Dror Bar-Natan: Papers: WKO:

The Kishino Braid

Pensieve Header: We verify that the "Kishino Braid" is non-trivial, yet its Alexander invariant is trivial.

Dear Jana, Karene, Lucy, Peter and Zsuzsi,

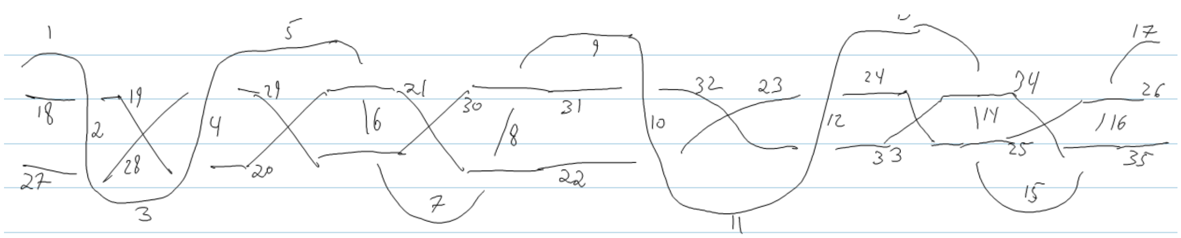
In the last Knot at Lunch meeting we mentioned the fact that probably some virtual braids cannot be seen by their action on Aut(F_n). Well, I've just verified that a braid that Karene and I were talking about, the "Kishino Braid" is non-trivial yet its projection into both w-braids and mirror-w-braids is trivial. So both actions of that braid on the free group are trivial.

See a mathematica notebook (also in pdf) at http://katlas.math.toronto.edu/drorbn/AcademicPensieve/2008-12/ (under "mathematica notebooks", not under "notebook pages"!).

The fact that the Kishino braid is both w-trivial and mirror-w-trivial is visibly obvious. So the only problem is to check that it is a non-trivial v-braid, and this I do by computing its Jones polynomial. As sanity checks I use Jana's thesis to compute its Alexander and the Alexander of the mirror (both are trivial, as expected). Another sanity check is to carry out the same computations for a trivial braid, and they all work out as they should.

Best,

Dror.



```
<< KnotTheory`
```

```
Loading KnotTheory` version of August 26, 2008, 13:57:21.1548.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
B = PD[
   X[18, 2, 19, 1], X[27, 3, 28, 2], X[19, 3, 20, 4], X[28, 4, 29, 5],
   X[5, 20, 6, 21], X[6, 29, 7, 30], X[7, 22, 8, 21], X[8, 31, 9, 30],
   X[31, 10, 32, 9], X[22, 11, 23, 10], X[32, 11, 33, 12], X[23, 12, 24, 13],
   X[13, 33, 14, 34], X[14, 24, 15, 25], X[15, 35, 16, 34], X[16, 26, 17, 25]
  ];
```

The Jones is non-trivial!

```
Jones[B][q] // Short
```

$$\frac{2\,P[1,27]\,P[17,35]\,P[18,26]}{1+q}+\frac{P[1,27]\,P[\ll 1\gg]\,P[18,26]}{q^4\,(1+q)}+\ll 180\gg+\frac{\ll 1\gg}{1+q}-\frac{\sqrt{q}\,P[1,17]\,P[18,26]\,P[27,35]}{1+q}$$

- **Yet the Alexander, also of the Mirror, is trivial:**

```
Import["http://www.math.toronto.edu/~drorbn/Talks/Sandbjerg-0810/pA.m"]
```

```
CircuitDiagram[B_PD] := CircuitDiagram @@ B /. {
   X[i_, j_, k_, l_] /; j > l :> Xp[i, j, k, l],
   X[i_, j_, k_, l_] /; j < l :> Xm[i, j, k, l]
  }
```

```
B1 = CircuitDiagram[B]
```

```
CircuitDiagram[Xp[18, 2, 19, 1], Xp[27, 3, 28, 2], Xm[19, 3, 20, 4],
 Xm[28, 4, 29, 5], Xm[5, 20, 6, 21], Xm[6, 29, 7, 30], Xp[7, 22, 8, 21], Xp[8, 31, 9, 30],
 Xp[31, 10, 32, 9], Xp[22, 11, 23, 10], Xm[32, 11, 33, 12], Xm[23, 12, 24, 13],
 Xm[13, 33, 14, 34], Xm[14, 24, 15, 25], Xp[15, 35, 16, 34], Xp[16, 26, 17, 25]]
```

```
pA[B1]
```

```
AHD[(t[18] == t[19] == t[20] == t[21] == t[22] == t[23] == t[24] == t[25] == t[26])
   (t[27] == t[28] == t[29] == t[30] == t[31] == t[32] == t[33] == t[34] == t[35])
   (t[1] == t[2] == t[3] == t[4] == t[5] == t[6] == t[7] == t[8] == t[9] == t[10] ==
     t[11] == t[12] == t[13] == t[14] == t[15] == t[16] == t[17]), {1, 18, 27},
 W[17, 26, 35], -t[1]^4 t[18]^2 t[27]^2 W[1, 18, 27] + t[1]^4 t[18]^2 t[27]^2 W[1, 18, 35] +
  t[1]^4 t[18]^2 t[27]^2 W[1, 26, 27] - t[1]^4 t[18]^2 t[27]^2 W[1, 26, 35] +
  t[1]^4 t[18]^2 t[27]^2 W[17, 18, 27] - t[1]^4 t[18]^2 t[27]^2 W[17, 18, 35] -
  t[1]^4 t[18]^2 t[27]^2 W[17, 26, 27] + t[1]^4 t[18]^2 t[27]^2 W[17, 26, 35]]
```

```
B2 = CircuitDiagram[Mirror[B]]
```

```
CircuitDiagram[Xm[1, 18, 2, 19], Xm[2, 27, 3, 28], Xp[3, 20, 4, 19],
 Xp[4, 29, 5, 28], Xp[20, 6, 21, 5], Xp[29, 7, 30, 6], Xm[21, 7, 22, 8], Xm[30, 8, 31, 9],
 Xm[9, 31, 10, 32], Xm[10, 22, 11, 23], Xp[11, 33, 12, 32], Xp[12, 24, 13, 23],
 Xp[33, 14, 34, 13], Xp[24, 15, 25, 14], Xm[34, 15, 35, 16], Xm[25, 16, 26, 17]]
```

```
pA[B2]
```

```
AHD[(t[18] == t[19] == t[20] == t[21] == t[22] == t[23] == t[24] == t[25] == t[26])
   (t[27] == t[28] == t[29] == t[30] == t[31] == t[32] == t[33] == t[34] == t[35])
   (t[1] == t[2] == t[3] == t[4] == t[5] == t[6] == t[7] == t[8] == t[9] == t[10] ==
     t[11] == t[12] == t[13] == t[14] == t[15] == t[16] == t[17]), {1, 18, 27},
 W[17, 26, 35], -t[1]^4 t[18]^2 t[27]^2 W[1, 18, 27] + t[1]^4 t[18]^2 t[27]^2 W[1, 18, 35] +
  t[1]^4 t[18]^2 t[27]^2 W[1, 26, 27] - t[1]^4 t[18]^2 t[27]^2 W[1, 26, 35] +
  t[1]^4 t[18]^2 t[27]^2 W[17, 18, 27] - t[1]^4 t[18]^2 t[27]^2 W[17, 18, 35] -
  t[1]^4 t[18]^2 t[27]^2 W[17, 26, 27] + t[1]^4 t[18]^2 t[27]^2 W[17, 26, 35]]
```

```
pA[B1] == pA[B2]
```

True

```
Last[pA[B1]] // Factor
```

$$-t[1]^4\, t[18]^2\, t[27]^2\, (W[1, 18, 27] - W[1, 18, 35] - W[1, 26, 27] + W[1, 26, 35] - W[17, 18, 27] + W[17, 18, 35] + W[17, 26, 27] - W[17, 26, 35])$$

Half of B

```
Jones[Take[B, 8]][q]
```

$$-\frac{P[1,27]\,P[9,31]\,P[18,22]}{1+q}-\frac{P[1,27]\,P[9,31]\,P[18,22]}{q\,(1+q)}-\frac{2\,P[1,27]\,P[9,31]\,P[18,22]}{\sqrt{q}\,(1+q)}-$$
$$\frac{\sqrt{q}\,P[1,27]\,P[9,31]\,P[18,22]}{1+q}-\frac{P[1,31]\,P[9,22]\,P[18,27]}{1+q}-\frac{P[1,31]\,P[9,22]\,P[18,27]}{q\,(1+q)}-$$
$$\frac{2\,P[1,31]\,P[9,22]\,P[18,27]}{\sqrt{q}\,(1+q)}-\frac{3\,P[1,22]\,P[9,31]\,P[18,27]}{1+q}-\frac{P[1,22]\,P[9,31]\,P[18,27]}{\sqrt{q}\,(1+q)}-$$
$$\frac{3\sqrt{q}\,P[1,22]\,P[9,31]\,P[18,27]}{1+q}-\frac{2\,q\,P[1,22]\,P[9,31]\,P[18,27]}{1+q}+$$
$$\frac{q^{3/2}\,P[1,22]\,P[9,31]\,P[18,27]}{1+q}+\frac{q^2\,P[1,22]\,P[9,31]\,P[18,27]}{1+q}-\frac{P[1,27]\,P[9,22]\,P[18,31]}{1+q}-$$
$$\frac{P[1,27]\,P[9,22]\,P[18,31]}{q^{3/2}\,(1+q)}-\frac{P[1,27]\,P[9,22]\,P[18,31]}{q\,(1+q)}-\frac{\sqrt{q}\,P[1,22]\,P[9,27]\,P[18,31]}{1+q}-$$
$$\frac{3\,P[1,18]\,P[9,31]\,P[22,27]}{1+q}-\frac{P[1,18]\,P[9,31]\,P[22,27]}{\sqrt{q}\,(1+q)}-\frac{3\sqrt{q}\,P[1,18]\,P[9,31]\,P[22,27]}{1+q}-$$
$$\frac{2\,q\,P[1,18]\,P[9,31]\,P[22,27]}{1+q}-\frac{q^{3/2}\,P[1,18]\,P[9,31]\,P[22,27]}{1+q}+\frac{q^2\,P[1,18]\,P[9,31]\,P[22,27]}{1+q}+$$
$$\frac{q^{5/2}\,P[1,18]\,P[9,31]\,P[22,27]}{1+q}-\frac{P[1,27]\,P[9,18]\,P[22,31]}{1+q}-\frac{P[1,27]\,P[9,18]\,P[22,31]}{q\,(1+q)}-$$
$$\frac{2\,P[1,27]\,P[9,18]\,P[22,31]}{\sqrt{q}\,(1+q)}-\frac{3\,P[1,18]\,P[9,27]\,P[22,31]}{1+q}-\frac{P[1,18]\,P[9,27]\,P[22,31]}{\sqrt{q}\,(1+q)}-$$
$$\frac{3\sqrt{q}\,P[1,18]\,P[9,27]\,P[22,31]}{1+q}-\frac{2\,q\,P[1,18]\,P[9,27]\,P[22,31]}{1+q}+$$
$$\frac{q^{3/2}\,P[1,18]\,P[9,27]\,P[22,31]}{1+q}+\frac{q^2\,P[1,18]\,P[9,27]\,P[22,31]}{1+q}-$$
$$\frac{3\,P[1,9]\,P[18,27]\,P[22,31]}{1+q}-\frac{P[1,9]\,P[18,27]\,P[22,31]}{\sqrt{q}\,(1+q)}-\frac{3\sqrt{q}\,P[1,9]\,P[18,27]\,P[22,31]}{1+q}+$$
$$\frac{q\,P[1,9]\,P[18,27]\,P[22,31]}{1+q}+\frac{q^{3/2}\,P[1,9]\,P[18,27]\,P[22,31]}{1+q}-\frac{P[1,18]\,P[9,22]\,P[27,31]}{1+q}-$$
$$\frac{P[1,18]\,P[9,22]\,P[27,31]}{q\,(1+q)}-\frac{2\,P[1,18]\,P[9,22]\,P[27,31]}{\sqrt{q}\,(1+q)}-\frac{\sqrt{q}\,P[1,18]\,P[9,22]\,P[27,31]}{1+q}$$

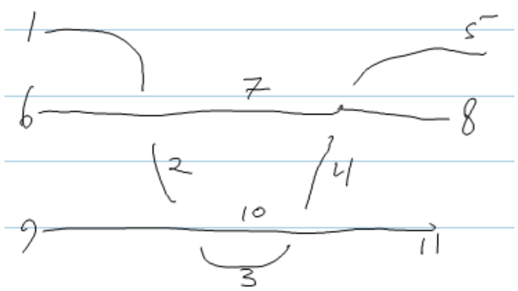
```
pA[CircuitDiagram[Take[B, 8]]] // Last // Factor
```

```
-t[1]^2 (t[18] t[27] W[1, 18, 27] - t[18] t[27] W[1, 18, 31] -
   t[18] t[27] W[1, 22, 27] + t[18] t[27] W[1, 22, 31] - t[18] t[27] W[9, 18, 27] +
   t[18] t[27] W[9, 18, 31] + t[18] t[27] W[9, 22, 27] - t[18] t[27] W[9, 22, 31] +
   W[18, 22, 27] - t[1] W[18, 22, 27] - t[27] W[18, 22, 27] + t[1] t[27] W[18, 22, 27] -
   W[18, 22, 31] + t[1] W[18, 22, 31] + t[27] W[18, 22, 31] - t[1] t[27] W[18, 22, 31] +
   W[18, 27, 31] - t[1] W[18, 27, 31] - t[18] W[18, 27, 31] + t[1] t[18] W[18, 27, 31] -
   W[22, 27, 31] + t[1] W[22, 27, 31] + t[18] W[22, 27, 31] - t[1] t[18] W[22, 27, 31])
```

```
pA[CircuitDiagram[Mirror[Take[B, 8]]]] // Last // Factor
```

```
-t[1]^2 (t[18] t[27] W[1, 18, 27] - t[18] t[27] W[1, 18, 31] -
   t[18] t[27] W[1, 22, 27] + t[18] t[27] W[1, 22, 31] - t[18] t[27] W[9, 18, 27] +
   t[18] t[27] W[9, 18, 31] + t[18] t[27] W[9, 22, 27] - t[18] t[27] W[9, 22, 31] +
   W[18, 22, 27] - t[1] W[18, 22, 27] - t[27] W[18, 22, 27] + t[1] t[27] W[18, 22, 27] -
   W[18, 22, 31] + t[1] W[18, 22, 31] + t[27] W[18, 22, 31] - t[1] t[27] W[18, 22, 31] +
   W[18, 27, 31] - t[1] W[18, 27, 31] - t[18] W[18, 27, 31] + t[1] t[18] W[18, 27, 31] -
   W[22, 27, 31] + t[1] W[22, 27, 31] + t[18] W[22, 27, 31] - t[1] t[18] W[22, 27, 31])
```

- **Sanity check on a trivial braid:**




```
TB = PD[X[1, 6, 2, 7], X[2, 9, 3, 10], X[3, 11, 4, 10], X[4, 8, 5, 7]]
```

```
PD[X[1, 6, 2, 7], X[2, 9, 3, 10], X[3, 11, 4, 10], X[4, 8, 5, 7]]
```

```
{Jones[TB][q], Jones[Mirror[TB]][q]}
```

$$\left\{-\frac{\sqrt{q}\ P[1,5]\ P[6,8]\ P[9,11]}{1+q},\ -\frac{\sqrt{q}\ P[1,5]\ P[6,8]\ P[9,11]}{1+q}\right\}$$

```
{TB1 = CircuitDiagram[TB], TB2 = CircuitDiagram[Mirror[TB]]}
```

```
{CircuitDiagram[Xm[1, 6, 2, 7], Xm[2, 9, 3, 10], Xp[3, 11, 4, 10], Xp[4, 8, 5, 7]],
 CircuitDiagram[Xp[6, 2, 7, 1], Xp[9, 3, 10, 2], Xm[10, 3, 11, 4], Xm[7, 4, 8, 5]]}
```

```
pA[TB1]
```

```
AHD[(t[6] == t[7] == t[8]) (t[9] == t[10] == t[11]) (t[1] == t[2] == t[3] == t[4] == t[5]),
 {1, 6, 9}, W[5, 8, 11],
 -t[6] t[9] W[1, 6, 9] + t[6] t[9] W[1, 6, 11] + t[6] t[9] W[1, 8, 9] - t[6] t[9] W[1, 8, 11] +
  t[6] t[9] W[5, 6, 9] - t[6] t[9] W[5, 6, 11] - t[6] t[9] W[5, 8, 9] + t[6] t[9] W[5, 8, 11]]
```

```
pA[TB2]
```

```
AHD[(t[6] == t[7] == t[8]) (t[9] == t[10] == t[11]) (t[1] == t[2] == t[3] == t[4] == t[5]),
 {1, 6, 9}, W[5, 8, 11], -t[1]^2 W[1, 6, 9] + t[1]^2 W[1, 6, 11] + t[1]^2 W[1, 8, 9] -
  t[1]^2 W[1, 8, 11] + t[1]^2 W[5, 6, 9] - t[1]^2 W[5, 6, 11] - t[1]^2 W[5, 8, 9] + t[1]^2 W[5, 8, 11]]
```

```
Last[pA[TB1]] // Factor
```

```
-t[6] t[9] (W[1, 6, 9] - W[1, 6, 11] - W[1, 8, 9] +
   W[1, 8, 11] - W[5, 6, 9] + W[5, 6, 11] + W[5, 8, 9] - W[5, 8, 11])
```

```
Last[pA[TB2]] // Factor
```

```
-t[1]^2 (W[1, 6, 9] - W[1, 6, 11] - W[1, 8, 9] +
   W[1, 8, 11] - W[5, 6, 9] + W[5, 6, 11] + W[5, 8, 9] - W[5, 8, 11])
```