REPORT ON 'FINITE TYPE INVARIANTS OF w-KNOTTED OBJECTS I: w-KNOTS AND THE ALEXANDER POLYNOMIAL' BY D. BAR-NATAN AND Z. DANCSO

1. Overview of the paper

The reviewed paper 'Finite type invariants of w-knotted objects I: w-knots and the Alexander polynomial' by D. Bar-Natan and Z. Dancso uses a combinatorial and topological approach to define Kontsevich integral-type invariants and universal finite type invariants, denoted by Z^w or Z, for what they call *w-knotted* objects.

These w-knotted objects are certain ribbon knotted objects in 4-dimensions and have a neat topological interpretation as the group of movies of flying rings in $\mathbb{R}^3$ or as certain classes of knotted surfaces in $\mathbb{R}^4$.

The paper 'Finite type invariants of w-knotted objects I: w-knots and the Alexander polynomial' is the first of a sequence of (possible) four papers (of which one is already available on the arXiv, see http://arxiv.org/abs/1405.1955) about how to use "chord diagram-like" approaches to relate questions from a combinatorial and topological side with equations on some algebraic side (usually something related to universal Lie theory). In particular, the authors relate w-knotted objects to solutions of the *Kashiwara-Vergne equation* (although not in the paper under review, but in the later papers).

On the other hand, the main goal of the first paper is to illustrate how their approach works for w-braids and (long) w-knots. In particular, the authors show that the invariant Z^w for (long) w-knots constructed from this approach coincides (in a very strong sense) with (the well-known) *Alexander polynomial* of (long) w-knots.

Moreover, they connect the combinatorial and topological world of w-knotted objects with finite dimensional, commutative Lie bi-algebras and associated graded spaces.

The main result of the paper 'Finite type invariants of w-knotted objects I: w-knots and the Alexander polynomial' is then Theorem 3.27 where the authors show that the invariant Z^w and the Alexander polynomial of a (long) w-knot contain "the same information".

The proof of Theorem 3.27 is based on a good idea (to compare certain derivatives) followed by a complicated and long calculation proving that signs and conventions work out as they should.

It should be noted that Theorem 3.27 was "known" before (in an *inexplicit* way) as the authors point out in Remark 3.28.

The paper is organized as follows.

- Section 1 is the introduction. This part is rather long due to the fact that the authors try to motivate the relevance of their approach already with respect to later papers of the sequence. Especially Figure 1 is very helpful to understand the "bigger picture".
- Section 2 contains the treatment of their approach with respect to w-braids. That is, after recalling the necessary definitions, they show how they can use their approach to define an expansion Z^w for w-braids.
- Section 3 contains the treatment of their approach with respect to (long) w-knots. The first part of this section is devoted to recall the necessary definitions. The second part, which is interesting in its own right, contains a combinatorial tool related to (long) w-knots akin to the classical Jacobi diagrams and relates this tool to the Lie algebraic side. The last part of this section contains the statement and the proof of Theorem 3.27.

- Section 4 is then a section about open questions, mostly with respect to possible computer calculations.

One should see this paper as a prologue to the story that the authors are trying to tell: how to connect topological questions to questions arising in universal Lie theory.

In fact, the paper 'Finite type invariants of w-knotted objects I: w-knots and the Alexander polynomial' seems to be more combinatorial and topological in nature than the follow-up paper on the arXiv cited above.

In particular, one purpose of the paper 'Finite type invariants of w-knotted objects I: w-knots and the Alexander polynomial' is how the approach of the authors is motivated from and related to classical weight systems, chord diagrams and finite type invariants.

2. Summary, opinion and potential issues

The reviewed paper provides an interesting, new way of relating topological and algebraical questions. As far as the language is concerned, the paper is mostly well understandable, but sometimes hard to read and to understand due to its combinatorial nature.

Most of the results in this paper are (smart) reformulations of classical construction in the theory of finite type invariants. Unfortunately, the really good, new and possible way-leading results are not in the paper 'Finite type invariants of w-knotted objects I: w-knots and the Alexander polynomial', but in its follow-up papers.

Although I do not think that the results in the paper 'Finite type invariants of w-knotted objects I: w-knots and the Alexander polynomial' are on the level of "Mathematische Annalen", I recommend the paper for publication, since I personally see it as the prologue or introduction to a new approach to solve old questions coming from a (at first and second sight) seemingly different part of mathematics, namely, universal Lie theory.

3. List of typos, questions and other issues

Here is the list of typos, comments and questions that I spotted while reading the paper.

(1) Page 1, Line 16: ...wider, but...
(2) Page 1, Line 17: ...hence, enlarging...
(3) Page 1, Line 17: ...the the... **change it to** ...the...
(4) Page 1, Line 28: **Suggestion**:...are algebraically...interesting (true, but does not follow from the above discussion. At this point of your abstract it seems to be more appropriate to mention combinatorial group theory. Mention relation to Lie algebras earlier would clear this point).
(5) Page 1, Line 35: ...algebras, but we...
(6) Page 2, Line 33: "closed form" in capital letters (following your convention).
(7) Page 2, Line 35: "notation" in capital letters (following your convention).
(8) Page 3, Line 4: Teichmüller.
(9) Page 3, Line 4: ...[Dr3], which...
(10) Page 3, Line 21: Thus, ...
(11) Page 3, Line 37: call here w-knotted **change it to** call w-knotted.
(12) Page 3, Line 53: your convention seems to be to order the references by date of appearance. If so, then this is not the correct order.
(13) Page 4, Line 7: **Suggestion**: Use directly usual instead of ordinary (because that is your convention for the rest of the paper).
(14) Page 4, Line 41: Thus, it...
(15) Page 4, Line 44: ...that, when...
(16) Page 4, Line 45: **Suggestion**: The notation Z is not explained to this point. Say something about the "classical" story related to the Kontsevich integral.

(17) Page 4, Line 56: order is again wrong.
(18) Page 4, Line 56: **Suggestion**: Better without brackets, just ", see [refs]."
(19) Page 5, Line 12: ...the u,v,w-knots...
(20) Page 5, Line 22: Thus, we...
(21) Page 5, Lines 37-57: **Question**: Any reason for the different typesetting?
(22) Page 5, Line 60: **Question-Footnote**: Is that easy to see?
(23) Page 6, Lines 8-end: **Question**: Any reason for the spacing?
(24) Page 7, Summary: **Suggestion**: Add Subsection before the references 2.1 etc.
(25) Page 7, Line 23: Hence, ...
(26) Page 7, Line 23: ...easy-at-first, but...
(27) Page 7, Line 27: ...digest, but...
(28) Page 7, Line 27: ...correct, "abstract"...
(29) Page 7, Line 39: ...and, if ... , then...
(30) Page 7, Line 41: **Suggestion**: A good place to recall/explain that all your pictures will be local.
(31) Page 7, Line 52: ...and, if ... , then...
(32) Page 7, Line 56: ...Subsection 2.1.1...
(33) Page 7, Line 58: ...Subsection 2.1.2...
(34) Page 8, Line 13: **Suggestion**: "mixed relations", that is,
(35) Page 8, Line 15: ...and, if ... , then...
(36) Page 8, Line 29: Thus, ...
(37) Page 8, Line 39: **Suggestion**: Not every reader (including the referee) is familiar with the fact that splitting short exact sequences in the category of groups lead to semi-direct products. The same happens in the category of Lie algebras. Thus, I think it could be worthwhile to add a comments about this, since it is important for you later on.
(38) Page 8, Line 42: Therefore, ...
(39) Page 9, Line 4: Thus, ...
(40) Page 9, Line 15: $\leq n$ **change it to** $\leq n-1$.
(41) Page 9, Line 28: over crossings **change it to** overcrossings.
(42) Page 9, Line 29: under crossings **change it to** undercrossings.
(43) Page 9, Line 29: ..."forbidden moves" in virtual...
(44) Page 9, Line 50: **Suggestion**: Use always (relation) as above and at several other points in the paper. See also my general remark about it.
(45) Page 9, Line 54: ...hence, we...
(46) Page 9, Line 54: **Question**: Do you really want to use the kernel symbol here?
(47) Page 9, Line 55: ...and thus, a semi-direct...
(48) Page 10, Line 8: **Suggestion**: The order of the product is different from (9) on the page before. Change the order.
(49) Page 10, Line 12: **Question**: You have 3 interpretations. Why do you say 2-3?
(50) Page 10, Line 15: **Suggestion**: The names in the title of Subsection 2.2.1 should be in capital letters due to your usual conventions.
(51) Page 10, Line 18: **Suggestion**: Centres in British English - centers is American English. I think you use AE in the rest of your paper.
(52) Page 10, Line 41: Thus, ...
(53) Page 10, Line 54: under-strand **change it to** strand going under in the local picture for σ_i.
(54) Page 10, Line 55: over-strand **change it to** strand going over (or: "over" strand as you use later).
(55) Page 10, Line 56: under-strand **change it to** strand going under.

(56) Page 10, Line 57: over-strand **change it to** strand going over.
(57) Page 10, Line 60: ...Subsection 2.4.5...
(58) Page 11, Line 14: $i = 1, \dots, n-1$.
(59) Page 11, Line 18: **Suggestion**: The names in the title of Subsection 2.2.2 should be in capital letters due to your usual conventions.
(60) Page 11, Line 40: ...thus, our...
(61) Page 11, Line 43: **Suggestion**: The names in the title of Subsection 2.2.3 should be in capital letters due to your usual conventions.
(62) Page 11, Line 43: ...(non-abelian)...
(63) Page 11, Line 48: ...multiplication, i.e...
(64) Page 11, Line 53: ...$a_i \in F_n$, so that...
(65) Page 11, Line 55: **Question**: $\xi_{\beta i}$ should be $\xi_{\beta(i)}$, right?
(66) Page 12, Lines 4-9: **Suggestion**: McCool's theorem is certainly not well-known. Maybe it is worthwhile to recall it. Explain the whole paragraph different or more detailed since it is hard to follow.
(67) Page 12, Line 14: **Question**: I do not understand your notation here. Where do the vectors come from?
(68) Page 12, Line 24: ...well-defined...
(69) Page 12, Line 25 and Line 37: **Suggestion**: Explain earlier the geometric picture. This makes it easier to see why i_u is really an inclusion.
(70) Page 12, Line 40: Thus, ...
(71) Page 12, Lines 41-42: avoid the overlap of math code.
(72) Page 12, Line 57: over **change it to** "over" (or the "under" from the line above to under).
(73) Page 13, Line 4: Thus, ...
(74) Page 13, Line 7: **Suggestion**: Use the shorthand notation (OC).
(75) Page 13, Line 8: Thus, ...
(76) Page 13, Line 11: Abelian **change it to** abelian.
(77) Page 13, Line 16: **Suggestion**: Use the shorthand notation (UC).
(78) Page 13, Line 23: **Suggestion**: Use (OC)/(UC) istead of OC/UC.
(79) Page 13, Line 27: ...well-known...
(80) Page 13, Line 27-28: ...braids", i.e. it...
(81) Page 13, Line 29: **Suggestion**: Help the reader and give a precise reference to a specific result in [BA].
(82) Page 13, Line 31-32: **Question**: I do not get the purpose of this remark. Do I miss something?
(83) Page 13, Line 31: ...[GK] found...
(84) Page 13, Line 31: ...(and thus, of w-braids)...
(85) Page 13, Line 41: **Suggestion**: The names in the title of Subsection 2.3.1 should be in capital letters due to your usual conventions.
(86) Page 13, Lines 42-43: your convention seems to be to order the references by date of appearance. If so, then this is not the correct order.
(87) Page 13, Line 56: ...braid, i.e. braids that...
(88) Page 13, Lines 56-57: avoid the overlap of math code.
(89) Page 14, Lines 5-14: **Question**: I think your example is incorrect, right?
(90) Page 14, Lines 18-19: **Question**: What do you mean by "picture and in algebraic notation"?
(91) Page 14, Line 20: ...and hence, the...
(92) Page 14, Line 47: ...permutation", see Figure...
(93) Page 14, Lines 48-52: **Suggestion**: It is very important for you to allow exp-like sums. You should stress this more. Maybe in an extra remark.

(94) Page 14, Line 51: Thus, ...
(95) Page 14, Line 57: use (OC) instead of "overcrossing commute". At least write "Overcrossing Commute" as before.
(96) Page 15, Line 15: Thus, ...
(97) Page 15, Line 26: ...v-braids (or w-braids)...
(98) Page 15, Line 29: ...v-braids (or w-braids)...
(99) Page 15, Line 46: As we shall see below,...**change it to** In contrast, as we will see below,...
(100) Page 15, Line 46: $\mathcal{A}$ **change it to** $\mathcal{A}^*$.
(101) Page 16, Lines 4-5: avoid the overlap of math code.
(102) Page 16, Line 20: **Suggestion**: It would be helpful to give the pictures again for the difference of virtual and usual crossings.
(103) Page 16, Line 25: **Question**: Why this name? Why not just associated graded (or something similar)?
(104) Page 16, Line 25: **Suggestion**: Use a different notation for the categories.
(105) Page 16, Line 30: ...manner: if the... (this would be the AE convention)
(106) Page 16, Line 31: ...permutations, then we...
(107) Page 16, Line 44: **Suggestion**: It is crucial for the reader to understand that your exponential expression makes sense due to the fact that you already work in the completion. It would be worthwhile to recall it here.
(108) Page 16, Line 60, Page 17, Line 4: avoid the overlap of math code.
(109) Page 17, Line 17: (Following [BP,AT]). For... **change it to** Following [BP,AT]: for...
(110) Page 17, Line 18: Overcrossings Commute **change it to** (OC).
(111) Page 17, Line 41: Tails Commute **change it to** (TC).
(112) Page 17, Line 42: Overcrossings Commute **change it to** (OC).
(113) Page 17, Lines 50-51: avoid the overlap of math code.
(114) Page 18, Line 11: ...F_n (Section 2.2.3). **change it to** ...F_n, see Section 2.2.3.
(115) Page 18, Line 13: **Suggestion**: The names in the title of Subsection 2.5.1.1 should be in capital letters due to your usual conventions.
(116) Page 18, Line 19: **Suggestion**: The names in the title of Subsection 2.5.1.2 should be in capital letters due to your usual conventions.
(117) Page 18, Lines 22-23: avoid the overlap of math code.
(118) Page 18, Line 45:...diagram, then $d_k D$...
(119) Page 18, Lines 4-49: **Question**: Are these Hopf algebraic structures? Or "just" bi-algebraic ones? Or something else?
(120) Page 19, Line 4: **Suggestion**: The names in the title of Subsection 2.5.1.5 should be in capital letters due to your usual conventions.
(121) Page 19, Line 6: ...free, associative...
(122) Page 19, Line 6: ...on the generators...
(123) Page 19, Line 12: **Suggestion**: Do not use a footnote right after math mode. It looks like $\mathcal{A}^{w15}$.
(124) Page 19, Lines 18-38: **Suggestion**: This whole paragraph is very interesting, I would try to highlight it more.
(125) Page 19, Line 42: your convention seems to be to order the references by date of appearance. If so, then this is not the correct order.
(126) Page 19, Lines 49-50: ...well-known...
(127) Page 19, Line 50: Indeed, if...
(128) Page 19, Line 51: **Suggestion**: [MKS] is an old book with about 450 pages. Are more precise reference to a particular statement would really help the reader.

(129) Page 19, Line 52: ...$\xi \in F_n$. Therefore...
(130) Page 19, Lines 56-57: avoid the overlap of math code.
(131) Page 20, Line 4: ...obvious, that is,...
(132) Page 20, Line 16: ...$Z_2 := (I + P) \circ Z_1$ **change it to** $Z_2 = (I + P) \circ Z_1$.
(133) Page 20, Line 30: (paragraph 2.5.1.2) **change it to** (Paragraph 2.5.1.2).
(134) Page 20, Line 30: (paragraph 2.5.1.3) **change it to** (Paragraph 2.5.1.3).
(135) Page 20, Lines 30-31: avoid the overlap of math code.
(136) Page 20, Line 37: **Suggestion**: Not every reader (including the referee) can recall the notion of primitives in a Lie algebra by heart. Since you use it again later, this seems to be a good place to recall the definition.
(137) Page 20, Line 42: **Suggestion**: The names in the title of Subsection 2.5.4 should be in capital letters due to your usual conventions.
(138) Page 20, Line 44: **Suggestion**: Centres in again British English.
(139) Page 20, Line 56: **Suggestion**: Move the condition $w_j\sigma_i = \sigma_i w_j$ to Line 56.
(140) Page 21, Line 23: ...that, if two...
(141) Page 21, Line 24: ...flips, then the...
(142) Page 22, Lines 20-end: **Question**: Any reason for the spacing?
(143) Page 23, Summary: **Suggestion**: Add Subsection before the references 3.2 etc.
(144) Page 23, Line 7: **Question**: Which map? Be more specific.
(145) Page 23, Line 12: ...(Subsection 3.6)...
(146) Page 23, Line 13: **Suggestion**: I would write "we easily construct..." instead of "with no difficulties at all".
(147) Page 23, Line 15: ...Subsection 3.7...
(148) Page 23, Line 17: **Suggestion**: The names in the title of this Section should be in capital letters due to your usual conventions.
(149) Page 23, Line 21: **Suggestion**: What about cookies? Then you have u,v,w and cakes, pastries, cookies.
(150) Page 23, Line 24: ...(hence, they...
(151) Page 23, Line 27: ...and thus, most...
(152) Page 23, Line 37: ...limited, but tasty, arena...
(153) Page 23, Line 42: **Suggestion**: Kuperberg is the right reference - sure. But I do not see a reason to cite Kauffman here.
(154) Page 23, Line 43: ...(see Remark 3.4)...
(155) Page 24, Lines 21-22: **Question**: Do you mean positive and negative crossings here?
(156) Page 24, Line 20: under-crossings **change it to** undercrossings.
(157) Page 24, Lines 23-24: **Question**: And some mirror-relations, right?
(158) Page 24, Line 26: Overcrossings Commute (OC) **change it to** (OC).
(159) Page 24, Line 28: Undercrossings Commute relation (UC) **change it to** relation (UC).
(160) Page 24, Lines 31-38: **Question**: Why? What is the reason? What is a reference?
(161) Page 24, Line 37: Abelian **change it to** abelian.
(162) Page 25, Line 15+several other lines on this page: **Suggestion**: Here your conventions collide: either write (R1) and so on, or change (OC) to OC etc. I would stick with the bracket notation due to easier readability.
(163) Page 25, Line 26: ...relations, but not...
(164) Page 25, Line 47: w-Knots **change it to** w-knots.
(165) Page 25, Line 48: ...v-knot; yet both...
(166) Page 26, Line 11: ...(see e.g. [Kup])...
(167) Page 26, Line 39: ...and hence, the...

(168) Page 26, Line 53: ...Section 2.2.1, a framing...
(169) Page 26, Line 56: **Suggestion**: Careful - there are some figures in between. Maybe it is better to give the figure a label and refer to it.
(170) Page 27, Figure 8: **Suggestion**: Explain notation more carefully, e.g. what do you mean by teh dotted arrows and the broken skeleton.
(171) Page 27, Figure 8: again missing brackets (relations). Similarly in other lines on this page.
(172) Page 27, Line 31: ...and hence, the...
(173) Page 27, Line 36: ...(see Section 2.3)...
(174) Page 27, Line 38: Abelian-group-valued invariant **change it to** invariant taking values in abelian groups.
(175) Page 27, Lines 47, 53-54: **Suggestion**: The name "long line" is misleading. Use maybe "long line skeleton".
(176) Page 27, Line 49: ...is shown in Figure...
(177) Page 27, Lines 54-55: ...is given in Figure...
(178) Page 27, Line 58: **Suggestion**: The footnote now really looks like an 8^{21}.
(179) Page 28, more than one Line: Convention about bracket for relations again.
(180) Page 28, Line 33: **Suggestion**: I would explain that $\mathcal{A}^{\text{more than one superscript}}(\uparrow)$ stands for any of the spaces under consideration. At least explain the notation $\mathcal{A}^{-}(\uparrow)$ in line 33.
(181) Page 28, Line 33: ...(see especially [ref])... (the bracket is missing).
(182) Page 28, Line 41: In summary, **change it to** In summary:.
(183) Page 28, Line 46: **Suggestion**: As suggested before, explain what primitive elements are.
(184) Page 28, Line 53: The bracketing in this sentence is confusing. Change it please.
(185) Page 29, whole page + parts of the next page: **Suggestion**: I do not see a point to put this table right in the middle of Section 3. In my humble opinion it disturbs the flow of reading. Nevertheless, I think it is interesting (in particular, the dimension difference between v and w), but maybe it is better to put it to the end of the paper or at least section and just refer here to it.
(186) Page 30, Line 18: ...knots (see e.g. [BN1])...
(187) Page 30, Lines 18+19: remove the brackets around the pictures.
(188) Page 30, Line 20: ...w-braids (as in Definition 2.11)...
(189) Page 30, Line 21: **Suggestion**: "ordinary knots" should be "u-knots" in your convention.
(190) Page 30, Lines 22-25: here you run into convention problems. You usually use "..." to indicate important definitions or concepts. But not here.
(191) Page 30, Lines 43+44: Tail Commute **change it to** (TC).
(192) Page 30, Lines 47-50: **Suggestion**: Since you refer to the (TC) relation, it would be helpful for the reader to give arrow diagrams here as well.
(193) Page 30, Line 52: ...case (see Theorem 2.15)...
(194) Page 30, Line 53: braids case **change it to** braid case.
(195) Page 31, Line 31: **Suggestion**: You are changing the combinatorial data in this subsection again. It would be helpful for the reader (who wants to understand why) to state immediately that, in the new combinatorial set-up, it is easier to see what the primitives are.
(196) Page 31, Definition 3.13: **Suggestion**: You say it, but I would stress that all trivalent vertices have two incoming and one outgoing edge. I would also say immediately that this corresponds to Lie algebras side (that is, Subsection 3.6.2) where you view these as maps $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$.
(197) Page 31, Line 52: **Suggestion**: Maybe not again "Note that" as in the sentence before.
(198) Page 31, Line 53-56: **Question**: How can I easily see that the number of trivalent vertices is even? I know the definition is as in the usual case, but still...

(199) Page 32, Some lines: **Suggestion**: Maybe it is a good idea to use (relations) for arrow diagram relations and $\overrightarrow{\text{relations}}$ for Jacobi diagram relations.
(200) Page 32, Some lines: **Suggestion**: Again convention problems with respect to "notation".
(201) Page 32, Line 26: **Question**: I think in the whole paper by formal linear combinations you mean $\mathbb{Q}$-linear, right? It is not so clear to me here.
(202) Page 32, Line 31: Thus,...
(203) Page 32, Line 39: ...theorem, i.e. Theorem 4.9...
(204) Page 32, Line 41: (bracket-rise) **change it to** (Bracket-rise).
(205) Page 32, Line 42: ...(see Definition 3.8)...
(206) Page 32, Lines 43-44: **Question**: You have more structure: an isomorphism of what?
(207) Page 33, Line 4: In fact,...
(208) Page 33, Line 4: **Suggestion**: If its so obvious (which I think its not), then why do you argue here at all?
(209) Page 33, Line 10: skeleton **change it to** long line skeleton.
(210) Page 33, Line 13: well defined **change it to** well-defined.
(211) Page 33, Line 13: ...and thus, descend...
(212) Page 33, Line 17: well defined **change it to** well-defined.
(213) Page 33, Line 20: ...and hence, the...
(214) Page 33, Line 20: well defined **change it to** well-defined.
(215) Page 33, Lines 33-34: well defined **change it to** well-defined.
(216) Page 33, Line 56: **Suggestion**: Say a little bit more why everything is oriented upwards.
(217) Page 33, Line 60: **Suggestion**: I like explicit examples and constructions, but this is confusing. It took me a while to realize that you do just an ordinary induction.
(218) Page 34, Lines 19-20: **Suggestion**: You have not said anything about the connections to Lie algebras up to this stage. So it does not make sense for the "naive reader" (for example, for the referee) at this point. I would mention the connection to Lie algebras earlier anyway (or refer to the corresponding section).
(219) Page 34, Lines 26-27: **Suggestion**: At this point the reader may wonder what the whole point of the translation is. You should stress the connection to the primitives here.
(220) Page 34, Line 35: Thus,...
(221) Page 34, Line 43: ...and hence, in...
(222) Page 34, Line 44: **Suggestion**: I think you should explain more carefully what trees and wheels are (since they are crucial for you).
(223) Page 35, Some lines: **Suggestion**: Relation notation again.
(224) Page 35, Line 12: **Suggestion**: Do you really what to call it a theorem?
(225) Page 35, Line 12: ...and hence, the...
(226) Page 35, Line 16: ...and hence, is...
(227) Page 35, Lines 19-20: **Suggestion**: A picture never hurts!
(228) Page 35, Line 30: ...and hence, the...
(229) Page 35, Lines 38: **Suggestion**: Explain what a Y is.
(230) Page 35, Line 49: classical **change it to** usual.
(231) Page 35, Line 49: classical **change it to** usual (yet again).
(232) Page 35, Line 55: **Suggestion**: As I said: I would stress this relation before.
(233) Page 35, Lines 59-60: **Suggestion**: Explain what you mean by "doubles of co-commutative Lie bialgebras".
(234) Page 36, Line 4: **Question**: Do you take complex Lie algebras?
(235) Page 36, Line 6: Abelian **change it to** abelian.

(236) Page 36, Line 11: **Question**: It seems like you are already using the notation in the universal enveloping algebras here, right?
(237) Page 36, Lines 12-13: the wording of "is itself $\mathbb{R}^3$" is strange.
(238) Page 36, Line 17: ...that, if $\mathfrak{g}$...
(239) Page 36, Line 17: ...bialgebra, then $I\mathfrak{g}$...
(240) Page 36, Line 22: **Suggestion**: You use construction twice in this sentence.
(241) Page 36, Line 31: **Question**: What is S^k_L?
(242) Page 36, Line 36: ...$\mathfrak{g}$, we construct...
(243) Page 36, Line 54: skeleton **change it to** long line skeleton.
(244) Page 37, Some lines: **Suggestion**: Relation notation again.
(245) Page 37, Lines 4-8: **Question**: Why do you use Greek letters all in a sudden?
(246) Page 37, Line 29: Abelian **change it to** abelian.
(247) Page 38, Line 22: **Suggestion**: "Recalled below" seems to be better.
(248) Page 38, Lines 26-27: avoid the overlap of math code.
(249) Page 38, Line 32: **Suggestion**: Recall the notion of a sign of a crossing.
(250) Page 38, Line 50: ...(see e.g. [Ro1]).
(251) Page 38, Line 57: ...(see Equation (22))...
(252) Page 38, Line 59: ...(see Equation (22)).
(253) Page 39, Line 6: **Suggestion**: w seems to be a bad name (you already have to many w's). Although, it is some kind of "wheeling" map.
(254) Page 39, Line 7: **Question**: "Long w-knot", right?
(255) Page 39, Line 18: ...with the vector...
(256) Page 39, Line 22: Hence, ...
(257) Page 39, Line 30: Thus, ...
(258) Page 39, Lines 36-end: **Question**: Any reason for the spacing?
(259) Page 40, Line 30: ...goal, that is, Equation...
(260) Page 40, Line 7: **Suggestion**: I think this is sometimes called Cauchy-Euler operator. Please check.
(261) Page 40, Lines 10-11: **Suggestion**: Slightly misleading formulation, since the logarithmic derivative is $\frac{f'}{f}$ and not, as you seem to say, $\frac{xf'}{f}$.
(262) Page 40, Line 11: ...(see Section 3.8.1)...
(263) Page 40, Line 12: ...(see Section 3.8.1)...
(264) Page 40, Line 25: ...and hence, there...
(265) Page 40, Line 31: Thus, ...
(266) Page 40, Lines 16-35: **Suggestion**: The whole part about the "bulk management" is hard to understand. Maybe a re-formulation would help.
(267) Page 40, Line 40: ...completed, graded algebra...
(268) Page 40, Lines 32-43: **Question**: I do not understand your notation here. What is the connection from $Ea = (\mathrm{deg}a)a$ and $Ef = xf'$?
(269) Page 40, Line 44: ...and hence, we...
(270) Page 40, Line 46: $Z^{-1} \cdot EZ$ **change it to** $Z^{-1}EZ$.
(271) Page 40, Line 52: **Suggestion**: Say a word why $\tilde{E}f$ is as you say (since it is important for the proof).
(272) Page 40, Line 56: ...and hence, ...
(273) Page 41, Line 5: Thus, ...
(274) Page 41, Line 5: ...goal, that is, Equation...
(275) Page 41, Line 7: ...and hence, ...

(276) Page 41, Line 16: **Suggestion**: The notation $w\colon A \to \mathcal{A}$ is a bad notation (to many A and $\mathcal{A}$'s in your paper).
(277) Page 41, Line 17: Then, by...
(278) Page 41, Line 18: ...(see Theorem 3.16)...
(279) Page 41, Line 23: ...determinant, see (21),...
(280) Page 41, Lines 31-32: Infinitesimal Alexander Module **change it to** "Infinitesimal Alexander Module" (following your conventions).
(281) Page 41, Line 39: well defined **change it to** well-defined.
(282) Page 41, Line 31-41: **Suggestion**: Hard to understand. Especially the stuff about sectors. You explain it way better on the next page.
(283) Page 41, Line 47: Hence, ...
(284) Page 41, Line 51: ...(see Remark 3.4)...
(285) Page 41, Line 55: **Question**: My standard confusion. Here you use $\mathbb{Z}$, but before you always seem to use $\mathbb{Q}$. Does it matter? If not, then why?
(286) Page 42, Figure 15: **Question**: What is the floating wheel indicating?
(287) Page 42, Some lines: **Question**: Again my standard confusion about $\mathbb{Z}$ or $\mathbb{Q}$. Does it matter?
(288) Page 42, Line 21: skeleton **change it to** long line skeleton.
(289) Page 42, Line 23: skeleton **change it to** long line skeleton.
(290) Page 42, Line 24: skeleton **change it to** long line skeleton.
(291) Page 42, Line 36: skeleton **change it to** long line skeleton.
(292) Page 42, Line 41: **Question**: How can it be $m(m-1)(\frac{1}{2}m+1)$? Your m is odd, right?
(293) Page 42, Line 48: $e^{\pm a}$ **change it to** $(e^{\pm a})$.
(294) Page 42, Line 52: **Suggestion**: Colours in again British English.
(295) Page 42, Line 57: ...occur. So we...
(296) Page 43, Line 40: **Question**: Why brackets (A)?
(297) Page 43, Line 41: **Suggestion**: Say explicitly what relation you mean (you have names in Figure 16).
(298) Page 43, Line 42: **Suggestion**: Again, say explicitly what relation you mean (you have names in Figure 16).
(299) Page 43, Line 52: Tail Commute **change it to** (TC) or to $\overrightarrow{STU}_3$.
(300) Page 43, Line 53: ...and hence,...
(301) Page 43, Line 55: ...relation, see Figure,...
(302) Page 43, Line 57: **Suggestion**: First appearance of ad, so explain it.
(303) Page 44, Line 18: RI **change it to** (RI).
(304) Page 44, Line 31: ...and hence,...
(305) Page 44, Lines 28 and 58: **Suggestion**: The two Propositions seem to more Lemmata for me (you use them to prove Theorem 3.27).
(306) Page 45, Figure 18: **Suggestion**: Indicate the red arrows, as before, with an "r" (for the reader without colors).
(307) Page 45, Line 30: Thus,...
(308) Page 45, Line 54: Thus,...
(309) Page 46, Lines 7-11: **Suggestion**: I would say this at a more prominent place since the signs are really a headache.
(310) Page 46, Line 54: Lescop's **change it to** Lescop's work, see.
(311) Page 46, Lines 32-38: **Suggestion**: Add something similar to "Thus, as expected, Z^w is way weaker (and easier to handle) than the Kontsevich integral Z^u".
(312) Page 47, Lines 6-end: **Question**: Any reason for the spacing?
(313) Page 48, Line 8: footnote 3 **change it to** Footnote 3.

(314) Page 48, Line 22: ...case, then many...
(315) Page 48, Line 24: Thus, ...
(316) Page 48, Line 32: Thus, ...
(317) Page 48, Line 47: **Suggestion**: "the "u" case has been known since long" cries for a reference.
(318) Page 48, Line 52: ...and thus, ...
(319) Page 49, Some lines: **Suggestion**: Relation notation again.
(320) Page 49, Line 17: ...and thus, ...
(321) Page 49, Line 17: ...and hence, ...
(322) Page 49, Lines 21-22: **Suggestion**: Put the last result in an extra line.
(323) Page 49, Lines 41-42: **Suggestion**: Put the "D-set" in an extra line.

4. General remarks

Some general comments.

- The abstract is long and complicated in my humble opinion. Try to shorten it.
- Throughout the paper: I personally think textit is better to illustrate important concepts and definitions than "something".
- Throughout the paper: sometimes you refer to relations with e.g. (OC) and sometimes with OC. Fix one convention and stay with it.
- Throughout the paper: sometimes you cite people as for example P.Lee (page 38) and sometimes just as for example Watanabe (page 35). Stay with one convention.
- Some of the figures use color (e.g. Figure 18 on page 45). Since colors may not appear in print, one should e.g. insert (darker shaded) or mark the corresponding arrows at least.
- Your list of references has some typos, for example [AT] and [AET] should be ordered vice versa, [AT] is Ann. of Math. 175-2, [BB] has already appeared and [BWC] is Math. Phys. 11-5.
- Use standard abbreviations for the journal references. For example, [AT] is Ann. of Math. instead of Annals of Mathematics. Other that I spotted are: [AET], [Ba], [BN2], [BGRT], [BHLR], [BLT], [BP], [BT], [Dye], [EK], [HS], [Joy], [KS], [KL], [Kur], [LM2], [MM], [Oh], [Sa] and [Wa].