5. w-Tangles

Section Summary. In 5.1 we introduce v-tangles and w-tangles, the obvious v- and w- counterparts of the standard u knot-theoretic notion of "tangles", and briefly discuss their finite type invariants and their associated spaces of "arrow diagrams", $\mathcal{A}^v(\uparrow_n)$ and $\mathcal{A}^w(\uparrow_n)$. We then construct a homomorphic expansion, or a "well-behaved" universal finite type invariant for w-tangles. Once again, the only algebraic tool we need to use is $\exp(a) := \sum a^n/n!$, and indeed, Section 5.1 is but a routine extension of parts of Section 3. We break away in 5.2 and show that almost canonically, $\mathcal{A}^w(\uparrow_n) \cong \mathcal{U}(\mathfrak{a}_n \oplus \mathfrak{tder}_n \ltimes \mathfrak{tr}_n)$, where $\mathfrak{a}_n$ is an Abelian algebra of rank n and where $\mathfrak{tder}_n$ and $\mathfrak{tr}_n$, two of the primary spaces used by Alekseev and Torossian [AT], have simple descriptions in terms of words and free Lie algebras. In 5.3 we discuss a subclass of w-tangles called "special" cyclic w-tangles, and relate them by similar means to Alekseev and Torossian's $\mathfrak{sder}_n$ and to "tree level" ordinary Vassiliev theory.

5.1. **v-Tangles and w-Tangles.** With the (surprisingly pleasant) task of defining circuit algebras completed in Section 4.4, the definition of v-tangles and w-tangles is simple.

Definition 5.1. The ($\mathcal{S}$-graded) circuit algebra $v\!T$ of v-tangles is the $\mathcal{S}$-graded directed circuit algebra generated by two generators in $C_{2,2}$ called the "positive crossing" and the "negative crossing", modulo the usual R2 and R3 moves as depicted in Figure 6 (these relations clearly make sense as circuit algebra relations between our two generators), with the obvious meaning for their skeleta. The circuit algebra $w\!T$ of w-tangles is the same, except we also mod out by the OC relation of Figure 6 (note that each side in that relation involves only two generators, with the apparent third crossing being merely a projection artifact).

Remark 5.2. One may also define v-tangles and w-tangles using the language of planar algebras, except then another generator is required (the "virtual crossing") and also a few further relations (VR1–VR3, M), and some of the operations (non-planar wirings) become less elegant to define.

Our next task is to study the projectivizations proj $v\!T$ and proj $w\!T$ of $v\!T$ and $w\!T$. Again, the language of circuit algebras makes it exceedingly simple.

Definition 5.3. The ($\mathcal{S}$-graded) circuit algebra $\mathcal{D}^v = \mathcal{D}^w$ of arrow diagrams is the graded and $\mathcal{S}$-graded directed circuit algebra generated by a single degree 1 generator a in $C_{2,2}$ called "the arrow" as shown on the right, with the obvious meaning for its skeleton. There are morphisms $\pi : \mathcal{D}^v \to v\!T$ and $\pi : \mathcal{D}^w \to w\!T$ defined by mapping the arrow to an overcrossing minus a no-crossing. (On the right some virtual crossings were added to make the skeleta match). Let $\mathcal{A}^v$ be $\mathcal{D}^v/6T$ and let $\mathcal{A}^w := \mathcal{A}^v/TC = \mathcal{D}^w/\overrightarrow{4T}, TC$, with $6T$, $\overrightarrow{4T}$, and TC being the same relation as in Figures 8 and 9 (allowing skeleta parts that are not explcitly connected to really lie on separate skeleton components).

Proposition 5.4. *The maps π above induce surjections $\pi : \mathcal{A}^v \to$ proj $v\!T$ and $\pi : \mathcal{A}^w \to$ proj $w\!T$. Hence in the language of Definition 4.7, $\mathcal{A}^v$ and $\mathcal{A}^w$ are candidate projectivizations of $v\!T$ and $w\!T$.*

Proof. MORE

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We do not know if $\mathcal{A}^v$ is indeed the projectivizations of $v\mathcal{T}$ (also see [BHLR]). Yet in the w case, the picture is simple:

Theorem 5.5. *The assignment $\times \mapsto e^a$ (with an obvious interpretation for e^a) extends to a well defined $Z : w\mathcal{T} \to \mathcal{A}^w$. The resulting map Z is a homomorphic $\mathcal{A}^w$-expansion, and in particular, $\mathcal{A}^w \cong \operatorname{proj} w\mathcal{T}$ and Z is a homomorphic expansion.*

Proof. There is nothing new here. Z is satisfies the Reidemeister moves for the same reasons as in Theorem 2.12 and Theorem 3.11 and as there it also satisfies the universallity property. The rest follows from Proposition 4.8. □

In a similar spirit to Definition 3.13, one may define a "w-Jacobi diagram" (often shorts to "arrow diagram") on an arbitrary skeleton. Denote the circuit algebra of formal linear combinations of arrow diagrams by $\mathcal{A}^{wt}$. We have the following bracket-rise theorem:

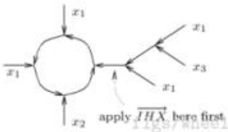
Theorem 5.6. *The obvious inclusion of diagrams induces a circuit algebra isomorphism $\mathcal{A}^w \cong \mathcal{A}^{wt}$. Furthermore, the $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations of Figure 12 hold in $\mathcal{A}^{wt}$.*

Proof. The proof of Theorem 3.15 can be repeated verbatim. Note that that proof does not make use of the connectivity of the skeleton.

Given the above theorem, we no longer keep the distinction between $\mathcal{A}^w$ and $\mathcal{A}^{wt}$.

5.2. $\mathcal{A}^w(\uparrow_n)$ and the Alekseev-Torossian Spaces.

Definition 5.7. Let $v\mathcal{T}(\uparrow_n)$ (likewise $w\mathcal{T}(\uparrow_n)$) be the set of v-tangles (w-tangles) whose skeleton is the disjoint union of n directed lines. Likewise let $\mathcal{A}^v(\uparrow_n)$ and $\mathcal{A}^w(\uparrow_n)$ be the parts of $\mathcal{A}^v$ and $\mathcal{A}^w$ in which the skeleton is the disjoint union of n directed lines.



In the same manner as in the case of knots (Theorem 3.16), $\mathcal{A}^w(\uparrow_n)$ is a bi-algebra isomorphic (via a diagrammatic PBW theorem, applied independently on each component of the skeleton) to a ~~space~~ coalgebra $\mathcal{B}^w(\star_n)$ of unitravelent diagrams with symmetrized ends coloured with colours in some n-element set (say $\{x_1, \dots, x_n\}$), modulo $\overrightarrow{AS}$ and $\overrightarrow{IHX}$. The primitives $\mathcal{P}_n$ of $\mathcal{B}^w(\star_n)$ are the connected diagrams (and hence the primitives of $\mathcal{A}^w(\uparrow_n)$ are the diagrams that remain connected even when the skeleton is removed). Given the "two in one out" rule for internal vertices, the diagrams in $\mathcal{P}_n$ can only be trees or wheels ("wheels of trees" can be reduced to simple wheels by repeatedly using $\overrightarrow{IHX}$, as on the right).

global: $P_n \to P\mathcal{D}^w(\uparrow_n)$

Thus $\mathcal{P}_n$ is easy to identify. It is a direct sum $\mathcal{P}_n = \langle\text{trees}\rangle \oplus \langle\text{wheels}\rangle$. The wheels part is simply the vector space generated by all cyclic words in the letters $x_1, \dots, x_n$. Alekseev and Torossian [AT] denote this space $\mathfrak{tr}_n$, and so shall we. The trees in $\mathcal{P}_n$ have leafs coloured $x_1, \dots, x_n$. Modulo $\overrightarrow{AS}$ and $\overrightarrow{IHX}$, they correspond to elements of the free Lie algebra $\mathfrak{Lie}_n$ on the generators $x_1, \dots, x_n$. But the root of each such tree also carries a label in $\{x_1, \dots, x_n\}$, hence there are n types of such trees as ~~separated~~ classified by their roots, and so $\mathcal{P}_n$ is isomorphic to the direct sum $\mathfrak{tr}_n \oplus \bigoplus_{i=1}^n \mathfrak{Lie}_n$ of $\mathfrak{tr}_n$ and n copies of $\mathfrak{Lie}_n$.

By the Milnor-Moore theorem [MM], $\mathcal{A}^w(\uparrow_n)$ is isomrphic to the universal enveloping algebra $\mathcal{U}(\mathcal{P}_n)$, with $\mathcal{P}_n$ identified as a subspace of $\mathcal{A}^w(\uparrow_n)$ using the PBW symmetrization map $\chi : \mathcal{B}(\star_n) \to \mathcal{A}^w(\uparrow_n)$. Thus in order to understand $\mathcal{A}^w(\uparrow_n)$ as an associative algebra, it is enough to understand the Lie algebra structure induced on $\mathcal{P}_n$ via the commutator bracket of $\mathcal{A}^w(\uparrow_n)$.

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First try...

$$\begin{tikzcd}
\mathcal{A}^w(\uparrow_n) \arrow[r] & \mathcal{A}^{w/\text{wheels}}(\uparrow_n) \arrow[d, dash] \\
\mathcal{D}^w \arrow[u, shift left=4, "\chi_S"] \arrow[u, "\chi_+"] \arrow[u, shift right=4, "\chi_-"'] \arrow[r, shift left, "p"] & \mathcal{D}^{w/\text{wheels}} = \mathcal{U}\left(\bigoplus \mathcal{L}ie_n\right) = \mathcal{U}(\mathfrak{a}_n \oplus \mathfrak{tder}_n) \arrow[l, shift left, "i"]
\end{tikzcd}$$

Second attempt...

$$\begin{tikzcd}
 & \mathcal{A}^w(\uparrow_n) \arrow[r] & \mathcal{A}^{w/\text{wheels}}(\uparrow_n) \arrow[d, dash, "\wr"] \\
\mathcal{U}(\mathfrak{tr}_n) = \mathcal{D}^{\text{wheels}} \arrow[r] & \mathcal{D}^w \arrow[u, shift left=4, "\chi_S"] \arrow[u, "\chi_+"] \arrow[u, shift right=4, "\chi_-"'] \arrow[r, shift left, "p"] & \mathcal{D}^{w/\text{wheels}} = \mathcal{U}\left(\bigoplus \mathcal{L}ie_n\right) = \mathcal{U}(\mathfrak{a}_n \oplus \mathfrak{tder}_n) \arrow[l, shift left, "i"]
\end{tikzcd}$$

$P \Big|$ $\quad\quad \Big\uparrow \mathcal{U}$

$$P\downarrow \qquad \uparrow U$$

$$\begin{tikzcd}
 & PA^w(\uparrow_n) \arrow[r] \arrow[dl, "div=i^{-1}\circ(x_+ - x_-)"'] & PA^{w/wheels}(\uparrow_n) \arrow[d, "s"] \\
tr_n = P\mathcal{D}^{wheels} \arrow[r] & P\mathcal{D}^w \arrow[u, "x_+"] \arrow[u, shift left, "x_s"'] \arrow[u, shift left=3, "x_-"'] \arrow[r, shift left] & P\mathcal{D}^{w/wheels} \arrow[l, shift left] = \oplus Lie_n = a_n \oplus tder_n
\end{tikzcd}$$

Fit div & j in $\mathcal{D}$

Third try.

$$\begin{tikzcd}
 & A^w(\uparrow_n) \arrow[r] & A^{w/wheels}(\uparrow_n) \arrow[d, "s"] \\
U(tr_n) = \mathcal{D}^{wheels} \arrow[r] & \mathcal{D}^w \arrow[u, "x_s"] \arrow[u, shift left, "x_+"'] \arrow[u, shift left=3, "x_-"'] \arrow[r, shift left, "p"] & \mathcal{D}^{w/wheels} \arrow[l, shift left, "i"] = U(\oplus Lie_n) = U(a_n \oplus tder_n)
\end{tikzcd}$$

$$P\downarrow \qquad \uparrow U$$

$$\begin{tikzcd}
 & PA^w(\uparrow_n) \arrow[r] \arrow[dl, "div=i^{-1}\circ(x_+ - x_-)"'] & PA^{w/wheels}(\uparrow_n) \arrow[d, "s"] & \circlearrowright PA^{w/w}(\uparrow_n \mid \uparrow) \\
tr_n = P\mathcal{D}^{wheels} \arrow[r] & P\mathcal{D}^w \arrow[u, "x_+"] \arrow[u, shift left, "x_s"'] \arrow[u, shift left=3, "x_-"'] \arrow[r, shift left] & P\mathcal{D}^{w/wheels} \arrow[l, shift left] = \oplus Lie_n = a_n \oplus tder_n
\end{tikzcd}$$