 1376 *for wTF^o , i.e. a group-like expansion $Z: wTF^o \rightarrow \mathcal{A}^{sw}$ which is a map of circuit*
 1377 *algebras and also intertwines the auxiliary operations of wTF^o with their arrow dia-*
 1378 *grammatic counterparts. In fact, there is a bijection between the set of solutions (F, a)*
 1379 *of the generalized KV problem (see Sect. 4.4) and the set of homomorphic expansions*
 1380 *for wTF^o which do not contain local arrows¹⁴ in the value V of the vertex.*

1381 Since both wTF^o and \mathcal{A}^{sw} are circuit algebras defined by generators and relations,
 1382 when looking for a suitable Z all we have to do is to find values for each of the
 1383 generators of wTF^o so that these satisfy (in \mathcal{A}^{sw}) the equations which arise from the
 1384 relations in wTF^o and the homomorphicity requirement. In this section we derive these
 1385 equations and in the next section we show that they are equivalent to the Alekseev–
 1386 Torossian version of the Kashiwara–Vergne equations [2]. In [3] Alekseev Enriquez
 1387 and Torossian construct explicit solutions to these equations using associators. In [39]
 1388 we will interpret these results in our context of homomorphic expansions for w-tangled
 1389 foams.

1390 Let $R := Z(\text{↗↘}) \in \mathcal{A}^{sw}(\uparrow_2)$. It follows from the Reidemeister 2 relation that
 1391 $Z(\text{↖↗}) = (R^{-1})^{21}$. As discussed in Sects. 3.1 and 3.5, Reidemeister 3 with group-
 1392 likeness and homomorphicity implies that $R = e^a$, where a is a single arrow pointing
 1393 from the over to the under strand. Let $C := Z(\uparrow) \in \mathcal{A}^{sw}(\uparrow)$. By Lemma 4.6, we
 1394 know that C is made up of wheels only. Finally, let
 1395 $V = V^+ := Z(\text{↗↘}) \in \mathcal{A}^{sw}(\text{↗↘}) \cong \mathcal{A}^{sw}(\uparrow_2)$, and

$$V^- := Z(\text{↖↗}) \in \mathcal{A}^{sw}(\text{↖↗}) \cong \mathcal{A}^{sw}(\uparrow_2).$$

1396
 1397 Before we translate each of the relations of Sect. 4.1.2 to equations let us slightly
 1398 extend the notation used in Sect. 3.5. Recall that R^{23} , for instance, meant “ R placed
 1399 on strands 2 and 3”. In this section we also need notation such as $R^{(23)1}$, which means
 1400 “ R with its first strand doubled, placed on strands 2, 3 and 1”.

1401 Now on to the relations, note that Reidemeister 2 and 3 and Overcrossings Commute
 1402 have already been dealt with. Of the two Reidemeister 4 relations, the first one induces
 1403 an equation that is automatically satisfied. Pictorially, the equation looks as follows:

¹³ The formal definition of the group-like property is along the lines of [38, Section 2.5.1.2]. In practice, it means that the Z -values of the vertices, crossings, and cap (denoted V , R and C below) are exponentials of linear combinations of connected diagrams.

¹⁴ For a detailed explanation of this minor point see the third paragraph of the proof.



1404

1405 In other words, we obtained the equation

1406

$$V^{12} R^{3(12)} = R^{32} R^{31} V^{12}.$$

1407 However, observe that by the “head-invariance” property of arrow diagrams (Remark
 1408 3.14) V^{12} and $R^{3(12)}$ commute on the left hand side. Hence the left hand side equals
 1409 $R^{3(12)} V^{12} = R^{32} R^{31} V^{12}$. Also, $R^{3(12)} = e^{a^{31}+a^{32}} = e^{a^{32}} e^{a^{31}} = R^{32} R^{31}$, where the
 1410 second step is due to the fact that a^{31} and a^{32} commute. Therefore, the equation is
 1411 true independently of the choice of V .

1412 We have no such luck with the second Reidemeister 4 relation, which, in the same
 1413 manner as in the paragraph above, translates to the equation

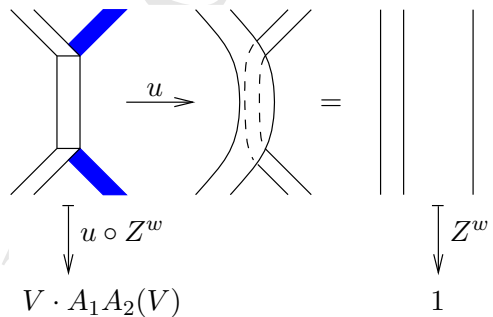
$$V^{12} R^{(12)3} = R^{23} R^{13} V^{12}. \tag{11}$$

1415 There is no “tail invariance” of arrow diagrams, so V and R do not commute on the left
 1416 hand side; also, $R^{(12)3} \neq R^{23} R^{13}$. As a result, this equation puts a genuine restriction
 1417 on the choice of V .

1418 The Cap Pull-out (CP) relation translates to the equation $R^{12} C^2 = C^2$. This is true
 1419 independently of the choice of C : by head-invariance, $R^{12} C^2 = C^2 R^{12}$. Now R^{12} is
 1420 just below the cap on strand 2, and the cap “kills heads”, in other words, every term of
 1421 R^{12} with an arrow head at the top of strand 2 is zero. Hence, the only surviving term
 1422 of R^{12} is 1 (the empty diagram), which makes the equation true.

1423 The homomorphicity of the orientation switch operation was used to prove the
 1424 uniqueness of R in Theorem 3.30. The homomorphicity of the adjoint leads to the
 1425 equation $V_- = A_1 A_2(V)$ (see Fig. 14), eliminating V_- as an unknown. Note that we
 1426 also silently assumed these homomorphicity properties when we did not introduce 32
 1427 different values of the vertex depending on the strand orientations.

1428 Homomorphicity of the (annular) unzip operation leads to an equation for V , which
 1429 we are going to refer to as “unitarity”. This is illustrated in the figure below. Recall
 1430 that A_1 and A_2 denote the adjoint (direction switch) operation on strand 1 and 2,
 1431 respectively.



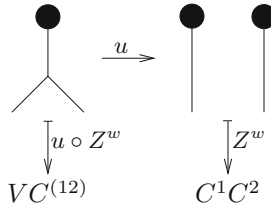
1432

Author Proof

Reading off the equation, we have

$$V \cdot A_1 A_2(V) = 1. \tag{12}$$

Homomorphicity of the disk unzip leads to an equation for C which we will refer to as the “cap equation”. The translation from homomorphicity to equation is shown in the figure on the right. C , as we introduced before, denotes the Z -value of the cap. Hence, the cap equation reads



$$V^{12}C^{(12)} = C^1 C^2 \quad \text{in } \mathcal{A}^{sw}(\uparrow_2) \tag{13}$$

The homomorphicity of deleting long strands does not lead to an equation on its own, however it was used to prove the uniqueness of R (Theorem 3.30).

To summarize, we have reduced the problem of finding a homomorphic expansion Z to finding the Z -values of the (positive) vertex and the cap, denoted V and C , subject to three equations: the “hard Reidemeister 4” equation (11); “unitarity of V ” Eq. (12); and the “cap equation” (13).

4.4 The equivalence with the Alekseev–Torossian equations

First let us recall Alekseev and Torossian’s formulation of the generalized Kashiwara–Vergne problem (see [2, Section 5.3]):

Generalized KV problem Find an element $F \in TAut_2$ with the properties

$$F(x + y) = \log(e^x e^y), \text{ and } j(F) \in \text{im}(\tilde{\delta}). \tag{14}$$

Here $\tilde{\delta}: \mathfrak{tr}_1 \rightarrow \mathfrak{tr}_2$ is defined by $(\tilde{\delta}a)(x, y) = a(x) + a(y) - a(\log(e^x e^y))$, where \mathfrak{tr}_2 is generated by cyclic words in the letters x and y . (See [2, Equation (8)]. Note that an element of \mathfrak{tr}_1 is a power series in one variable with no constant term. In other words, the second condition says that there exists $a \in \mathfrak{tr}_1$ such that $jF = a(x) + a(y) - a(\log(e^x e^y))$.

Proof of Theorem 4.9 We have reduced the problem of finding a homomorphic expansion to finding group-like solutions V and C to the hard Reidemeister 4 equation (11), the unitarity equation (12), and the cap equation (13).

Suppose we have found such solutions and write $V = e^b e^{uD}$, where $b \in \mathfrak{tr}_2^s$, $D \in \mathfrak{tdet}_2 \oplus \mathfrak{a}_2$, and where u is the map $u: \mathfrak{tdet}_2 \rightarrow \mathcal{A}^{sw}(\uparrow_2)$ which plants the head of a tree above all of its tails, as introduced in Sect. 3.2. V can be written in this form

Author Proof

1463 without loss of generality because wheels can always be commuted to the bottom of a
 1464 diagram (at the possible cost of more wheels). Furthermore, V is group-like and hence
 1465 it can be written in exponential form. Similarly, write $C = e^c$ with $c \in \mathfrak{tr}_1^s$.

1466 Note that $u(\mathfrak{a}_2)$ is central in $\mathcal{A}^{sw}(\uparrow_2)$ and that replacing a solution (V, C) by
 1467 $(e^{u(a)}V, C)$ for any $a \in \mathfrak{a}_2$ does not interfere with any of the Eqs. (11), (12) or (13).
 1468 Hence we may assume that D does not contain any single arrows, that is, $D \in \mathfrak{td}\mathfrak{tr}_2$.
 1469 Also, a solution (V, C) in \mathcal{A}^{sw} can be lifted to a solution in \mathcal{A}^w by simply setting the
 1470 degree one terms of b and c to be zero. It is easy to check that this $b \in \mathfrak{tr}_2$ and $c \in \mathfrak{tr}_1$
 1471 along with D still satisfy the equations. (In fact, in \mathcal{A}^w (12) and (13) respectively
 1472 imply that b is zero in degree 1, and that the degree 1 term of c is arbitrary, so we may
 1473 as well assume it to be zero.) In light of this we declare that $b \in \mathfrak{tr}_2$ and $c \in \mathfrak{tr}_1$.

1474 The hard Reidemeister 4 equation (11) reads $V^{12}R^{(12)3} = R^{23}R^{13}V^{12}$. Denote the
 1475 arrow from strand 1 to strand 3 by x , and the arrow from strand 2 to strand 3 by y .
 1476 Substituting the known value for R and rearranging, we get

$$1477 \quad e^b e^{uD} e^{x+y} e^{-uD} e^{-b} = e^y e^x.$$

1478 Equivalently, $e^{uD} e^{x+y} e^{-uD} = e^{-b} e^y e^x e^b$. Now on the right side there are only tails
 1479 on the first two strands, hence e^b commutes with $e^y e^x$, so $e^{-b} e^b$ cancels. Taking
 1480 logarithm of both sides we obtain $e^{uD}(x+y)e^{-uD} = \log e^y e^x$. Now for notational
 1481 alignment with [2] we switch strands 1 and 2, which exchanges x and y so we obtain:

$$1482 \quad e^{uD^{21}}(x+y)e^{-uD^{21}} = \log e^x e^y. \tag{15}$$

1483 The unitarity of V (Eq. (12)) translates to $1 = e^b e^{uD} (e^b e^{uD})^*$, where $*$ denotes
 1484 the adjoint map (Definition 3.26). Note that the adjoint switches the order of a product
 1485 and acts trivially on wheels. Also, $e^{uD} (e^{uD})^* = J(e^D) = e^{j(e^D)}$, by Proposition 3.27.
 1486 So we have $1 = e^b e^{j(e^D)} e^b$. Multiplying by e^{-b} on the right and by e^b on the left, we
 1487 get $1 = e^{2b} e^{j(e^D)}$, and again by switching strand 1 and 2 we arrive at

$$1488 \quad 1 = e^{2b^{21}} e^{j(e^{D^{21}})}. \tag{16}$$

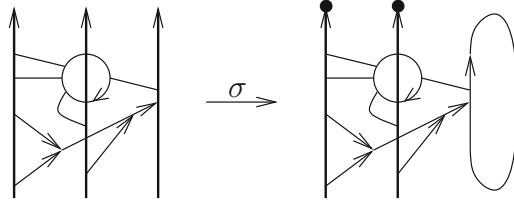
1489 As for the cap equation, if $C^1 = e^{c(x)}$ and $C^2 = e^{c(y)}$, then $C^{12} = e^{c(x+y)}$. Note
 1490 that wheels on different strands commute, hence $e^{c(x)} e^{c(y)} = e^{c(x)+c(y)}$, so the cap
 1491 equation reads

$$1492 \quad e^b e^{uD} e^{c(x+y)} = e^{c(x)+c(y)}.$$

1493 As this equation lives in the space of arrow diagrams on two *capped* strands, we can
 1494 multiply the left side on the right by e^{-uD} : uD has its head at the top, so it is 0 by the
 1495 Cap relation, hence $e^{uD} = 1$ near the cap. Hence,

$$1496 \quad e^b e^{uD} e^{c(x+y)} e^{-uD} = e^{c(x)+c(y)}.$$

1497 On the right side of the equation above $e^{uD}e^{c(x+y)}e^{-uD}$ reminds us of Eq. (15),
 1498 however we cannot use (15) directly as we are working in a different space now. In
 1499 particular, x there meant an arrow from strand 1 to strand 3, while here it means a
 1500 one-wheel on (capped) strand 1, and similarly for y . Fortunately, there is a map σ :
 1501 $\mathcal{A}^{sw}(\uparrow_3) \rightarrow \mathcal{A}^{sw}(\uparrow_2)$, where σ “closes the third strand and turns it into a chord (or
 1502 internal) strand, and caps the first two strands”, as shown on the right. This map is
 1503 well defined (in fact, it kills almost all relations, and turns one \overrightarrow{STU} into an $\overrightarrow{IH\dot{X}}$).
 1504 Under this map, using our abusive notation, $\sigma(x) = x$ and $\sigma(y) = y$.



1505
 1506 Now we can apply Eq. (15) to get $e^{uD}e^{c(x+y)}e^{-uD} = e^{c(\log e^y e^x)}$. Substituting this
 1507 into the cap equation we obtain $e^b e^{c(\log e^y e^x)} = e^{c(x)+c(y)}$, which, using that tails
 1508 commute, implies $b = c(x) + c(y) - c(\log e^y e^x)$. Switching strands 1 and 2, we
 1509 obtain

$$1510 \quad b^{21} = c(x) + c(y) - c(\log e^x e^y) \quad (17)$$

1511 In summary, we can use (V, C) to produce $F := e^{D^{21}}$ (sorry¹⁵) and $a := -2c$
 1512 which satisfy the Alekseev–Torossian equations (14), as follows: $e^{D^{21}}$ acts on \mathfrak{lie}_2 by
 1513 conjugation by $e^{uD^{21}}$, so the first part of (14) is implied by (15). The second half of
 1514 (14) is true due to (16) and (17).

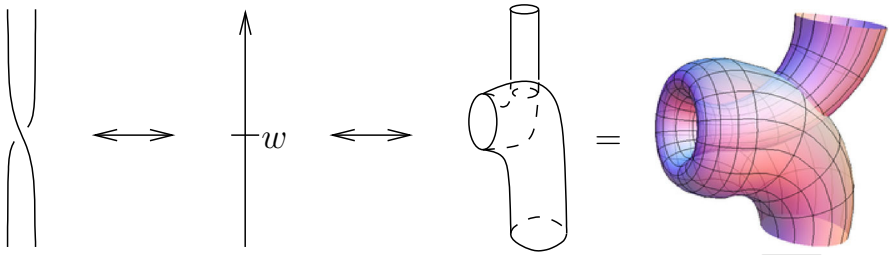
1515 On the other hand, suppose that we have found $F \in TAut_2$ and $a \in \text{tr}_1$ satisfying
 1516 (14). Then set $D^{21} := \log F$, $b^{21} := \frac{-j(e^{D^{21}})}{2}$, and $c \in \tilde{\delta}^{-1}(b^{21})$, in particular $c =$
 1517 $-\frac{a}{2}$ works. Then $V = e^b e^{uD}$ and $C = e^c$ satisfy the equations for homomorphic
 1518 expansions (11), (12) and (13).

1519 Furthermore, the two maps between solutions of the KV problem and homomorphic
 1520 expansions for wTF^o defined in the last two paragraphs are obviously inverses of each
 1521 other, and hence they provide a bijection between these sets as stated. \square

1522 4.5 The wen

1523 A topological feature of w -tangled foams which we excluded from the theory so far is
 1524 the wen w . The wen is a Klein bottle cut apart (as mentioned in [38, Section 2.5.4]);
 1525 in other words it amounts to changing the 2D orientation of a tube, as shown in the
 1526 picture below:

¹⁵ We apologize for the annoying $2 \leftrightarrow 1$ transposition in this equation, which makes some later equations, especially (22), uglier than they could have been. There is no depth here, just mis-matching conventions between us and Alekseev–Torossian.



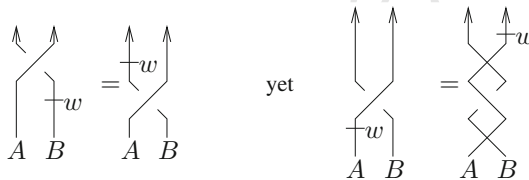
1527

1528 In this section we study the circuit algebra of w -Tangled Foams with the wen
 1529 included as a generator, and denote this space by wTF . The wen is also added to the
 1530 circuit algebra of skeleta. We will find that homomorphic expansions for wTF are in
 1531 bijection with solutions to the KV problem with “even Duflo function”, as explained
 1532 below.

1533 *4.5.1 The relations and auxiliary operations of wTF*

1534 Adding the wen as a generator means we have to impose additional relations involving
 1535 the wen to keep our topological heuristics intact, as follows:

1536 The interaction of a wen and a crossing is described by the following relation
 1537 (cf. [38, Section 2.5.4]):

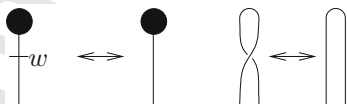


1538

1539 To explain this relation note that in flying ring language, a wen is a ring that flips
 1540 over. It does not matter whether ring B flips first and then flies through ring A or vice
 1541 versa. However, the movies in which ring A first flips and then ring B flies through
 1542 it, or B flies through A first and then A flips differ in the fly-through direction of B
 1543 through A, which is cancelled by virtual crossings, as in the figure above. We will
 1544 refer to these relations as the Flip Relations, and abbreviate them by FR.

1545 A double flip is homotopic to no flip, in other words two consecutive wens equal
 1546 no wen. Let us denote this relation by W^2 , for Wen squared. Note that this relation
 1547 explains why there are no “left and right wens”.

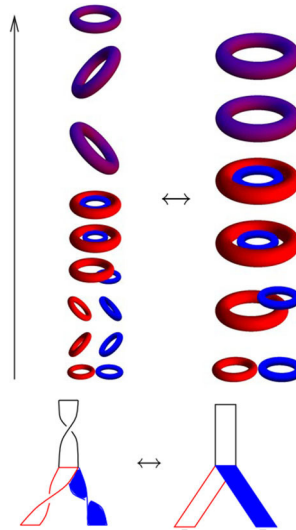
1548 A cap can slide through a wen, hence a capped wen disappears, as shown on the
 1549 right, to be denoted CW.



1550

1551 The last wen relation describes the interaction of wens and vertices. Recall that there
 1552 are four types of vertices with the same strand orientation: among the bottom two bands
 1553 (in the pictures on the left) there is a non-filled and a filled band (corresponding to

1554 over/under in the strand diagrams), meaning the “large” ring and the “small” one which
 1555 flies into it before they merge. Furthermore, there is a top and a bottom band (among
 1556 these bottom two, with apologies for the ambiguity in overusing the word bottom): this
 1557 denotes the fly-in direction (flying in from below or from above). Conjugating a vertex
 1558 by three wens switches the top and bottom bands, as shown in the figure on the left: if
 1559 both rings flip, then merge, and then the merged ring flips again, this is homotopic to
 1560 no flips, except the fly-in direction (from below or from above) has changed. We are
 1561 going to denote this relation by TV, for “twisted vertex”.



1562

1563 The auxiliary operations are the same as for wTF^o : orientation switches, adjoints,
 1564 deletion of long linear strands, cap unzips and unzips.¹⁶

1565 **Definition 4.10** Summarizing the above, we have

$$1566 \quad wTF = CA \left\langle \begin{array}{c} \nearrow, \nwarrow, \bullet, \curvearrowright, \curvearrowleft, \uparrow, \downarrow \\ \nwarrow, \nearrow, \uparrow, \downarrow, \uparrow, \downarrow \end{array} \middle| \begin{array}{c} R1^s, R2, R3, R4, OC, CP, \\ FR, W^2, CW, TV \end{array} \middle| S_e, A_e, u_e, d_e \right\rangle$$

1567 **4.5.2 The associated graded structure**

1568 The associated graded structure of wTF (still denoted \mathcal{A}^{sw}) is the same as the associated
 1569 graded for wTF^o but with the wen added as a generator (a degree 0 skeleton feature),
 1570 and with extra relations describing the behaviour of the wen. Of course, the relations
 1571 describing the interaction of wens with the other skeleton features (W^2 , TV, and CW)
 1572 still apply, as well as the old $\overrightarrow{4T}$, TC, VI, CP, and RI relations.

1573 In addition, the Flip Relations FR imply that wens “commute” with arrow heads,
 1574 but “anti-commute” with tails. We also call these FR relations:

¹⁶ We need not specify how to unzip an edge e that carries a wen. To unzip such e , first use the TV relation to slide the wen off e .

$$\text{FR: } \begin{array}{c} | \\ \leftarrow w \\ | \end{array} = \begin{array}{c} \leftarrow \\ | \\ w \\ | \end{array}, \quad \begin{array}{c} | \\ w \\ \rightarrow \\ | \end{array} = - \begin{array}{c} | \\ \rightarrow \\ w \\ | \end{array}$$

That is,

$$\mathcal{A}^{sw} = \text{CA} \left\langle \uparrow, \downarrow, \curvearrowright, \curvearrowleft, \curvearrowup, \curvearrowdown, \uparrow w, \downarrow w \mid \begin{array}{l} W^2, \text{TW}, \text{CW}, \overline{4T}, \text{TC}, \\ \text{VI}, \text{CP}, \text{RI}, \text{FR} \end{array} \mid S_e, A_e, u_e, d_e \right\rangle$$

4.5.3 The homomorphic expansion

The goal of this section is to prove that there exists a homomorphic expansion Z for wTF . This involves solving a similar system of equations to Sect. 4.3, but with an added unknown for the value of the wen, as well as added equations arising from the wen relations. Let $W \in \mathcal{A}^{sw}(\uparrow_1)$ denote the Z -value of the wen, and let us agree that the arrow diagram W always appears just above the wen on the skeleton. In fact, we are going to show that $W = 1$ for any homomorphic expansion.

As two consecutive wens on the skeleton cancel, we obtain the equation shown in the picture and explained below:

$$\begin{array}{c} \uparrow \\ \boxed{W} \\ | \\ \leftarrow w \\ | \\ \boxed{W} \\ | \\ \leftarrow w \\ | \end{array} = \begin{array}{c} \uparrow \\ \boxed{W} \\ | \\ \overline{\boxed{W}} \\ | \\ \leftarrow w \\ | \\ \leftarrow w \\ | \end{array} = \begin{array}{c} \uparrow \\ \boxed{W} \\ | \\ \overline{\boxed{W}} \\ | \end{array}$$

The Z -value of two consecutive wens on a strand is a skeleton wen followed by W followed by a skeleton wen and another W . Sliding the bottom W through the skeleton wen “multiplies each tail by (-1) ”. Let us denote this operation by “bar”, i.e. for an arrow diagram D , $\overline{D} = D \cdot (-1)^{\# \text{ of tails in } D}$. Cancelling the two skeleton wens, we obtain $\overline{W}W = 1$. Recall that $\mathcal{A}^{sw}(\uparrow_1)$ consists only of wheels and single arrows. Since we are looking for a group-like Z , we can assume that $W = e^w$ and $\overline{W}W = 1$ means that w is a linear combination of odd wheels and possibly single arrows.

Now recall the Twisted Vertex relation of Sect. 4.5.1. Note that the Z -value of the negative vertex on the right hand side of the relation can be written as $S_1 S_2 A_1 A_2(V) = \overline{V}$ (cf Remark 4.3). On the other hand, applying Z to the left hand side of the relation we obtain:

$$\begin{array}{c} \boxed{W} \\ | \\ \leftarrow w \\ | \\ \text{V} \\ \swarrow \quad \searrow \\ \boxed{W} \quad \boxed{W} \\ | \quad | \\ \leftarrow w \quad \leftarrow w \end{array} = \begin{array}{c} \boxed{W} \\ | \\ \leftarrow w \\ | \\ \overline{\text{V}} \\ \swarrow \quad \searrow \\ \boxed{\overline{W}} \quad \boxed{\overline{W}} \\ | \quad | \\ \leftarrow w \quad \leftarrow w \end{array} = \begin{array}{c} \uparrow \\ \text{u(W)} \\ \swarrow \quad \searrow \\ \boxed{\overline{W}} \quad \boxed{\overline{W}} \\ | \quad | \\ \leftarrow w \quad \leftarrow w \end{array} \longleftrightarrow \begin{array}{c} \uparrow \quad \uparrow \\ \text{u(W)} \\ | \quad | \\ \boxed{\overline{W}} \quad \boxed{\overline{W}} \\ | \quad | \\ \leftarrow w \quad \leftarrow w \end{array} = 1$$

Hence, using that $\overline{W} = W^{-1}$, the twisted vertex relation induces the equation $W^1 W^2 = W^{(12)}$ in $\mathcal{A}^{sw}(\uparrow_2)$. One can verify degree by degree, using that W can be written as an exponential, that the only solution to this equation is $W = 1$.

We have yet to verify that the CW relation (i.e., a cap can slide through a wen) can be satisfied with $W = 1$. Keep in mind that the wen as a skeleton feature anti-commutes with tails (this is the Flip Relation of Sect. 4.2.1). The value of the cap C is a combination of only wheels (Lemma 4.6), hence the CW relation translates to the equation $\overline{C} = C$, which is equivalent to saying that C consists only of even wheels.

The fact that Z can be chosen to have this property follows from Proposition 6.2 of [2]: the value of the cap is $C = e^c$, where c can be set to $c = -\frac{a}{2}$, as explained in the proof of Theorem 4.9. Here a is such that $\tilde{\delta}(a) = jF$ as in Eq. (14). A power series f so that $a = \text{tr } f$ (where tr is the trace which turns words into cyclic words) is called the Duflo function of F . In Proposition 6.2 Alekseev and Torossian show that the even part of f is $\frac{1}{2} \frac{\log(e^{x/2} - e^{-x/2})}{x}$, and that for any f with this even part there is a corresponding solution F of the generalized KV problem. In particular, f can be assumed to be even, namely the power series above, and hence it can be guaranteed that C consists of even wheels only. Thus we have proven the following:

Theorem 4.11 *Group-like homomorphic expansions $Z : wTF \rightarrow \mathcal{A}^{sw}$ (with no local arrows in the value of V) are in one-to one correspondence with solutions to the KV problem with an even Duflo function.*

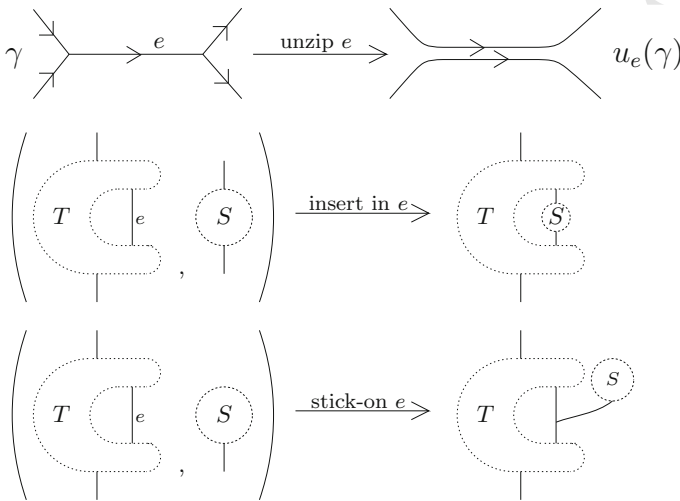
4.6 Interlude: u -knotted trivalent graphs

The “usual”, or classical knot-theoretical objects corresponding to wTF are loosely speaking Knotted Trivalent Graphs, or KTGs. We give a brief introduction/review of this structure before studying the relationship between their homomorphic expansions and homomorphic expansions for wTF . The last goal of this paper is to show that the topological relationship between the two spaces explains the relationship between the KV problem and Drinfel’d associators.

A trivalent graph is a graph with three edges meeting at each vertex, equipped with a cyclic orientation of the three half-edges at each vertex. KTGs are framed embeddings of trivalent graphs into \mathbb{R}^3 , regarded up to isotopies. The skeleton of a KTG is the trivalent graph (as a combinatorial object) behind it. For a detailed introduction to KTGs see for example [8]. Here we only recall the most important facts. The reader might recall that in Sect. 3, the w -knot section, of [38] we only dealt with long w -knots, as the w -theory of round knots is essentially trivial (see [38, Theorem 3.17]). A similar issue arises with “ w -knotted trivalent graphs”. Hence, the space we are really interested in is “long KTGs”, meaning, trivalent tangles with 1 or 2 ends.

Long KTGs form an algebraic structure with operations as follows. *Orientation switch* reverses the orientation of a specified edge. *Edge unzip* doubles a specified edge as shown on the right. *Tangle insertion* is inserting a small copy of a $(1, 1)$ -tangle S into the middle of some specified strand of a tangle T , as shown in the second row on the right (tangle composition is a special case of this). The *stick-on* operation “sticks a 1-tangle S onto a specified edge of another tangle T ”, as shown. (In the figures T is

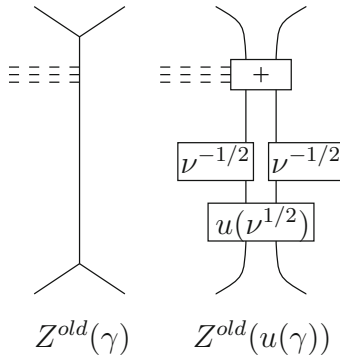
1642 a 2-tangle, but this is irrelevant.) *Disjoint union* of two 1-tangles produces a 2-tangle.
 1643 Insertion, disjoint union and stick-on are a slightly weaker set of operations than the
 1644 connected sum of [8].



1645

1646 The associated graded structure of the algebraic structure of long KTGs is the graded
 1647 space \mathcal{A}^u of chord diagrams on trivalent graph skeleta, modulo the $4T$ and vertex
 1648 invariance (VI) relations. The induced operations on \mathcal{A}^u are as expected: orientation
 1649 switch multiplies a chord diagram by (-1) to the number of chord endings on the edge.
 1650 The edge unzip u_e maps a chord diagram with k chord endings on the edge e to a sum
 1651 of 2^k diagrams where each chord ending has a choice between the two daughter edges.
 1652 Finally, tangle insertion, stick-on and disjoint union induces the insertion, sticking on
 1653 and disjoint union of chord diagrams, respectively.

1654 In [8] the authors prove that there is no *homomorphic* expansion for KTGs. This
 1655 theorem, as well as the proof, applies to long KTGs with slight modifications. However
 1656 there are well-known—and nearly homomorphic—expansions constructed by extend-
 1657 ing the Kontsevich integral to KTGs, or from Drinfel’d associators. There are several
 1658 such constructions ([14, 17, 35]). For now, let us denote any one of these expansions by
 1659 Z^{old} . All Z^{old} are “almost homomorphic”: they intertwine every operation except for
 1660 edge unzip with their chord-diagrammatic counterparts; but commutativity with unzip
 1661 fails by a controlled amount, as shown on the right. Here v denotes the “invariant of
 1662 the unknot”, the value of which was conjectured in [9] and proven in [11].

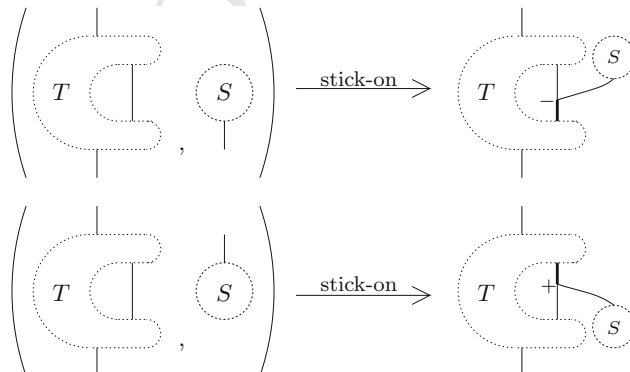


1663

1664 In [8] the authors fix this anomaly by slightly changing the space of KTGs and
 1665 adding some extra combinatorics (“dots” on the edges), and construct a homomorphic
 1666 expansion for this new space by a slight adjustment of Z^{old} . Here we are going to use
 1667 a similar but different adjustment of the space of trivalent 1- and 2-tangles. Namely
 1668 we break the symmetry of the vertices and restrict the set of allowed unzips.

1669 **Definition 4.12** A “signed KTG”, denoted $sKTG$, is a trivalent oriented 1- or 2-tangle
 1670 embedded in \mathbb{R}^3 with a cyclic orientation of edges meeting at each vertex, and in
 1671 addition each vertex is equipped with a sign and one of the three incident edges is
 1672 marked as distinguished (sometimes denoted by a thicker line). Our pictorial conven-
 1673 tion will be that a vertex drawn in a “ λ ” shape with all strands oriented up and the top
 1674 strand distinguished is always positive and a vertex drawn in a “ Y ” shape with strands
 1675 oriented up and the bottom strand distinguished is always negative (see Fig. 18).

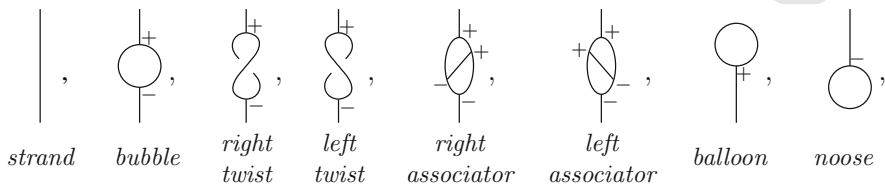
1676 The algebraic structure $sKTG$ has one kind of objects for each skeleton (a skeleton
 1677 is a uni-trivalent graph with signed vertices but no embedding), as well as several
 1678 operations: orientation switch, edge unzip, tangle insertion, disjoint union of 1-tangles,
 1679 and stick-on. Orientation switch of either of the non-distinguished strands changes the
 1680 sign of the vertex, switching the orientation of the distinguished strand does not. Unzip
 1681 of an edge is only allowed if the edge is distinguished at both of its ends and the vertices
 1682 at either end are of opposite signs. The stick-on operation can be done in either one
 1683 of the two ways shown on the right (i.e., the stuck-on edge can be attached at a vertex
 1684 of either sign, but it can not become the distinguished edge of that vertex).



1685

1686 To consider expansions of $sKTG$, and ultimately the compatibility of these with
 1687 Z^w , we first note that $sKTG$ is finitely generated (and therefore any expansion Z^u is
 1688 determined by its values on finitely many generators). The proof of this is not hard but
 1689 somewhat lengthy, so we postpone it to Sect. 5.2.

1690 **Proposition 4.13** *The algebraic structure $sKTG$ is finitely generated by the following*
 1691 *list of elements:*

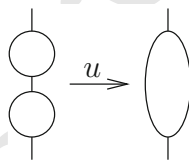


1693 Note that we ignore strand orientations for simplicity in the statement of this propo-
 1694 sition; this is not a problem as orientation switches are allowed in $sKTG$ without
 1695 restriction.

1696 **4.6.1 Homomorphic expansions for $sKTG$**

1697 Suppose that $Z^u : sKTG \rightarrow \mathcal{A}^u$ is a homomorphic expansion. We hope to determine
 1698 the value of Z^u on each of the generators.

1699 The value of Z^u on the single strand is an element of $\mathcal{A}^u(\uparrow)$ whose square is itself,
 1700 hence it is 1. The value of the bubble is an element $x \in \mathcal{A}^u(\uparrow_2)$, as all chords can be
 1701 pushed to the “bubble” part using the VI relation. Two bubbles can be composed and
 1702 unzipped to produce a single bubble (see on the right), hence we have $x^2 = x$, which
 1703 implies $x = 1$ in $\mathcal{A}^u(\uparrow_2)$.



1705 Recall that a Drinfel’d associator is a group-like element $\Phi \in \mathcal{A}^u(\uparrow_3)$ along with
 1706 a group-like element $R^u \in \mathcal{A}^u(\uparrow_2)$ satisfying the so-called pentagon and positive and
 1707 negative hexagon equations, as well as a non-degeneracy and mirror skew-symmetry
 1708 property. For a detailed explanation see Section 4 of [8]; associators were first defined
 1709 in [19]. We claim that the Z^u -value Φ of the right associator, along with the value
 1710 R^u of the right twist forms a Drinfel’d associator pair. The proof of this statement is
 1711 the same as the proof of Theorem 4.2 of [8], with minor modifications (making heavy
 1712 use of the assumption that Z^u is homomorphic). It is easy to check by composition
 1713 and unzips that the value of the left associator and the left twist are Φ^{-1} and $(R^u)^{-1}$,
 1714 respectively. Note that if Φ is required to be a *horizontal chord* associator (i.e., all
 1715 the chords of Φ are horizontal on three strands) then R^u is forced to be $e^{c/2}$ where

1716 c denotes a single chord. Note that the reverse is not true: there exist non-horizontal
 1717 chord associators Φ that satisfy the hexagon equations with $R^u = e^{c/2}$.

1718 Let b and n denote the Z^u -values of the balloon and the noose, respectively. Note
 1719 that using the VI relation all chord endings can be pushed to the “looped” strands,
 1720 so b and n live in $\mathcal{A}^u(\uparrow)$, as seen in Fig. 16. The argument in that figure shows that
 1721 $n \cdot b$ is the inverse in $\mathcal{A}^u(\uparrow)$ of “an associator on a squiggly strand”, as shown on the
 1722 right. In Fig. 16 we start with the $sKTG$ on the top left and either apply Z^u followed
 1723 by unzipping the edges marked by stars, or first unzip the same edges and then apply
 1724 Z^u . Since Z^u is homomorphic, the two results in the bottom right corner must agree.
 1725 (Note that two of the four unzips we perform are “illegal”, as the strand directions
 1726 can’t match. However, it is easy to get around this issue by inserting small bubbles
 1727 at the top of the balloon and the bottom of the noose, and switching the appropriate edge
 1728 orientations before and after the unzips. The Z^u -value of a bubble is 1, hence this will
 1729 not effect the computation and so we ignore the issue for simplicity.)

$$n \cdot b = \left(\text{Diagram of a squiggly strand with a box } \Phi \text{ and a loop} \right)^{-1}$$

1730

1731 In addition, it follows from Theorem 4.2 of [8] via deleting two edges that the
 1732 inverse of an “associator on a squiggly strand” is ν , the invariant of the unknot. To
 1733 summarize, we have proven the following:

1734 **Lemma 4.14** *If Z^u is a homomorphic expansion then the Z^u values of the strand and*
 1735 *the bubble are 1, the values of the right associator and right twist form an associator*
 1736 *pair (Φ, R^u) , and the values of the left twist and left associator are inverses of these.*
 1737 *With n and b denoting the value of the noose and the balloon, respectively, and ν being*
 1738 *the invariant of the unknot, we have $n \cdot b = \nu$ in $\mathcal{A}^u(\uparrow)$.*

1739 The natural question to ask is whether any triple (Φ, R^u, n) gives rise to a homo-
 1740 morphic expansion. We don’t know whether this is true, but we do know that any pair
 1741 (Φ, R^u) gives rise to a “nearly homomorphic” expansion of KTGs [14, 17, 35], and
 1742 we can construct a homomorphic expansion for $sKTG$ from any of these (as shown
 1743 below). However, all of these expansions take the same specific value on the noose and
 1744 the balloon (also see below). We don’t know whether there really is a one parameter
 1745 family of homomorphic expansions Z^u for each choice of (Φ, R^u) or if we are simply
 1746 missing a relation.

1747 We now construct explicit homomorphic expansions $Z^u : sKTG \rightarrow \mathcal{A}^u$ from any
 1748 Z^{old} (where Z^{old} stands for an “almost homomorphic” expansion of KTGs) as follows.
 1749 First of all we need to interpret Z^{old} as an invariant of 2-tangles. This can be done
 1750 by connecting the top and bottom ends by a non-interacting long strand followed by
 1751 a normalization, as shown on the right. By “multiplying by ν^{-1} ” we mean that after
 1752 computing Z^{old} we insert ν^{-1} on the long strand (recall that ν is the “invariant of the
 1753 unknot”). We interpret Z^{old} of a 1-tangle as follows: stick the 1-tangle onto a single
 1754 strand to obtain a 2-tangle, then proceed as above. The result will only have chords on

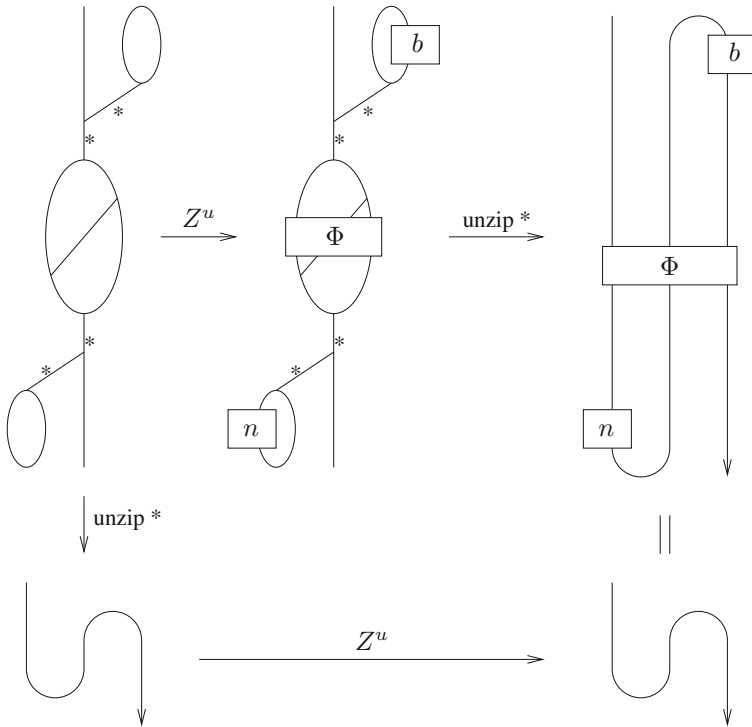


Fig. 16 Unzipping a noose and a balloon to a squiggle

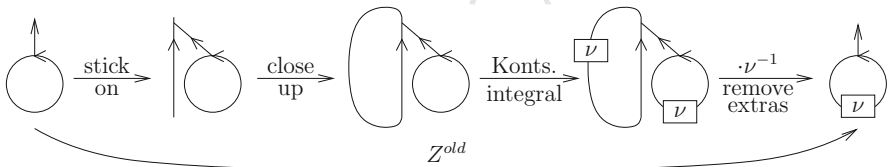


Fig. 17 Computing the Z^{old} value of the noose. The third step uses that the Kontsevich integral of KTGs is homomorphic with respect to the “connected sum” operation and that the value of the unknot is ν (see [8] for an explanation of both of these facts)

1755 the 1-tangle (using that the extensions of the Kontsevich Integral are homomorphic
 1756 with respect to “connected sums”), so we define the result to be the value of Z^{old} on
 1757 the 1-tangle. As an example, we compute the value of Z^{old} for the noose in Fig. 17
 1758 (note that the computation for the balloon is the same).

$$Z^{old} \left(\begin{array}{c} | \\ | \\ \boxed{T} \\ | \\ | \end{array} \right) := \nu^{-1} \cdot Z^{old} \left(\begin{array}{c} \text{loop} \\ | \\ \boxed{T} \\ | \end{array} \right)$$

1759

1760 Now to construct a homomorphic Z^u from Z^{old} we add normalizations near the
 1761 vertices, as in Fig. 18, where c denotes a single chord. Checking that Z^u is a homo-

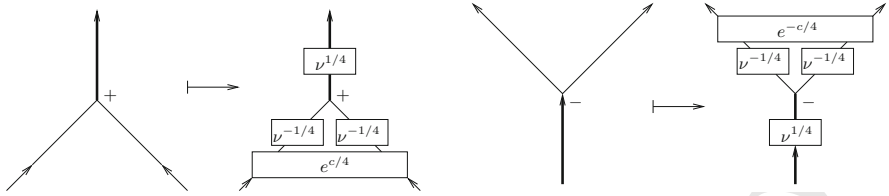
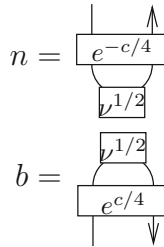


Fig. 18 Normalizations for Z^u at the vertices

1762 morphic expansion is a simple calculation using the almost homomorphicity of Z^{old} ,
 1763 which we leave to the reader. The reader can also verify that Z^u of the strand and the
 1764 bubble is 1 as it should be. Z^u of the right twist is $e^{c/2}$ and Z^u of the right associator
 1765 is a Drinfel'd associator Φ (note that Φ depends on which Z^{old} was used). From the
 1766 calculation of Fig. 17 it follows that the Z^u value of the balloon and the noose (for
 1767 any Z^{old}) are as shown on the right, and indeed $n \cdot b = v$.



1768

1769 **4.7 The relationship between sKTG and wTF**

1770 We move on to the question of compatibility between the homomorphic expansions
 1771 Z^u and Z^w (from now on we are going to refer to the homomorphic expansion of
 1772 wTF—called Z in the previous section—as Z^w to avoid confusion).

1773 There is a map $a : sKTG \rightarrow wTF$, given by interpreting sKTG diagrams as wTF
 1774 diagrams. In particular, positive vertices (of edge orientations as shown in Fig. 18) are
 1775 interpreted as the positive wTF vertex ↗ ↘ and negative vertices as the negative vertex ↖ ↙.
 1776 (The map a can also be interpreted topologically as Satoh's tubing map.) The
 1777 induced map $\alpha : \mathcal{A}^u \rightarrow \mathcal{A}^{sw}$ is as defined in Sect. 3.3, that is, α maps each chord to
 1778 the sum of its two possible orientations. Hence we can ask whether the two expansions
 1779 are compatible (or can be chosen to be compatible), which takes us to the main result
 1780 of this section:

1781 **Theorem 4.15** *Let Z^u be a homomorphic expansion for sKTG with the properties that*
 1782 *Φ is a horizontal chord associator and $n = e^{-c/4}v^{1/2}$ in the sense of Sect. 4.6.1.¹⁷*
 1783 *Then there exists a homomorphic expansion Z^w for wTF compatible with Z^u in the*
 1784 *sense that the square on the right commutes.*

¹⁷ It will become apparent that in the proof we only use slightly weaker but less aesthetic conditions on Z^u .

Author Proof

1785

$$\begin{array}{ccc}
 sKTG & \xrightarrow{a} & wTF \\
 \downarrow Z^u & & \downarrow Z^w \\
 \mathcal{A}^u & \xrightarrow{\alpha} & \mathcal{A}^{sw}
 \end{array} \tag{18}$$

1786 Furthermore, such Z^w are in one to one correspondence¹⁸ with “symmetric solu-
 1787 tions of the KV problem” satisfying the KV equations (14), the “twist equation” (20)
 1788 and the associator equation (22).

1789 Before moving on to the proof let us state and prove the following Lemma, to be
 1790 used repeatedly in the proof of the theorem.

1791 **Lemma 4.16** *If a and b are group-like elements in $\mathcal{A}^{sw}(\uparrow_n)$, then $a = b$ if and only*
 1792 *if $\pi(a) = \pi(b)$ and $aa^* = bb^*$. Here π is the projection induced by $\pi : \mathcal{P}^w(\uparrow_n) \rightarrow$*
 1793 *$\mathfrak{td}\mathfrak{e}\mathfrak{r}_n \oplus \mathfrak{a}_n$ (see Sect. 3.2), and $*$ refers to the adjoint map of Definition 3.26. In the*
 1794 *notation of this section $*$ is applying the adjoint A on all strands.*

1795 *Proof* Write $a = e^w e^{uD}$ and $b = e^{w'} e^{uD'}$, where $w \in \mathfrak{t}\mathfrak{r}_n$, $D \in \mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n \oplus \mathfrak{a}_n$ and
 1796 $u : \mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n \oplus \mathfrak{a}_n \rightarrow \mathcal{P}_n$ is the “upper” map of Sect. 3.2. Assume that $\pi(a) = \pi(b)$ and
 1797 $aa^* = bb^*$. Since $\pi(a) = e^D$ and $\pi(b) = e^{D'}$, we conclude that $D = D'$. Now
 1798 we compute $aa^* = e^w e^{uD} e^{-lD} e^w = e^w e^{j(D)} e^w$, where $j : \mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n \rightarrow \mathfrak{t}\mathfrak{r}_n$ is the map
 1799 defined in Section 5.1 of [2] and discussed in 3.27 of this paper. Now note that both
 1800 w and $j(D)$ are elements of $\mathfrak{t}\mathfrak{r}_n$, hence they commute, so $aa^* = e^{2w+j(D)}$. Thus,
 1801 $aa^* = bb^*$ means that $e^{2w+j(D)} = e^{2w'+j(D)}$, which implies that $w = w'$ and $a = b$.
 1802 □

1803 *Proof of Theorem 4.15* In addition to being a homomorphic expansion for wTF , Z^w
 1804 has to satisfy an the added condition of being compatible with Z^u . Since $sKTG$ is
 1805 finitely generated, this translates to one additional equation for each generator of
 1806 $sKTG$, some of which are automatically satisfied. To deal with the others, we use the
 1807 machinery established in the previous sections to translate these equations to conditions
 1808 on F , and they turn out to be the properties studied in [2] which link solutions of the
 1809 KV problem with Drinfel’d associators.

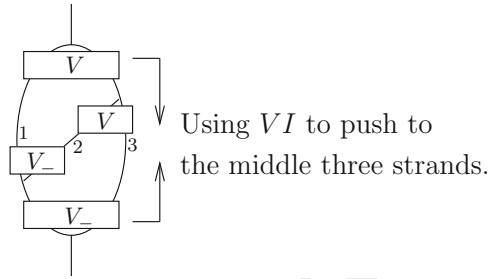
1810 To start, note that for the single strand and the bubble the commutativity of the
 1811 square (18) is satisfied with any Z^w : both the Z^u and Z^w values are 1 (note that
 1812 the Z^w value of the bubble is 1 due to the unitarity (12) of Z^w). Each of the other
 1813 generators will require more study. □

1814 *Commutativity of (18) for the twists* Recall that the Z^u -value of the right twist (for a
 1815 Z^u with horizontal chord Φ) is $R^u = e^{c/2}$; and note that its Z^w -value is $V^{-1}RV^{21}$,
 1816 where $R = e^{a_{12}}$ is the Z^w -value of the crossing (and a_{12} is a single arrow pointing from
 1817 strand 1 to strand 2). Hence the commutativity of (18) for the right twist is equivalent

¹⁸ An even nicer theorem would be a classification of homomorphic expansions for the combined algebraic structure $(sKTG \xrightarrow{a} wTF)$ in terms of solutions of the KV problem. The two obstacles to this are clarifying whether there is a free choice of n for Z^u , and — probably much harder—how much of the horizontal chord condition is necessary for a compatible Z^w to exist.

Author Proof

Fig. 19 The Z^w -value of the right associator



1818 to the “Twist Equation” $\alpha(R^u) = V^{-1}RV^{21}$. By definition of α , $\alpha(R^u) = e^{\frac{1}{2}(a_{12}+a_{21})}$,
 1819 where a_{12} and a_{21} are single arrows pointing from strand 1 to 2 and 2 to 1, respectively.
 1820 Hence we have

$$1821 \quad e^{\frac{1}{2}(a_{12}+a_{21})} = V^{-1}RV^{21}. \tag{19}$$

1822 To translate this to the language of [2], we use Lemma 4.16, which implies that it is
 1823 enough for V to satisfy the Twist Equation “on tree level” (i.e., after applying π), and
 1824 for which the adjoint condition of the Lemma holds.

1825 We first prove that the adjoint condition holds for any homomorphic expansion of
 1826 wTF . Multiplying the left hand side of the Twist Equation by its adjoint, we get

$$1827 \quad e^{\frac{1}{2}(a_{12}+a_{21})} (e^{\frac{1}{2}(a_{12}+a_{21})})^* = e^{\frac{1}{2}(a_{12}+a_{21})} e^{-\frac{1}{2}(a_{12}+a_{21})} = 1.$$

1828 As for the right hand side, we have to compute $V^{-1}RV^{21}(V^{21})^*R^*(V^{-1})^*$. Since V is
 1829 unitary (Eq. (12)), $VV^* = V \cdot A_1A_2(V) = 1$. Now $R = e^{a_{12}}$, so $R^* = e^{-a_{12}} = R^{-1}$,
 1830 hence the expression on the right hand side also simplifies to 1, as needed.

1831 As for the “tree level” of the Twist Equation, recall that in Sect. 4.3 we used
 1832 Alekseev and Torossian’s solution $F \in TAut_2$ to the Kashiwara–Vergne equations
 1833 [2] to find solutions V to Eqs. (11), (12) and (13). We produced V from F by setting
 1834 $F = e^{D^{21}}$ with $D \in \mathfrak{tder}_2^s$, $b := \frac{-j(F)}{2} \in \mathfrak{tr}_2$ and $V := e^b e^{uD}$, so F is “the tree part”
 1835 of V , up to re-numbering strands. Hence, the tree level Twist Equation translates to
 1836 a new equation for F . Substituting $V = e^b e^{uD}$ into the Twist Equation we obtain
 1837 $e^{\frac{1}{2}(a_{12}+a_{21})} = e^{-uD} e^{-b} e^{a_{12}} e^{b^{21}} e^{uD^{21}}$, and applying π , we get

$$1838 \quad e^{\frac{1}{2}(a_{12}+a_{21})} = (F^{21})^{-1} e^{a_{12}} F. \tag{20}$$

1839 In [2] the solutions F of the KV equations which also satisfy this equation are called
 1840 “symmetric solutions of the Kashiwara–Vergne problem” discussed in Sections 8.2
 1841 and 8.3. (Note that in [2] R denotes $e^{a_{21}}$).

1842 *Commutativity of (18) for the associators* Recall that the Z^u value of the right asso-
 1843 ciator is a Drinfel’d associator $\Phi \in \mathcal{A}^u(\uparrow_3)$; for the Z^w value see Fig. 19. Hence the
 1844 new condition on V is the following:

$$1845 \quad \alpha(\Phi) = V_-^{(12)3} V_-^{12} V^{23} V^{1(23)} \quad \text{in} \quad \mathcal{A}^{sw}(\uparrow_3) \tag{21}$$

Author Proof

1846 Again we treat the “tree and wheel parts” separately using Lemma 4.16. As Φ is
 1847 by definition group-like, let us denote $\Phi =: e^\phi$. We first verify that the “wheel part”
 1848 or adjoint condition of the Lemma holds. Starting with the right hand side of Eq. (21),
 1849 the unitarity $V V^* = 1$ of V implies that

$$1850 \quad V_-^{(12)3} V_-^{12} V^{23} V^{1(23)} (V^{1(23)})^* (V^{23})^* (V_-^{12})^* (V_-^{(12)3})^* = 1.$$

1851 For the left hand side of (21) we need to show that $e^{\alpha(\phi)} (e^{\alpha(\phi)})^* = 1$ as well, and
 1852 this is true for any *horizontal chord* associator. Indeed, restricted to the α -images of
 1853 horizontal chords $*$ is multiplication by -1 , and as it is an anti-Lie morphism, this
 1854 fact extends to the Lie algebra generated by α -images of horizontal chords. Hence
 1855 $e^{\alpha(\phi)} (e^{\alpha(\phi)})^* = e^{\alpha(\phi)} e^{\alpha(\phi)^*} = e^{\alpha(\phi)} e^{-\alpha(\phi)} = 1$.

1856 On to the tree part, applying π to Eq. (21) and keeping in mind that $V_- = V^{-1}$ by
 1857 the unitarity of V , we obtain

$$1858 \quad e^{\pi\alpha(\phi)} = (F^{3(12)})^{-1} (F^{21})^{-1} F^{32} F^{(23)} 1 = e^{-D^{(12)3}} e^{-D^{12}} e^{D^{23}} e^{D^{1(23)}}$$

1859 in $\text{SAut}_3 := \exp(\mathfrak{sdet}_3) \subset \text{TAut}_3. \tag{22}$

1860 This is Equation (26) of [2], up to re-numbering strands 1 and 2 as 2 and 1¹⁹. The
 1861 following fact from [2] (their Theorem 7.5, Propositions 9.2 and 9.3 combined) implies
 1862 that there is a solution F to the KV equations (14) which also satisfies (20) and
 1863 (22).

1864 **Fact 4.17** *If $\Phi' = e^{\phi'}$ is an associator in SAut_3 so that $j(\Phi') = 0$ ²⁰ then Eq. (22)*
 1865 *has a solution $F = e^{D^{21}}$ which is also a solution to the KV equations, and all such*
 1866 *solutions are symmetric (i.e. verify the Twist Equation (20)).* □

1867 To use this Fact, we need to show that $\Phi' := \pi\alpha(\Phi)$ is an associator in SAut_3 and
 1868 that $j(\Phi') = j(\pi\alpha(\Phi)) = 0$. The latter is the unitarity of Φ which is already proven.
 1869 The former follows from the fact that Φ is an associator and the fact (Theorem 3.28)
 1870 that the image of $\pi\alpha$ is contained in \mathfrak{sdet} (ignoring degree 1 terms, which are not
 1871 present in an associator anyway).

1872 In summary, the condition of the Fact are satisfied and so there exists a solution F
 1873 which in turn induces a Z^w which is compatible with Z^u for the strand, the bubble, the
 1874 twists and the associators. That is, all generators of $sKTG$ except possibly the balloon
 1875 and the noose. As the last step of the proof of Theorem 4.15 we show that any such
 1876 Z^w also automatically make (18) commutative for the balloon and the noose.

1877 *Commutativity of (18) for the balloon and the noose* Since we know the Z^u -values
 1878 B and n of the balloon and the noose, we start by computing Z^w of the noose. Z^w
 1879 assigns a V value to the vertex with the first strand orientation switched as shown in

¹⁹ Note that in [2] “ Φ' is an associator” means that Φ' satisfies the pentagon equation, mirror skew-symmetry, and positive and negative hexagon equations in the space SAut_3 . These equations are stated in [2] as equations (25), (29), (30), and (31), and the hexagon equations are stated with strands 1 and 2 re-named to 2 and 1 as compared to [8, 19]. This is consistent with $F = e^{D^{21}}$.

²⁰ The condition $j(\phi') = 0$ is equivalent to the condition $\Phi \in \text{KR}V_3^0$ in [2]. The relevant definitions in [2] can be found in Remark 4.2 and at the bottom of page 434 (before Section 5.2).

1880 the figure on the right. The balloon is the same, except with a negative vertex and the
 1881 second strand reversed. Hence what we need to show is that the two equations below
 1882 hold:

$$Z^w \left(\text{noose with arrow} \right) = \boxed{S_1(V)}$$

$$\boxed{S_1(V)} = \boxed{\alpha(\nu)^{1/2}} \boxed{e^{-\frac{D_A}{2}}} = \boxed{S_2(V_-)} = \boxed{e^{\frac{D_A}{2}}} \boxed{\alpha(\nu)^{1/2}}$$

1885 Let us denote the left hand side of the first equation above by n^w and b^w (the
 1886 Z^w value of the noose and the balloon, respectively). We will start by proving that
 1887 the product of these two equations holds, namely that $n^w b^w = \alpha(\nu)$. (We used that
 1888 any local (small) arrow diagram on a single strand is central in $\mathcal{A}^{sw}(\uparrow_n)$, hence the
 1889 cancellations.) This product equation is satisfied due to an argument identical to that
 1890 of Fig. 16, but carried out in wTF , and using that by the compatibility with associators,
 1891 Z^w of an associator is $\alpha(\Phi)$.

1892 What remains is to show that the noose and balloon equations hold individually. In
 1893 light of the results so far, it is sufficient to show that

$$n^w = b^w \cdot e^{-D_A}, \tag{23}$$

1895 where D_A stands for a single arrow on one strand (whose direction doesn't matter
 1896 due to the RI relation. As stated in [38, Theorem 3.15], $\mathcal{A}^{sw}(\uparrow_1)$ is the polynomial
 1897 algebra freely generated by the arrow D_A and wheels of degrees 2 and higher. Since V
 1898 is group-like, n^w (resp. b^w) is an exponential e^{A_1} (resp. e^{A_2}) with $A_1, A_2 \in \mathcal{A}^{sw}(\uparrow_1)$.
 1899 We want to show that $e^{A_1} = e^{A_2} \cdot e^{-D_A}$, equivalently that $A_1 = A_2 - D_A$.

1900 In degree 1, this can be done by explicit verification. Let $A_1^{\geq 2}$ and $A_2^{\geq 2}$ denote
 1901 the degree 2 and higher parts of A_1 and A_2 , respectively. We claim that capping the
 1902 strand at both its top and its bottom takes e^{A_1} to $e^{A_1^{\geq 2}}$, and similarly e^{A_2} to $e^{A_2^{\geq 2}}$. (In
 1903 other words, capping kills arrows but leaves wheels un-changed.) This can be proven
 1904 similarly to the proof of Lemma 4.6, but using

$$F' := \sum_{k_1, k_2=0}^{\infty} \frac{(-1)^{k_1+k_2}}{k_1!k_2!} D_A^{k_1+k_2} S_L^{k_1} S_R^{k_2}$$

1906 in place of F in the proof. What we need to prove, then, is the following equality, and
 1907 the proof is shown in Fig. 20.

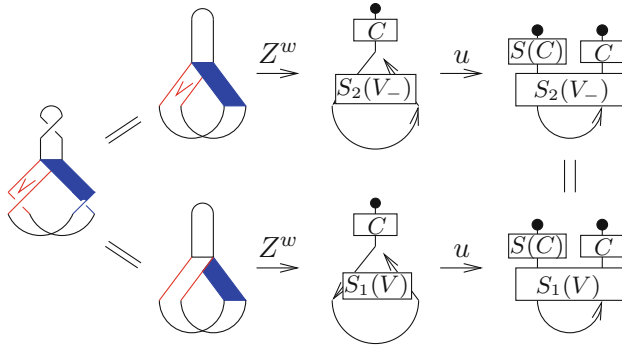


Fig. 20 The proof of Eq. (24). Note that the unzips are “illegal”, as the strand directions don’t match. This can be fixed by inserting a small bubble at the bottom of the noose and doing a number of orientation switches. As this doesn’t change the result or the main argument, we suppress the issue for simplicity. Equation (24) is obtained from this result by multiplying by $S(C)^{-1}$ on the *bottom* and by C^{-1} on the *top*

$$\begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \boxed{S_1(V)} \\ \curvearrowright \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \boxed{S_2(V_-)} \\ \curvearrowright \end{array} . \tag{24}$$

1908

1909 This concludes the proof of Theorem 4.15.

1910 Recall from Sect. 3.3 that there is no commutative square linking $Z^u : uT \rightarrow \mathcal{A}^u$
 1911 and $Z^w : wT \rightarrow \mathcal{A}^{sw}$, for the simple reason that the Kontsevich integral for tangles Z^u
 1912 is not canonical, but depends on a choice of parenthesizations for the “bottom” and the
 1913 “top” strands of a tangle T . Yet given such choices, a tangle T can be “closed up with
 1914 trees” as within the proof of Proposition 4.13 (see Sect. 5) into an sKTG which we will
 1915 denote G . For G a commutativity statement does hold as we have just proven. The Z^u
 1916 and Z^w invariants of T and of G differ only by a number of vertex-normalizations and
 1917 vertex-values on skeleton-trees at the bottom or at the top of G , and using VI, these
 1918 values can slide so they are placed on the original skeleton of T . This is summarized
 1919 as the following proposition:

1920 **Proposition 4.18** *Let n and n' be natural numbers. Given choices c and c' of parenthe-*
 1921 *sizations of n and n' strands respectively, there exists invertible elements $C \in \mathcal{A}^{sw}(\uparrow_n)$*
 1922 *and $C' \in \mathcal{A}^{sw}(\uparrow_{n'})$ so that for any u -tangle T with n “bottom” ends and n' “top”*
 1923 *ends we have*

$$\alpha Z_{c,c'}^u(T) = C^{-1} Z^w(aT) C', \tag{24}$$

1924

1925 where $Z_{c,c'}^u$ denotes the usual Kontsevich integral of T with bottom and top parenthe-
 1926 sizations c and c' .

1927 For u-braids the above proposition may be stated with $c = c'$ and then C and C'
 1928 are the same.

1929 **5 Odds and ends**1930 **5.1 Motivation for circuit algebras: electronic circuits**

1931 Electronic circuits are made of “components” that can be wired together in many
 1932 ways. On a logical level, we only care to know which pin of which component is
 1933 connected with which other pin of the same or other component. On a logical level,
 1934 we don’t really need to know how the wires between those pins are embedded in space
 1935 (see Figs. 21, 22). “Printed Circuit Boards” (PCBs) are operators that make smaller
 1936 components (“chips”) into bigger ones (“circuits”)—logically speaking, a PCB is
 1937 simply a set of “wiring instructions”, telling us which pins on which components
 1938 are made to connect (and again, we never care precisely how the wires are routed
 1939 provided they reach their intended destinations, and ever since the invention of multi-
 1940 layered PCBs, all conceivable topologies for wiring are actually realizable). PCBs can
 1941 be composed (think “plugging a graphics card onto a motherboard”); the result of a
 1942 composition of PCBs, logically speaking, is simply a larger PCB which takes a larger
 1943 number of components as inputs and outputs a larger circuit. Finally, it doesn’t matter
 1944 if several PCB are connected together and then the chips are placed on them, or if the
 1945 chips are placed first and the PCBs are connected later; the resulting overall circuit
 1946 remains the same.

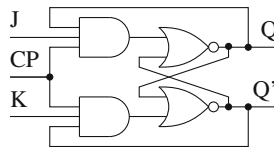


Fig. 21 The J-K flip flop, a very basic memory cell, is an electronic circuit that can be realized using 9 components—two triple-input “and” gates, two standard “nor” gates, and 5 “junctions” in which 3 wires connect (many engineers would not consider the junctions to be real components, but we do). Note that the “crossing” in the middle of the figure is merely a projection artifact and does not indicate an electrical connection, and that electronically speaking, we need not specify how this crossing may be implemented in \mathbb{R}^3 . The J-K flip flop has 5 external connections (labelled J, K, CP, Q, and Q’) and hence in the circuit algebra of computer parts, it lives in C_5 . In the directed circuit algebra of computer parts it would be in $C_{3,2}$ as it has 3 incoming wires (J, CP, and K) and two outgoing wires (Q and Q’)

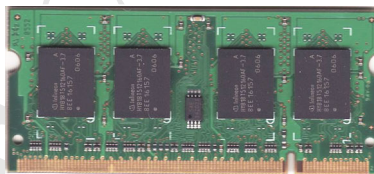


Fig. 22 The circuit algebra product of 4 big black components and 1 small black component carried out using a green wiring diagram, is an even bigger component that has many golden connections (at bottom). When plugged into a yet bigger circuit, the CPU board of a laptop, our circuit functions as 4,294,967,296 binary memory cells

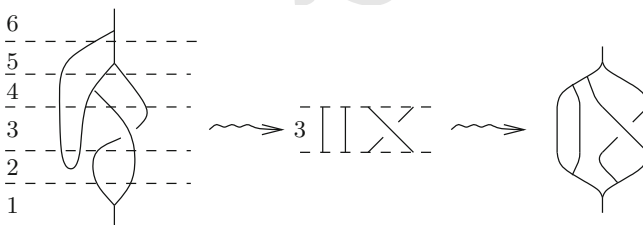
1947 **5.2 Proof of Proposition 4.13**

1948 We are going to ignore strand orientations throughout this proof for simplicity. This
 1949 is not an issue as orientation switches are allowed in $sKTG$ without restriction. We are
 1950 also going to omit vertex signs from the pictures given the pictorial convention stated
 1951 in Sect. 4.6.

1952 We need to prove that any $sKTG$ (call it G) can be built from the generators listed
 1953 in the statement of the proposition, using $sKTG$ operations. To show this, consider a
 1954 Morse drawing of G , that is, a planar projection of G with a height function so that all
 1955 singularities along the strands are Morse and so that every “feature” of the projection
 1956 (local minima and maxima, crossings and vertices) occurs at a different height.

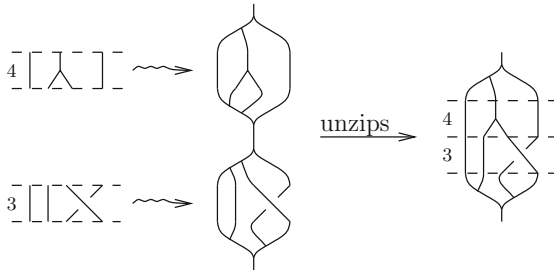
1957 The idea in short is to decompose G into levels of this Morse drawing where at each
 1958 level only one “feature” occurs. The levels themselves are not $sKTG$'s, but we show
 1959 that the composition of the levels can be achieved by composing their “closed-up”
 1960 $sKTG$ versions followed by some unzips. Each feature gives rise to a generator by
 1961 “closing up” extra ends at its top and bottom. We then show that we can construct each
 1962 level using the generators and the tangle insert operation.

1963 So let us decompose G into a composition of trivalent tangles (“levels”), each of
 1964 which has one “feature” and (possibly) some straight vertical strands. Note that by
 1965 isotopy we can make sure that every level has strands ending at both its bottom and
 1966 top, except for the first or the last level in the case of 1-tangles. An example of level
 1967 decomposition is shown in the figure below. Note that the levels are generally not
 1968 elements of $sKTG$ (have too many ends). However, we can turn each of them into a
 1969 $(1, 1)$ -tangle (or a 1-tangle in case of the aforementioned top first or last levels) by
 1970 “closing up” their tops and bottoms by arbitrary trees. In the example below we show
 1971 this for one level of the Morse-drawn $sKTG$ containing a crossing and two vertical
 1972 strands.



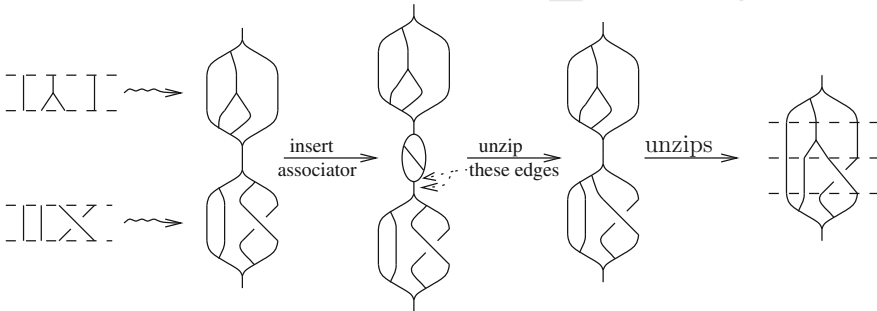
1973

1974 Now we can compose the $sKTG$'s obtained from closing up each level. Each tree
 1975 that we used to close up the tops and bottoms of levels determines a “parenthesization”
 1976 of the strand endings. If these parenthesizations match on the top of each level with the
 1977 bottom of the next, then we can recreate tangle composition of the levels by composing
 1978 their closed versions followed by a number of unzips performed on the connecting
 1979 trees. This is illustrated in the example below, for two consecutive levels of the $sKTG$
 1980 of the previous example.



1981

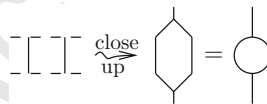
1982 If the trees used to close up consecutive levels correspond to different parenthesiza-
 1983 tions, then we can use insertion of the left and right associators (the 5th and 6th pictures
 1984 of the list of generators in the statement of the theorem) to change one parenthesization
 1985 to match the other. This is illustrated in the figure below.



1986

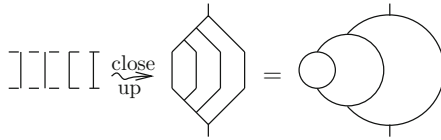
1987 So far we have shown that G can be assembled from closed versions of the levels in
 1988 its Morse drawing. The closed versions of the levels of G are simpler $sKTG$'s, and it
 1989 remains to show that these can be obtained from the generators using $sKTG$ operations.

1990 Let us examine what each level might look like. First of all, in the absence of any
 1991 "features" a level might be a single strand, in which case it is the first generator itself.
 1992 Two parallel strands when closed up become the "bubble", as shown on the right.



1993

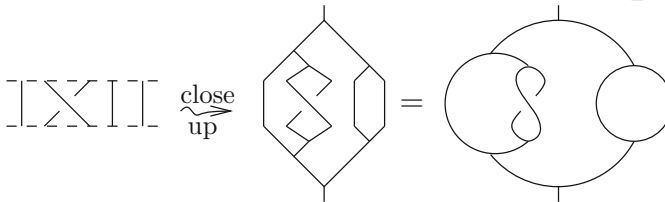
1994 Now suppose that a level consists of n parallel strands, and that the trees used to
 1995 close it up on the top and bottom are horizontal mirror images of each other, as shown
 1996 below (if not, then this can be achieved by associator insertions and unzips). We want
 1997 to show that this $sKTG$ can be obtained from the generators using $sKTG$ operations.
 1998 Indeed, this can be achieved by repeatedly inserting bubbles into a bubble, as shown:



1999

2000 A level consisting of a single crossing becomes a left or right twist when closed up
 2001 (depending on the sign of the crossing). Similarly, a single vertex becomes a bubble.
 2002 A single minimum or maximum becomes a noose or a balloon, respectively.

2003 It remains to see that the *sKTG*'s obtained when closing up simple features accom-
 2004 panied by more through strands can be built from the generators. A minimum
 2005 accompanied by an extra strand gives rise to the *sKTG* obtained by sticking a noose
 2006 onto a vertical strand (similarly, a balloon for a maximum). In the case of all the
 2007 other simple features and for minima and maxima accompanied by more strands, we
 2008 inserting the already generated elements into nested bubbles (bubbles inserted into
 2009 bubbles), as in the example shown below. This completes the proof.



2010

2011 6 Glossary of notation

2012 Greek letters, then Latin, then symbols:

- 2013 δ Satoh's tube map [3.4](#)
- 2014 Δ Co-product [3.2](#)
- 2015 ι Inclusion $\mathfrak{t}_n \rightarrow \mathcal{P}^w(\uparrow_n)$ [3.2](#)
- 2016 ν The invariant of the unknot [4.6](#)
- 2017 π The projection $\mathcal{P}^w(\uparrow_n) \rightarrow \mathfrak{a}_n \oplus \mathfrak{tdet}_n$ [3.2](#)
- 2018 ϕ Log of an associator [4.6](#)
- 2019 Φ An associator [4.6](#)
- 2020 ψ_β "operations" [2.1](#)
- 2021 \mathfrak{a}_n n -Dimensional Abelian Lie algebra [3.2](#)
- 2022 \mathcal{A} A candidate associated graded structure [2.3](#)
- 2023 \mathcal{A}^{sv} \mathcal{D}^v mod 6T, RI [3.1](#)
- 2024 \mathcal{A}^{sw} \mathcal{D}^w mod $\overrightarrow{4T}$, TC, RI [3.1](#)
- 2025 \mathcal{A}^{sw} grad wTF^o [4.2](#)
- 2026 \mathcal{A}^{sw} grad wTF [4.5.2](#)
- 2027 $\mathcal{A}^{(s)w}$ \mathcal{A}^w and/or \mathcal{A}^{sw} [4.2](#)
- 2028 \mathcal{A}^u Chord diagrams mod rels for KTGs [4.6](#)
- 2029 \mathcal{A}^v \mathcal{D}^v mod 6T [3.1](#)
- 2030 \mathcal{A}^w \mathcal{D}^w mod $\overrightarrow{4T}$, TC [3.1](#)

2031	\mathcal{A}^w	grad wTF^o without RI 4.2
2032	$\mathcal{A}^-(\uparrow_n)$	\mathcal{A}^- for pure n -tangles 3.2
2033	A_e	1D orientation reversal 4.1.3
2034	Ass	Associative words 3.2
2035	Ass ⁺	Non-empty associative words 3.2
2036	\mathcal{B}_n^w	n -coloured univalent arrow diagrams 3.2
2037		
2038	C	The invariant of a cap 4.3
2039	CP	The Cap-Pull relation 4.1.2, 4.2
2040	CW	Cap-Wen relations 4.5.1
2041	c	A chord in \mathcal{A}^u 4.6
2042	der	Lie-algebra derivations 3.2
2043	$\mathcal{D}^v, \mathcal{D}^w$	Arrow diagrams for v/w -tangles 3.1
2044	div	The “divergence” 3.2
2045	F	A map $\mathcal{A}^w \rightarrow \mathcal{A}^w$ 4.2
2046	F	The main [2] unknown 4.4
2047	FR	Flip Relations 4.5.1, 4.5.2
2048	fil	A filtered structure 2.3
2049	\mathcal{I}	Augmentation ideal 2.2
2050	J	A map $T\text{Aut}_n \rightarrow \exp(\text{tr}_n)$ 3.2
2051	j	A map $T\text{Aut}_n \rightarrow \text{tr}_n$ 3.2
2052	KTG	Knotted Trivalent Graphs 4.6
2053	lie_n	Free Lie algebra 3.2
2054	l	A map $\text{tder}_n \rightarrow \mathcal{P}^w(\uparrow_n)$ 3.2
2055	\mathcal{O}	An “algebraic structure” 2.1
2056	\mathcal{P}_n^w	Primitives of \mathcal{B}_n^w 3.2
2057	$\mathcal{P}^-(\uparrow_n)$	Primitives of $\mathcal{A}^-(\uparrow_n)$ 3.2
2058	grad	Associated graded structure 2.2
2059	R	The invariant of a crossing 4.3
2060	R4	A Reidemeister move for foams/graphs 4.1.2
2061		
2062	sder	Special derivations 3.3
2063	\mathcal{S}	The circuit algebra of skeletons 2.4
2064	SAut_n	The group $\exp(\text{sder}_n)$ 4.6
2065	S_k	Complete orientation reversal 3.5
2066	S_e	Complete orientation reversal 4.1.3
2067	$sKTG$	Signed long KTGs 4.6
2068	TV	Twisted Vertex relations 4.5.1
2069	tder	Tangential derivations 3.2
2070	tr_n	Cyclic words 3.2
2071	tr_n^s	Cyclic words mod degree 1 3.2
2072	$T\text{Aut}_n$	The group $\exp(\text{tder}_n)$ 3.2
2073	u	A map $\text{tder}_n \rightarrow \mathcal{P}^w(\uparrow_n)$ 3.2
2074	u_e	Strand unzips 4.1.3
2075	$u\mathcal{I}$	U-tangles 3.3
2076	V, V^+	The invariant of a (positive) vertex 4.3

2077	V^-	The invariant of a negative vertex 4.3
2078	Π	Vertex Invariance 4.2
2079	vT	v-tangles 3.1
2080	W	$Z(w)$ 4.5.3
2081	W^2	Wen squared 4.5.1
2082	w	The wen 4.5
2083	wT	w-tangles 3.1
2084	wTF	w-tangled foams with wens 4.5
2085	wTF^o	Orientable w-tangled foams 4.1
2086	Z	Expansions throughout
2087	$Z_{\mathcal{A}}$	An \mathcal{A} -expansion 2.3
2088	$4T$	$4T$ relations 4.6
2089	\uparrow	A “long” strand throughout
2090	\uparrow	The quandle operation 2.1
2091	$*$	The adjoint on $\mathcal{A}^w(\uparrow_n)$ 3.2

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2095 References

- 2096 1. Alekseev, A., Meinrenken, E.: The Kashiwara–Vergne conjecture and Drinfel’d’s associators. *Inven-*
 2097 *tiones Mathematicae* **164**, 615–634 (2006). [arXiv:0506499](https://arxiv.org/abs/0506499)
- 2098 2. Alekseev, A., Torossian, C.: The Kashiwara–Vergne conjecture and Drinfel’d’s associators. *Ann. Math.*
 2099 **175**, 415–463 (2012). [arXiv:0802.4300](https://arxiv.org/abs/0802.4300)
- 2100 3. Alekseev, A., Enriquez, B., Torossian, C.: Drinfel’d’s associators, braid groups and an explicit solution
 2101 of the Kashiwara–Vergne equations. *Publications Mathématiques de L’IHÉS* **112–1**, 143–189 (2010).
 2102 [arXiv:0903.4067](https://arxiv.org/abs/0903.4067)
- 2103 4. Bar-Natan, D.: On the Vassiliev knot invariants. *Topology* **34**, 423–472 (1995)
- 2104 5. Bar-Natan, D.: Non-associative tangles. In: Kazez, H. (ed.) *Geometric Topology. Proceedings of the*
 2105 *Georgia International Topology Conference*, pp. 139–183. Amer. Math. Soc. and International Press,
 2106 Providence (1997)
- 2107 6. Bar-Natan, D.: On Associators and the Grothendieck–Teichmüller Group I. *Selecta Mathematica New*
 2108 *Ser.* **4**, 183–212 (1998)
- 2109 7. Bar-Natan, D.: Algebraic Knot Theory—A Call for Action (2006). <http://www.math.toronto.edu/~drorbn/papers/AKT-CFA.html>
- 2110 8. Bar-Natan, D., Dancso, Z.: Homomorphic expansions for knotted trivalent graphs. *J. Knot Theory*
 2111 *Ramif.* **22**(1) (2013). [arXiv:1103.1896](https://arxiv.org/abs/1103.1896)
- 2112 9. Bar-Natan, D., Garoufalidis, S., Rozansky, L., Thurston, D.P.: Wheels, wheeling, and the Kontsevich
 2113 integral of the unknot. *Israel J. Math.* **119**, 217–237 (2000). [arXiv:q-alg/9703025](https://arxiv.org/abs/q-alg/9703025)
- 2114 10. Bar-Natan, D., Halacheva, I., Leung, L., Roukema, F.: Some Dimensions of Spaces of Finite Type
 2115 Invariants of Virtual Knots (**submitted**)
- 2116 11. Bar-Natan, D., Le, T.Q.T., Thurston, D.P.: Two applications of elementary knot theory to Lie algebras
 2117 and Vassiliev invariants. *Geom. Topol.* **7–1**, 1–31 (2003). [arXiv:math.QA/0204311](https://arxiv.org/abs/math.QA/0204311)
- 2118 12. Berceanu, B., Papadima, S.: Universal representations of Braid and Braid-permutation groups. *J. Knot*
 2119 *Theory Ramif.* **18–7**, 973–983 (2009). [arXiv:0708.0634](https://arxiv.org/abs/0708.0634)
- 2120 13. Brendle, T., Hatcher, T.: Configuration Spaces of Rings and Wickets. [arXiv:0805.4354](https://arxiv.org/abs/0805.4354)
- 2121 14. Cheptea, D., Le, T.Q.T.: A TQFT associated to the LMO invariant of three-dimensional manifolds.
 2122 *Commun. Math. Phys.* **272**, 601–634 (2007)
- 2123 15. Carter, J.S., Saito, M.: Knotted surfaces and their diagrams. In: *Mathematical Surveys and Monographs*,
 2124 vol. 55. American Mathematical Society, Providence (1998)

- 2126 16. Dahm, D.M.: A Generalization of braid theory. PhD Thesis, Princeton University (1962)
- 2127 17. Dancso, Z.: On a Kontsevich integral for knotted trivalent graphs. *Algebr. Geom. Topol.* **10**, 1317–1365
- 2128 (2010). [arXiv:0811.4615](https://arxiv.org/abs/0811.4615)
- 2129 18. Drinfel'd, V.G.: Quantum groups. In: Proceedings of the International Congress of Mathematicians,
- 2130 pp. 798–820. Berkeley (1986)
- 2131 19. Drinfel'd, V.G.: Quasi-Hopf Algebras. *Leningrad Math. J.* **1**, 1419–1457 (1990)
- 2132 20. Drinfel'd, V.G.: On quasitriangular quasi-hopf algebras and a group closely connected with $\text{Gal}(\mathbb{Q}/\mathbb{Q})$.
- 2133 *Leningrad Math. J.* **2**, 829–860 (1991)
- 2134 21. Etingof, P., Kazhdan, D.: Quantization of Lie Bialgebras, I. *Selecta Mathematica N. Ser.* **2**, 1–41 (1996).
- 2135 [arXiv:q-alg/9506005](https://arxiv.org/abs/q-alg/9506005)
- 2136 22. Fenn, R., Rimanyi, R., Rourke, C.: The braid-permutation group. *Topology* **36**, 123–135 (1997)
- 2137 23. Goldsmith, D.L.: The theory of motion groups. *Mich. Math. J.* **28-1**, 3–17 (1981)
- 2138 24. Jones, V.: Planar algebras, I. *N. Zeal. J. Math.* [arXiv:math.QA/9909027](https://arxiv.org/abs/math.QA/9909027) (to appear)
- 2139 25. Joyce, D.: A classifying invariant of knots, the knot quandle. *J. Pure Appl. Algebra* **23**, 37–65 (1982)
- 2140 26. Kashiwara, M., Vergne, M.: The Campbell–Hausdorff formula and invariant hyperfunctions. *Invent.*
- 2141 *Math.* **47**, 249–272 (1978)
- 2142 27. Kauffman, L.H.: Virtual knot theory. *Eur. J. Comb.* **20**, 663–690 (1999). [arXiv:math.GT/9811028](https://arxiv.org/abs/math.GT/9811028)
- 2143 28. Kuperberg, G.: What is a virtual link? *Algebr. Geom. Topol.* **3**, 587–591 (2003).
- 2144 [arXiv:math.GT/0208039](https://arxiv.org/abs/math.GT/0208039)
- 2145 29. Le, T.Q.T., Murakami, J.: The universal Vassiliev–Kontsevich invariant for framed oriented links. *Compositio Math.* **102**, 41–64 (1996). [arXiv:hep-th/9401016](https://arxiv.org/abs/hep-th/9401016)
- 2146 30. Leinster, T.: Higher operads, higher categories. In: London Mathematical Society Lecture Note Series,
- 2147 vol. 298, Cambridge University Press, Cambridge. ISBN 0-521-53215-9. [arXiv:math.CT/0305049](https://arxiv.org/abs/math.CT/0305049)
- 2148 31. Levine, J.: Addendum and correction to: “homology cylinders: an enlargement of the mapping class
- 2149 group”. *Alg. Geom. Top.* **2**, 1197–1204 (2002). [arXiv:math.GT/0207290](https://arxiv.org/abs/math.GT/0207290)
- 2150 32. Loday, J.-L.: Une version non commutative des algebres de Lie: des algebres de Leibniz. *Enseign.*
- 2151 *Math. (2)* **39**(3–4), 269–293 (1975)
- 2152 33. McCool, J.: On basis-conjugating automorphisms of free groups. *Can. J. Math.* **38-6**, 1525–1529
- 2153 (1986)
- 2154 34. Milnor, J., Moore, J.: On the structure of Hopf algebras. *Ann. Math.* **81**, 211–264 (1965)
- 2155 35. Murakami, J., Ohtsuki, T.: Topological quantum field theory for the universal quantum invariant.
- 2156 *Commun. Math. Phys.* **188**(3), 501–520 (1997)
- 2157 36. Satoh, S.: Virtual knot presentations of ribbon torus knots. *J. Knot Theory Ramif.* **9-4**, 531–542 (2000)
- 2158 37. Bar-Natan, D., Dancso, Z.: Finite type invariants of w-knotted objects: from Alexander to Kashiwara
- 2159 and Vergne, earlier web version of the first two papers of this series in one. Paper, videos (wClips) and
- 2160 related files at <http://www.math.toronto.edu/~drorbn/papers/WKO/>. [arXiv:1309.7155](https://arxiv.org/abs/1309.7155) edition may be
- 2161 older
- 2162 38. Bar-Natan, D., Dancso, Z.: Finite type invariants of W-knotted objects I: braids, knots and the Alexander
- 2163 polynomial. <http://www.math.toronto.edu/drorbn/LOP.html#WKO1>. [arXiv:1405.1956](https://arxiv.org/abs/1405.1956)
- 2164 39. Bar-Natan, D., Dancso, Z.: Finite Type Invariants of W-Knotted Objects III: the Double Tree Con-
- 2165 struction (in preparation)
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