FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS: FROM ALEXANDER TO KASHIWARA AND VERGNE

DROR BAR-NATAN AND ZSUZSANNA DANCSO

Abstract. w-Knots, and more generally, w-knotted objects (w-braids, w-tangles, etc.) make a class of knotted objects which is wider but weaker than their "usual" counterparts. To get (say) w-knots from u-knots, one has to allow non-planar "virtual" knot diagrams, hence enlarging the the base set of knots. But then one imposes a new relation, the "over-crossings commute" relation, further beyond the ordinary collection of Reidemeister moves, making w-knotted objects a bit weaker once again.

The group of w-braids was studied (under the name "welded braids") by Fenn, Rimanyi and Rourke [FRR] and was shown to be isomorphic to the McCool group [Mc] of "basis-conjugating" automorphisms of a free group F_n — the smallest subgroup of $\mathrm{Aut}(F_n)$ that contains both braids and permutations. Brendle and Hatcher [BH], in work that traces back to Goldsmith [Gol], have shown this group to be a group of movies of flying rings in $\mathbb{R}^3$. Satoh [Sa] studied several classes of w-knotted objects (under the name "weakly-virtual") and has shown them to be closely related to certain classes of knotted surfaces in $\mathbb{R}^4$. So w-knotted objects are algebraically and topologically interesting.

In this article we study finite type invariants of several classes of w-knotted objects. Following Berceanu and Papadima [BP], we construct homomorphic universal finite type invariants of w-braids and of w-tangles. We find that the universal finite type invariant of w-knots is more or less the Alexander polynomial (details inside).

Much as the spaces $\mathcal{A}$ of chord diagrams for ordinary knotted objects are related to metrized Lie algebras, we find that the spaces $\mathcal{A}^w$ of "arrow diagrams" for w-knotted objects are related to not-necessarily-metrized Lie algebras. Many questions concerning w-knotted objects turn out to be equivalent to questions about Lie algebras. Most notably we find that a homomorphic universal finite type invariant of w-knotted foams is essentially the same as a solution of the Kashiwara-Vergne [KV] conjecture and much of the Alekseev-Torossian [AT] work on Drinfel'd associators and Kashiwara-Vergne can be re-interpreted as a study of w-knotted trivalent graphs.

The true value of w-knots, though, is likely to emerge later, for we expect them to serve as a warmup example for what we expect will be even more interesting — the study of virtual knots, or v-knots. We expect v-knotted objects to provide the global context whose projectivization (or "associated graded structure") will be the Etingof-Kazhdan theory of deformation quantization of Lie bialgebras [EK].

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Contents

1. Introduction

1.1. **Dreams.** We have a dream[1], at least partially founded on reality, that many of the difficult algebraic equations in mathematics, especially those that are written in graded spaces, more especially those that are related in one way or another to quantum groups [Dr1], and even more especially those related to the work of Etingof and Kazhdan [EK], can be understood, and indeed, would appear more natural, in terms of finite type invariants of various topological objects.

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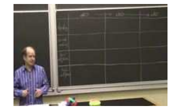

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wClips are explained in Section 1.6.

We believe this is the case for Drinfel'd's theory of associators [Dr2], which can be interpreted as a theory of well-behaved universal finite type invariants of parenthesized tangles[2] [LM2, BN3], and even more elegantly, as a theory of universal finite type invariants of knotted trivalent graphs [Da].

We believe this is the case for Drinfel'd's "Grothendieck-Teichmuller group" [Dr3] which is better understood as a group of automorphisms of a certain algebraic structure, also related to universal finite type invariants of parenthesized tangles [BN6].

And we're optimistic, indeed we believe, that sooner or later the work of Etingof and Kazhdan [EK] on quantization of Lie bialgebras will be re-interpreted as a construction of a well-behaved universal finite type invariant of virtual knots [Ka2] or of some other class of virtually knotted objects. Some steps in that direction were taken by Haviv [Hav].

We have another dream, to construct a useful "Algebraic Knot Theory". As at least a partial writeup exists [BN8], we'll only state that an important ingredient necessary to fulfil that dream would be a "closed form"[3] formula for an associator, at least in some reduced sense. Formulae for associators or reduced associators were in themselves the goal of several studies undertaken for various other reasons [LM1, Lie, Kur, Lee1].

1.2. **Stories.** Thus the first named author, DBN, was absolutely delighted when in January 2008 Anton Alekseev described to him his joint work [AT] with Charles Torossian — Anton told DBN that they found a relationship between the Kashiwara-Vergne conjecture [KV], a cousin of the Duflo isomorphism (which DBN already knew to be knot-theoretic [BLT]), and associators taking values in a space called $\mathfrak{sder}$, which he could identify as "tree-level Jacobi diagrams", also a knot-theoretic space related to the Milnor invariants [BN2, HM]. What's more, Anton told DBN that in certain quotient spaces the Kashiwara-Vergne conjecture can be solved explicitly; this should lead to some explicit associators!

So DBN spent the following several months trying to understand [AT], and this paper is a summary of these efforts. The main thing we learned is that the Alekseev-Torossian paper, and with it the Kashiwara-Vergne conjecture, fit very nicely with our first dream recalled above, about interpreting algebra in terms of knot theory. Indeed much of [AT] can be reformulated as a construction and a discussion of a well-behaved universal finite type invariant Z of a certain class of knotted objects (which we will call here w-knotted), a certain natural quotient of the space of virtual knots (more precisely, virtual trivalent tangles). And our hopes remain high that later we (or somebody else) will be able to exploit this relationship

[1] Understanding the authors' history and psychology ought never be necessary to understand their papers, yet it may be helpful. Nothing material in the rest of this paper relies on Section 1.1.

[2] "q-tangles" in [LM2], "non-associative tangles" in [BN3].

[3] The phrase "closed form" in itself requires an explanation. See Section 7.1.

in directions compatible with our second dream recalled above, on the construction of an "algebraic knot theory".

The story, in fact, is prettier than we were hoping for, for it has the following additional qualities:

- w-Knotted objects are quite interesting in themselves: as stated in the abstract, they are related to combinatorial group theory via "basis-conjugating" automorphisms of a free group F_n, to groups of movies of flying rings in $\mathbb{R}^3$, and more generally, to certain classes of knotted surfaces in $\mathbb{R}^4$. The references include [BH, FRR, Gol, Mc, Sa].
- The "chord diagrams" for w-knotted objects (really, these are "arrow diagrams") describe formulae for invariant tensors in spaces pertaining to not-necessarily-metrized Lie algebras in much of the same way as ordinary chord diagrams for ordinary knotted objects describe formulae for invariant tensors in spaces pertaining to metrized Lie algebras. This observation is bound to have further implications.
- Arrow diagrams also describe the Feynman diagrams of topological BF theory [CCM, CCFM] and of a certain class of Chern-Simons theories [Na]. Thus it is likely that our story is directly related to quantum field theory[4].
- When composed with the map from knots to w-knots, Z becomes the Alexander polynomial. For links, it becomes an invariant stronger than the multi-variable Alexander polynomial which contains the multi-variable Alexander polynomial as an easily identifiable reduction. On other w-knotted objects Z has easily identifiable reductions that can be considered as "Alexander polynomials" with good behaviour relative to various knot-theoretic operations — cablings, compositions of tangles, etc. There is also a certain specific reduction of Z that can be considered as the "ultimate Alexander polynomial" — in the appropriate sense, it is the minimal extension of the Alexander polynomial to other knotted objects which is well behaved under a whole slew of knot theoretic operations, including the ones named above.

1.3. **The Bigger Picture.** Parallel to the w-story run the possibly more significant u-story and v-story. The u-story is about u-knots, or more generally, u-knotted objects (braids, links, tangles, etc.), where "u" stands for <u>u</u>sual; hence the u-story is about ordinary knot theory. The v-story is about v-knots, or more generally, v-knotted objects, where "v" stands for <u>v</u>irtual, in the sense of Kauffman [Ka2].

The three stories, u, v, and w, are different from each other. Yet they can be told along similar lines — first the knots (topology), then their finite type invariants and their "chord diagrams" (combinatorics), then those map into certain universal enveloping algebras and similar spaces associated with various classes of Lie algebras (low algebra), and finally, in order to construct a "good" universal finite type invariant, in each case one has to confront a certain deeper algebraic subject (high algebra). These stories are summarized in a table form in Figure 1.

u-Knots map into v-knots, and v-knots map into w-knots[5]. The other parts of our stories, the "combinatorics" and "low algebra" and "high algebra" rows of Figure 1, are likewise related, and this relationship is a crucial part of our overall theme. Thus we cannot and will

[4]Some non-perturbative relations between BF theory and w-knots was discussed by Baez, Wise and Crans [BWC].

[5]Though the composition "$u \to v \to w$" is not 0. In fact, the composed map $u \to w$ is injective.

	u-Knots $\longrightarrow$	v-Knots $\longrightarrow$	w-Knots
Topology	Ordinary (usual) knotted objects in 3D — braids, knots, links, tangles, knotted graphs, etc.	Virtual knotted objects — "algebraic" knotted objects, or "not specifically embedded" knotted objects; knots drawn on a surface, modulo stabilization.	Ribbon knotted objects in 4D; "flying rings". Like v, but also with "overcrossings commute".
Combinatorics	Chord diagrams and Jacobi diagrams, modulo $4T$, STU, IHX, etc.	Arrow diagrams and v-Jacobi diagrams, modulo $6T$ and various "directed" STUs and IHXs, etc.	Like v, but also with "tails commute". Only "two in one out" internal vertices.
Low Algebra	Finite dimensional metrized Lie algebras, representations, and associated spaces.	Finite dimensional Lie bi-algebras, representations, and associated spaces.	Finite dimensional co-commutative Lie bi-algebras (i.e., $\mathfrak{g} \ltimes \mathfrak{g}^*$), representations, and associated spaces.
High Algebra	The Drinfel'd theory of associators.	Likely, quantum groups and the Etingof-Kazhdan theory of quantization of Lie bi-algebras.	The Kashiwara-Vergne-Alekseev-Torossian theory of convolutions on Lie groups and Lie algebras.

Figure 1. The u-v-w Stories

not tell the w-story in isolation, and while it is central to this article, we will necessarily also include some episodes from the u and v series. **wClip 120111-1 ends**

1.4. **Plans.** Our order of proceedings is: w-braids, w-knots, generalities, w-tangles, w-tangled foams. For more detailed information consult the "Section Summary" paragraphs below and at the beginning of each of the sections. An "odds and ends" section follows on page 93, and a glossary of notation is on page 95.

Section 2, w-Braids. (page 7) This section is largely a compilation of existing literature, though we also introduce the language of arrow diagrams that we use throughout the rest of the paper. In 2.1 and 2.2 we define v-braids and then w-braids and survey their relationship with basis-conjugating automorphisms of free groups and with "the group of (horizontal) flying rings in $\mathbb{R}^3$" (really, a group of knotted tubes in $\mathbb{R}^4$). In 2.3 we play the usual game of introducing finite type invariants, weight systems, chord diagrams (arrow diagrams, for this case), and 4T-like relations. In 2.4 we define and construct a universal finite type invariant Z for w-braids — it turns out that the only algebraic tool we need to use is the formal exponential function $\exp(a) := \sum a^n/n!$. In 2.5 we study some good algebraic properties of Z, its injectivity, and its uniqueness, and we conclude with the slight modifications needed for the study of non-horizontal flying rings.

Section 3, w-Knots. (page 23) In 3.1 we define v-knots and w-knots (long v-knots and long w-knots, to be precise) and discuss the map

$v \to w$. In 3.2 we determine the space of "chord diagrams" for w-knots to be the space $\mathcal{A}^w(\uparrow)$ of arrow diagrams modulo $\overrightarrow{4T}$ and TC relations and in 3.3 we compute some relevant dimensions. In 3.5 we show that $\mathcal{A}^w(\uparrow)$ can be re-interpreted as a space of trivalent graphs modulo STU- and IHX-like relations, and is therefore related to Lie algebras (Sec. 3.6). This allows us to completely determine $\mathcal{A}^w(\uparrow)$. With no difficulty at all in 3.4 we construct a universal finite type invariant for w-knots. With a bit of further difficulty we show in Sec. 3.7 that it is essentially equal to the Alexander polynomial.

Section 4, Algebraic Structures, Projectivizations, Expansions, Circuit Algebras. (page 48) In this section we define the "projectivization" (Sec. 4.2) of an arbitrary algebraic structure (4.1) and introduce the notions of "expansions" and "homomorphic expansions" (4.3) for such projectivizations. Everything is so general that practically anything is an example. The baby-example of quandles is built in into the section; the braid groups and w-braid groups appeared already in Section 2, yet our main goal is to set the language for the examples of w-tangles and w-tangled foams, which appear later in this paper. Both of these examples are types of "circuit algebras", and hence we end this section with a general discussion of circuit algebras (Sec. 4.4).

Section 5, w-Tangles. (page 56) In Sec. 5.1 we introduce v-tangles and w-tangles, the obvious v- and w- counterparts of the standard knot-theoretic notion of "tangles", and briefly discuss their finite type invariants and their associated spaces of "arrow diagrams", $\mathcal{A}^v(\uparrow_n)$ and $\mathcal{A}^w(\uparrow_n)$. We then construct a homomorphic expansion Z, or a "well-behaved" universal finite type invariant for w-tangles. Once again, the only algebraic tool we need to use is $\exp(a) := \sum a^n/n!$, and indeed, Sec. 5.1 is but a routine extension of parts of Section 3. We break away in Sec. 5.2 and show that $\mathcal{A}^w(\uparrow_n) \cong \mathcal{U}(\mathfrak{a}_n \oplus \mathfrak{tder}_n \ltimes \mathfrak{tr}_n)$, where $\mathfrak{a}_n$ is an Abelian algebra of rank n and where $\mathfrak{tder}_n$ and $\mathfrak{tr}_n$, two of the primary spaces used by Alekseev and Torossian [AT], have simple descriptions in terms of words and free Lie algebras. We also show that some functionals studied in [AT], div and j, have a natural interpretation in our language. In 5.3 we discuss a subclass of w-tangles called "special" w-tangles, and relate them by similar means to Alekseev and Torossian's $\mathfrak{sder}_n$ and to "tree level" ordinary Vassiliev theory. Some conventions are described in Sec. 5.4 and the uniqueness of Z is studied in Sec.5.5.

Section 6, w-Tangled Foams. (page 69) If you have come this far, you must have noticed the approximate Bolero spirit of this article. In every chapter a new instrument comes to play; the overall theme remains the same, but the composition is more and more intricate. In this chapter we add "foam vertices" to w-tangles (and a few lesser things as well) and ask the same questions we asked before; primarily, "is there a homomorphic expansion?". As we shall see, in the current context this question is equivalent to the Alekseev-Torossian [AT] version of the Kashiwara-Vergne [KV] problem and explains the relationship between these topics and Drinfel'd's theory of associators.

1.5. **Acknowledgement.** We wish to thank Anton Alekseev, Jana Archibald, Scott Carter, Karene Chu, Iva Halacheva, Joel Kamnitzer, Lou Kauffman, Peter Lee, Louis Leung, Dylan Thurston, Lucy Zhang, and Jean-Baptiste Meilhan for comments and suggestions. +Huan Vo

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1.6. **wClips.** Alongside this paper there is a series of video clips explaining parts of it. The series as a whole can be found at [BND2]; references to specific clips and specific times within clips appear at the margin of this paper. We thank Peter Lee for contributing wClip:120201 and Karene Chu for contributing wClip:120314.

2. w-Braids

Section Summary. This section is largely a compilation of existing literature, though we also introduce the language of arrow diagrams that we use throughout the rest of the paper. In 2.1 and 2.2 we define v-braids and then w-braids and survey their relationship with basis-conjugating automorphisms of free groups and with "the group of (horizontal) flying rings in $\mathbb{R}^3$" (really, a group of knotted tubes in $\mathbb{R}^4$). In 2.3 we play the usual game of introducing finite type invariants, weight systems, chord diagrams (arrow diagrams, for this case), and 4T-like relations. In 2.4 we define and construct a universal finite type invariant Z for w-braids — it turns out that the only algebraic tool we need to use is the formal exponential function $\exp(a) := \sum a^n/n!$. In 2.5 we study some good algebraic properties of Z, its injectivity, and its uniqueness, and we conclude with the slight modifications needed for the study of non-horizontal flying rings.

2.1. **Preliminary: Virtual Braids, or v-Braids.** Our main object of study for this section, w-braids, are best viewed as "virtual braids" [Ba, KL, BB], or v-braids, modulo one additional relation. Hence we start with v-braids.

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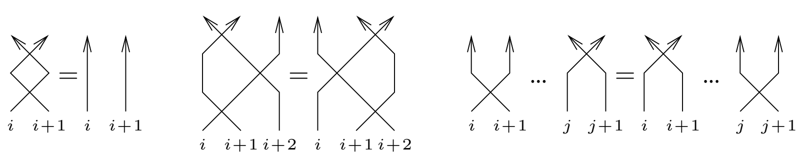
It is simplest to define v-braids in terms of generators and relations, either algebraically or pictorially. This can be done in at least two ways — the easier-at-first but philosophically-less-satisfactory "planar" way, and the harder to digest but morally more correct "abstract" way.[6]

2.1.1. *The "Planar" Way.* For a natural number n set vB_n to be the group generated by symbols σ_i ($1 \leq i \leq n-1$), called "crossings" and graphically represented by an overcrossing "between strand i and strand $i+1$" (with inverse)[7], and s_i, called "virtual crossings" and graphically represented by a non-crossing, , also "between strand i and strand $i+1$", subject to the following relations:

- The subgroup of vB_n generated by the virtual crossings s_i is the symmetric group S_n, and the s_i's correspond to the transpositions $(i, i+1)$. That is, we have
$$s_i^2 = 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad \text{and if } |i-j| > 1 \text{ then} \qquad s_i s_j = s_j s_i. \tag{1}$$
In pictures, this is



(2)

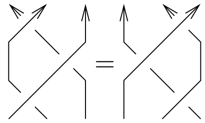
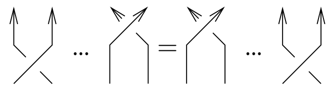
Note that we read our braids from bottom to top.

- The subgroup of vB_n generated by the crossings σ_i's is the usual braid group uB_n, and σ_i corresponds to the braiding of strand i over strand $i+1$. That is, we have
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad \text{and if } |i-j| > 1 \text{ then} \qquad \sigma_i \sigma_j = \sigma_j \sigma_i. \tag{3}$$

[6]Compare with a similar choice that exists in the definition of manifolds, as either appropriate subsets of some ambient Euclidean spaces (module some equivalences) or as abstract gluings of coordinate patches (modulo some other equivalences). Here in the "planar" approach of Section 2.1.1 we consider v-braids as "planar" objects, and in the "abstract approach" of Section 2.1.2 they are just "gluings" of abstract "crossings", not drawn anywhere in particular.

[7]We sometimes refer to as a "positive crossing" and to as a "negative crossing".

In pictures, dropping the indices, this is

 and  (4)

The first of these relations is the "Reidemeister 3 move"[8] of knot theory. The second is sometimes called "locality in space" [BN3].

- Some "mixed relations",

$$s_i\sigma_{i+1}^{\pm 1}s_i = s_{i+1}\sigma_i^{\pm 1}s_{i+1}, \qquad \text{and if } |i-j|>1 \text{ then} \qquad s_i\sigma_j = \sigma_j s_i. \tag{5}$$

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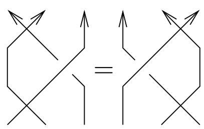
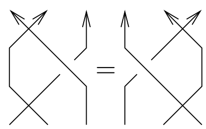
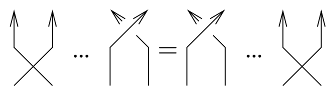
 ,  and  (6)

Remark 2.1. The "skeleton" of a v-braid B is the set of strands appearing in it, retaining the association between their beginning and ends but ignoring all the crossing information. More precisely, it is the permutation induced by tracing along B, and even more precisely it is the image of B via the "skeleton morphism" $\varsigma\colon vB_n \to S_n$ defined by $\varsigma(\sigma_i) = \varsigma(s_i) = s_i$ (or pictorially, by $\varsigma(\nearrow\!\!\!\!\nwarrow) = \varsigma(\times) = \times$). Thus the symmetric group S_n is both a subgroup and a quotient group of vB_n.

Like there are pure braids to accompany braids, there are pure virtual braids as well:

Definition 2.2. A pure v-braid is a v-braid whose skeleton is the identity permutation; the group PvB_n of all pure v-braids is simply the kernel of the skeleton morphism $\varsigma\colon vB_n \to S_n$.

We note the sequence of group homomorphisms

$$1 \longrightarrow PvB_n \hookrightarrow vB_n \xrightarrow{\varsigma} S_n \longrightarrow 1. \tag{7}$$

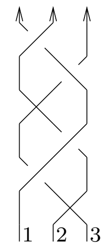
This sequence is exact and split, with the splitting given by the inclusion $S_n \hookrightarrow vB_n$ mentioned above (1). Therefore we have that

$$vB_n = PvB_n \rtimes S_n. \tag{8}$$

2.1.2. *The "Abstract" Way.* The relations (2) and (6) that govern the behaviour of virtual crossings precisely say that virtual crossings really are "virtual" — if a piece of strand is routed within a braid so that there are only virtual crossings around it, it can be rerouted in any other "virtual only" way, provided the ends remain fixed (this is Kauffman's "detour move" [Ka2, KL]). Since a v-braid B is independent of the routing of virtual pieces of strand, we may as well never supply this routing information.

[8]The Reidemeister 2 move is the relations $\sigma_i\sigma_i^{-1} = 1$ which is part of the definition of "a group". There is no Reidemeister 1 move in the theory of braids.

Thus for example, a perfectly fair verbal description of the (pure!) v-braid on the right is "strand 1 goes over strand 3 by a positive crossing then likewise positively over strand 2 then negatively over 3 then 2 goes positively over 1". We don't need to specify how strand 1 got to be near strand 3 so it can go over it — it got there by means of virtual crossings, and it doesn't matter how. Hence we arrive at the following "abstract" presentation of PvB_n and vB_n:



Proposition 2.3. *(E.g.* [Ba]*)*

(1) *The group PvB_n of pure v-braids is isomorphic to the group generated by symbols σ_{ij} for $1 \leq i \neq j \leq n$ (meaning "strand i crosses over strand j at a positive crossing"*[9]*), subject to the third Reidemeister move and to locality in space (compare with* (3) *and* (4)*):*

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$$\sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \qquad \text{whenever} \qquad |\{i,j,k\}| = 3,$$
$$\sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \qquad \text{whenever} \qquad |\{i,j,k,l\}| = 4.$$

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(2) *If $\tau \in S_n$, then with the action $\sigma_{ij}^{\tau} := \sigma_{\tau i, \tau j}$ we recover the semi-direct product decomposition $vB_n = PvB_n \rtimes S_n$.* □

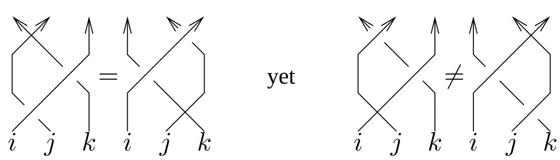
2.2. **On to w-Braids.** To define w-braids, we break the symmetry between over crossings and under crossings by imposing one of the "forbidden moves" virtual knot theory, but not the other:

$$\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}, \qquad \text{yet} \qquad s_i\sigma_{i+1}\sigma_i \neq \sigma_{i+1}\sigma_i s_{i+1}. \tag{9}$$

Alternatively,

$$\sigma_{ij}\sigma_{ik} = \sigma_{ik}\sigma_{ij}, \qquad \text{yet} \qquad \sigma_{ik}\sigma_{jk} \neq \sigma_{jk}\sigma_{ik}.$$

In pictures, this is



(10)

The relation we have just imposed may be called the "unforbidden relation", or, perhaps more appropriately, the "overcrossings commute" relation (OC). Ignoring the non-crossings[10] $\times$, the OC relation says that it is the same if strand i first crosses over strand j and then over strand k, or if it first crosses over strand k and then over strand j. The "undercrossings commute" relation UC, the one we do not impose in (9), would say the same except with "under" replacing "over".

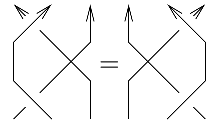
Definition 2.4. The group of w-braids is $wB_n := vB_n/OC$. Note that ς descends to wB_n and hence we can define the group of pure w-braids to be $PwB_n := \ker \varsigma\colon wB_n \to S_n$. We still have a split exact sequence as at (7) and a semi-direct product decomposition $wB_n = PwB_n \rtimes S_n$.

[9]The inverse, σ_{ij}^{-1}, is "strand i crosses over strand j at a negative crossing"

[10]Why this is appropriate was explained in the previous section.

Exercise 2.5. Show that the OC relation is equivalent to the relation

$$\sigma_i^{-1} s_{i+1} \sigma_i = \sigma_{i+1} s_i \sigma_{i+1}^{-1} \qquad \text{or}$$

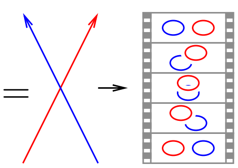


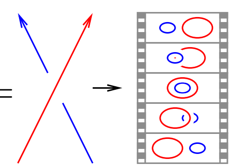
While mostly in this paper the pictorial / algebraic definition of w-braids (and other w-knotted objects) will suffice, we ought describe at least briefly 2-3 further interpretations of wB_n:

2.2.1. *The group of flying rings.* Let X_n be the space of all placements of n numbered disjoint geometric circles in $\mathbb{R}^3$, such that all circles are parallel to the xy plane. Such placements will be called horizontal[11]. A horizontal placement is determined by the centres in $\mathbb{R}^3$ of the n circles and by n radii, so $\dim X_n = 3n + n = 4n$. The permutation group S_n acts on X_n by permuting the circles, and one may think of the quotient $\tilde{X}_n := X_n/S_n$ as the space of all horizontal placements of n unmarked circles in $\mathbb{R}^3$. The fundamental group $\pi_1(\tilde{X}_n)$ is a group of paths traced by n disjoint horizontal circles (modulo homotopy), so it is fair to think of it as "the group of flying rings".

Theorem 2.6. *The group of pure w-braids PwB_n is isomorphic to the group of flying rings $\pi_1(X_n)$. The group wB_n is isomorphic to the group of unmarked flying rings $\pi_1(\tilde{X}_n)$.*

For the proof of this theorem, see [Gol, Sa] and especially [BH]. Here we will contend ourselves with pictures describing the images of the generators of wB_n in $\pi_1(\tilde{X}_n)$ and a few comments:

$s_i =$ 

$\sigma_i =$ 

Thus we map the permutation s_i to the movie clip in which ring number i trades its place with ring number $i+1$ by having the two flying around each other. This acrobatic feat is performed in $\mathbb{R}^3$ and it does not matter if ring number i goes "above" or "below" or "left" or "right" of ring number $i+1$ when they trade places, as all of these possibilities are homotopic. More interestingly, we map the braiding σ_i to the movie clip in which ring $i+1$ shrinks a bit and flies through ring i. It is a worthwhile exercise for the reader to verify that the relations in the definition of wB_n become homotopies of movie clips. Of these relations it is most interesting to see why the "overcrossings commute" relation $\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}$ holds, yet the "undercrossings commute" relation $\sigma_i^{-1}\sigma_{i+1}^{-1}s_i = s_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}$ doesn't.

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Exercise 2.7. To be perfectly precise, we have to specify the fly-through direction. In our notation, σ_i means that the ring corresponding to the under-strand approaches the bigger ring representing the over-strand from below, flies through it and exists above. For σ_i^{-1} we are "playing the movie backwards", i.e., the ring of the under-strand comes from above and exits below the ring of the over-strand.

[11] For the group of non-horizontal flying rings see Section 2.5.4

Let “the signed w braid group”, swB_n, be the group of horizontal flying rings where both fly-through directions are allowed. This introduces a “sign” for each crossing σ_i:

In other words, swB_n is generated by s_i, σ_{i+} and σ_{i-}, for $i = 1, ..., n$. Check that in swB_n $\sigma_{i-} = s_i\sigma_{i+}^{-1}s_i$, and this, along with the other obvious relations implies $swB_n \cong wB_n$.

For a rigorous discussion of orientations and signs, see Section 5.4.

2.2.2. *Certain ribbon tubes in* $\mathbb{R}^4$. With time as the added dimension, a flying ring in $\mathbb{R}^3$ traces a tube (an annulus) in $\mathbb{R}^4$, as shown in the picture below:

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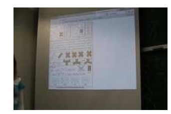

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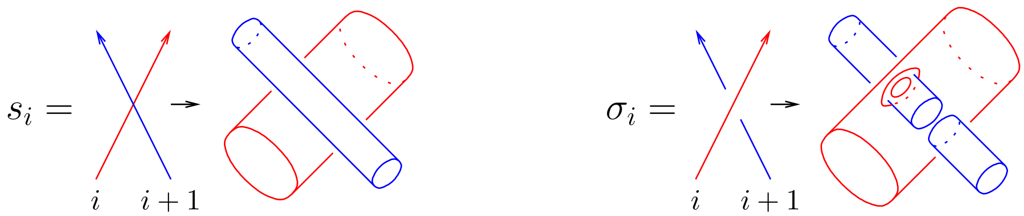

Note that we adopt here the drawing conventions of Carter and Saito [CS] — we draw surfaces as if they were projected from $\mathbb{R}^4$ to $\mathbb{R}^3$, and we cut them open whenever they are “hidden” by something with a higher fourth coordinate.

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continues Sec. 2.5.4

Note also that the tubes we get in $\mathbb{R}^4$ always bound natural 3D “solids” — their “insides”, in the pictures above. These solids are disjoint in the case of s_i and have a very specific kind of intersection in the case of σ_i — these are transverse intersections with no triple points, and their inverse images are a meridional disk on the “thin” solid tube and an interior disk on the “thick” one. By analogy with the case of ribbon knots and ribbon singularities in $\mathbb{R}^3$ (e.g. [Ka1, Chapter V]) and following Satoh [Sa], we call this kind if intersections of solids in $\mathbb{R}^4$ “ribbon singularities” and thus our tubes in $\mathbb{R}^4$ are always “ribbon tubes”.

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2.2.3. *Basis conjugating automorphisms of* F_n. Let F_n be the free (non-Abelian) group with generators $\xi_1, \ldots, \xi_n$. Artin’s theorem (Theorems 15 and 16 of [Ar]) says that the (usual) braid group uB_n (equivalently, the subgroup of wB_n generated by the σ_i’s) has a faithful right action on F_n. In other words, uB_n is isomorphic to a subgroup H of $\mathrm{Aut}^{\mathrm{op}}(F_n)$ (the group of automorphisms of F_n with opposite multiplication; $\psi_1\psi_2 := \psi_2 \circ \psi_1$). Precisely, using $(\xi, B) \mapsto \xi /\!\!/ B$ to denote the right action of $\mathrm{Aut}^{\mathrm{op}}(F_n)$ on F_n, the subgroup H consists of those automorphisms $B\colon F_n \to F_n$ of F_n that satisfy the following two conditions:

(1) B maps any generator ξ_i to a conjugate of a generator (possibly different). That is, there is a permutation $\beta \in S_n$ and elements $a_i \in F_n$ so that for every i,
$$\xi_i /\!\!/ B = a_i^{-1}\xi_{\beta i}a_i. \tag{11}$$

(2) B fixes the ordered product of the generators of F_n,
$$\xi_1\xi_2\cdots\xi_n /\!\!/ B = \xi_1\xi_2\cdots\xi_n.$$

McCool's theorem [Mc] says that the same holds true[12] if one replaces the braid group uB_n with the bigger group wB_n and drops the second condition above. So wB_n is precisely the group of "basis-conjugating" automorphisms of the free group F_n, the group of those automorphisms which map any "basis element" in $\{\xi_1, \dots, \xi_n\}$ to a conjugate of a (possibly different) basis element.

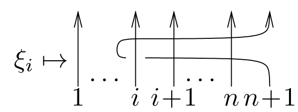
The relevant action is explicitly defined on the generators of wB_n and F_n as follows (with the omitted generators of F_n always fixed):

$$(\xi_i, \xi_{i+1}) \,/\!\!/\, s_i = (\xi_{i+1}, \xi_i) \qquad (\xi_i, \xi_{i+1}) \,/\!\!/\, \sigma_i = (\xi_{i+1}, \xi_{i+1}\xi_i\xi_{i+1}^{-1}) \qquad \xi_j \,/\!\!/\, \sigma_{ij} = \xi_i\xi_j\xi_i^{-1} \tag{12}$$

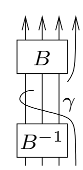
It is a worthwhile exercise to verify that $/\!\!/$ respects the relations in the definition of wB_n and that the permutation β in (11) is the skeleton $\varsigma(B)$.

There is a more conceptual description of $/\!\!/$, in terms of the structure of wB_{n+1}. Consider the inclusions

$$wB_n \overset{\iota}{\hookrightarrow} wB_{n+1} \overset{i_u}{\hookleftarrow} F_n. \tag{13}$$



Here ι is the inclusion of wB_n into wB_{n+1} by adding an inert $(n+1)$–st strand (it is injective as it has a well defined one sided inverse — the deletion of the $(n+1)$-st strand). The inclusion i_u of the free group F_n into wB_{n+1} is defined by $i_u(\xi_i) := \sigma_{i,n+1}$. The image $i_u(F_n) \subset wB_{n+1}$ is the set of all w-braids whose first n strands are straight and vertical, and whose $(n+1)$-st strand wanders among the first n strands mostly virtually (i.e., mostly using virtual crossings), occasionally slipping under one of those n strands, but never going over anything. In the "flying rings" picture of Section 2.2.1, the image $i_u(F_n) \subset wB_{n+1}$ can be interpreted as the fundamental group of the complement in $\mathbb{R}^3$ of n stationary rings (which is indeed F_n) — in $i_u(F_n)$ the only ring in motion is the last, and it only goes under, or "through", other rings, so it can be replaced by a point object whose path is an element of the fundamental group. The injectivity of i_u follows from this geometric picture.



One may explicitly verify that $i_u(F_n)$ is normalized by $\iota(wB_n)$ in wB_{n+1} (that is, the set $i_u(F_n)$ is preserved by conjugation by elements of $\iota(wB_n)$). Thus the following definition (also shown as a picture on the right) makes sense, for $B \in wB_n \subset wB_{n+1}$ and for $\gamma \in F_n \subset wB_{n+1}$:

$$\gamma \,/\!\!/\, B := i_u^{-1}(B^{-1}\gamma B) \tag{14}$$

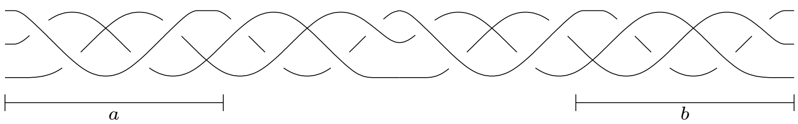
It is a worthwhile exercise to recover the explicit formulae in (12) from the above definition.

Warning 2.8. People familiar with the Artin story for ordinary braids should be warned that even though wB_n acts on F_n and the action is induced from the inclusions in (13) in much of the same way as the Artin action is induced by inclusions $uB_n \overset{\iota}{\hookrightarrow} uB_{n+1} \overset{i}{\hookleftarrow} F_n$, there are also some differences, and some further warnings apply:

- In the ordinary Artin story, $i(F_n)$ is the set of braids in uB_{n+1} whose first n strands are unbraided (that is, whose image in uB_n via "dropping the last strand" is the identity). This is not true for w-braids. For w-braids, in $i_u(F_n)$ the last strand always goes "under" all other strands (or just virtually crosses them), but never over.

[12]Though see Warning 2.8.

- Thus unlike the isomorphism $PuB_{n+1} \cong PuB_n \ltimes F_n$, it is not true that PwB_{n+1} is isomorphic to $PwB_n \ltimes F_n$.
- The Overcrossings Commute relation imposed in wB breaks the symmetry between overcrossings and undercrossings. Thus let $i_o\colon F_n \to wB_n$ be the "opposite" of i_u, mapping into braids in which the last strand is always "over" or virtual. Then i_o is not injective (its image is in fact Abelian) and its image is not normalized by $\iota(wB_n)$. So there is no "second" action of wB_n on F_n defined using i_o.
- For v-braids, both i_u and i_o are injective and there are two actions of vB_n on F_n — one defined by first projecting into w-braids, and the other defined by first projecting into v-braids modulo "Undercrossings Commute". Yet v-braids contain more information than these two actions can see. The "Kishino" v-braid below, for example, is visibly trivial if either overcrossings or undercrossings are made to commute, yet by computing its Kauffman bracket we know it is non-trivial as a v-braid [BND2, "The Kishino Braid"]:



$\left(\text{The commutator } ab^{-1}a^{-1}b \text{ of v-braids } a, b \text{ annihilated by OC/UC, respectively, with a minor cancellation.}\right)$

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Problem 2.9. Is PwB_n a semi-direct product of free groups? Note that both PuB_n and PvB_n are such semi-direct products: For PuB_n, this is the well known "combing of braids"; it follows from $PuB_n \cong PuB_{n-1} \ltimes F_{n-1}$ and induction. For PvB_n, it is a result stated in [Ba] (though our own understanding of [Ba] is incomplete).

Remark 2.10. Note that Gutiérrez and Krstić [GK] find "normal forms" for the elements of PwB_n, yet they do not decide whether PwB_n is "automatic" in the sense of [Ep].

2.3. **Finite Type Invariants of v-Braids and w-Braids.** Just as we had two definitions for v-braids (and thus w-braids) in Section 2.1, we will give two (obviously equivalent) developments of the theory of finite type invariants of v-braids and w-braids — a pictorial/topological version in Section 2.3.1, and a more abstract algebraic version in Section 2.3.2.

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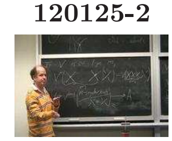

describes the standard theory, briefly

2.3.1. *Finite Type Invariants, the Pictorial Approach.* In the standard theory of finite type invariants of knots (also known as Vassiliev or Goussarov-Vassiliev invariants) [Gou1, Vas, BN1, BN7] one progresses from the definition of finite type via iterated differences to chord diagrams and weight systems, to $4T$ (and other) relations, to the definition of universal finite type invariants, and beyond. The exact same progression (with different objects playing similar roles, and sometimes, when yet insufficiently studied, with the last step or two missing) is also seen in the theories of finite type invariants of braids [BN5], 3-manifolds [Oh, LMO, Le], virtual knots [GPV, Po] and of several other classes of objects. We thus assume that the reader has familiarity with these basic ideas, and we only indicate briefly how they are implemented in the case of v-braids and w-braids.

Much like the formula $\doublepoint \to \overcrossing - \undercrossing$ of the Vassiliev-Goussarov fame, given a v-braid invariant $V\colon vB_n \to A$ valued in some Abelian group A, we extend it to "singular" v-braids, braids that contain "semi-virtual crossings" like $\semivirtualover$ and $\semivirtualunder$ using the formulae $V(\semivirtualover) := V(\overcrossing) - V(\virtualcrossing)$ and $V(\semivirtualunder) := V(\undercrossing) - V(\virtualcrossing)$ (see [GPV, Po, BHLR]). We say that "V is of type m" if its extension vanishes on singular v-braids having more than m semi-virtual crossings.

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inconsistent.

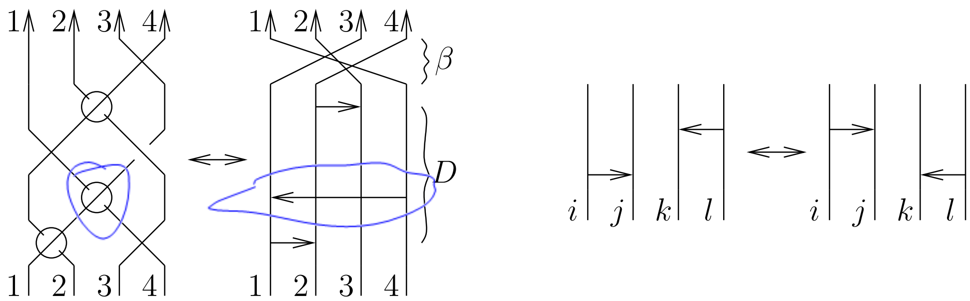
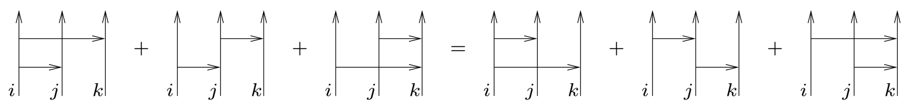


Figure 2. On the left, a 3-singular v-braid and its corresponding 3-arrow diagram. A self-explanatory algebraic notation for this arrow diagram is $(a_{12}a_{41}a_{23}, 3421)$. picture and in algebraic notation. Note that we regard arrow diagrams as graph-theoretic objects, and hence the two arrow diagrams on the right, whose underlying graphs are the same, are regarded as equal. In algebraic notation this means that we always impose the relation $a_{ij}a_{kl} = a_{kl}a_{ij}$ when the indices i, j, k, and l are all distinct.



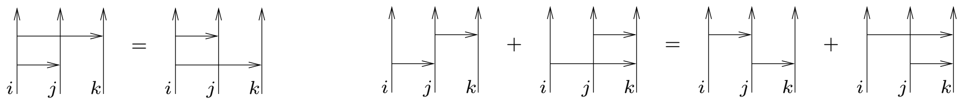
$$a_{ij}a_{ik} + a_{ij}a_{jk} + a_{ik}a_{jk} = a_{ik}a_{ij} + a_{jk}a_{ij} + a_{jk}a_{ik}$$
$$\text{or} \qquad [a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}] = 0$$

Figure 3. The $6T$ relation. Standard knot theoretic conventions apply — only the relevant parts of each diagram is shown; in reality each diagram may have further vertical strands and horizontal arrows, provided the extras are the same in all 6 diagrams. Also, the vertical strands are in no particular order — other valid $6T$ relations are obtained when those strands are permuted in other ways.

Up to invariants of lower type, an invariant of type m is determined by its "weight system", which is a functional $W = W_m(V)$ defined on "m-singular v-braids modulo $\overcrossing = \virtualcrossing = \undercrossing$". Let us denote the vector space of all formal linear combinations of such equivalence classes by $\mathcal{G}_m\mathcal{D}^v_n$. Much as m-singular knots modulo $\overcrossing = \undercrossing$ can be identified with chord diagrams, the basis elements of $\mathcal{G}_m\mathcal{D}^v_n$ can be identified with pairs (D, β), where D is a horizontal arrow diagram and β is a "skeleton permutation". See Figure 2.

We assemble the spaces $\mathcal{G}_m\mathcal{D}^v_n$ together to form a single graded space, $\mathcal{D}^v_n := \oplus_{m=0}^{\infty}\mathcal{G}_m\mathcal{D}^v_n$. Note that throughout this paper, whenever we write an infinite direct sum, we automatically complete it. Thus in $\mathcal{D}^v_n$ we allow infinite sums with one term in each homogeneous piece $\mathcal{G}_m\mathcal{D}^v_n$.

In the standard finite-type theory for knots, weight systems always satisfy the $4T$ relation, and are therefore functionals on $\mathcal{A} := \mathcal{D}/4T$. Likewise, in the case of v-braids, weight systems satisfy the "$6T$ relation" of [GPV, Po, BHLR], shown in Figure 3, and are therefore functionals on $\mathcal{A}^v_n := \mathcal{D}^v_n/6T$. In the case of w-braids, the "overcrossings commute" relation (9) implies the "Tails Commute" (TC) relation on the level of arrow diagrams, and in the presence of the TC relation, two of the terms in the $6T$ relation drop out, and what



$$a_{ij}a_{ik} = a_{ik}a_{ij} \qquad a_{ij}a_{jk} + a_{ik}a_{jk} = a_{jk}a_{ij} + a_{jk}a_{ik}$$

$$\text{or} \quad [a_{ij}, a_{ik}] = 0 \qquad \text{or} \quad [a_{ij} + a_{ik}, a_{jk}] = 0$$

Figure 4. The TC and the $\overrightarrow{4T}$ relations.

remains is the "$\overrightarrow{4T}$" relation. These relations are shown in Figure 4. Thus weight systems of finite type invariants of w-braids are linear functionals on $\mathcal{A}_n^w := \mathcal{D}_n^v/TC, \overrightarrow{4T}$.

The next question that arises is whether we have already found *all* the relations that weight systems always satisfy. More precisely, given a degree m linear functional on $\mathcal{A}_n^v = \mathcal{D}_n^v/6T$ (or on $\mathcal{A}_n^w = \mathcal{D}_n^v/TC, \overrightarrow{4T}$), is it always the weight system of some type m invariant V of v-braids (or w-braids)? As in every other theory of finite type invariants, the answer to this question is affirmative if and only if there exists a "universal finite type invariant" (or simply, an "expansion") of v-braids (w-braids):

Definition 2.11. An expansion for v-braids (w-braids) is an invariant $Z\colon vB_n \to \mathcal{A}_n^v$ (or $Z\colon wB_n \to \mathcal{A}_n^w$) satisfying the following "universality condition":

- If B is an m-singular v-braid (w-braid) and $D \in \mathcal{G}_m\mathcal{D}_n^v$ is its underlying arrow diagram as in Figure 2, then

$$Z(B) = D + (\text{terms of degree } > m).$$

Indeed if Z is an expansion and $W \in \mathcal{G}_m\mathcal{A}^\star$,[13] the universality condition implies that $W \circ Z$ is a finite type invariant whose weight system is W. To go the other way, if (D_i) is a basis of $\mathcal{A}$ consisting of homogeneous elements, if (W_i) is the dual basis of $\mathcal{A}^\star$ and (V_i) are finite type invariants whose weight systems are the W_i's, then $Z(B) := \sum_i D_iV_i(B)$ defines an expansion.

In general, constructing a universal finite type invariant is a hard task. For knots, one uses either the Kontsevich integral or perturbative Chern-Simons theory (also known as "configuration space integrals" [BT] or "tinker-toy towers" [Th]) or the rather fancy algebraic theory of "Drinfel'd associators" (a summary of all those approaches is at [BS]). For homology spheres, this is the "LMO invariant" [LMO, Le] (also the "Århus integral" [BGRT2]). For v-braids, we still don't know if an expansion exists. As we shall see below, the construction of an expansion for w-braids is quite easy.

2.3.2. *Finite Type Invariants, the Algebraic Approach.* For any group G, one can form the group algebra $\mathbb{F}G$ for some field $\mathbb{F}$ by allowing formal linear combinations of group elements and extending multiplication linearly. The *augmentation ideal* is the ideal generated by differences, or equivalently, the set of linear combinations of group elements whose coefficients sum to zero:

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$$\mathcal{I} := \left\{ \sum_{i=1}^k a_ig_i\colon a_i \in \mathbb{F}, g_i \in G, \sum_{i=1}^k a_i = 0 \right\}.$$

[13] $\mathcal{A}$ here denotes either $\mathcal{A}_n^v$ or $\mathcal{A}_n^w$, or in fact, any of many similar spaces that we will discuss later on.

Powers of the augmentation ideal provide a filtration of the group algebra. Let $\mathcal{A}(G) := \bigoplus_{m\geq 0}\mathcal{I}^m/\mathcal{I}^{m+1}$ be the associated graded space corresponding to this filtration.

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is much more detailed on these matters

Definition 2.12. An expansion for the group G is a map $Z\colon G \to \mathcal{A}(G)$, such that the linear extension $Z\colon \mathbb{F}G \to \mathcal{A}(G)$ is filtration preserving and the induced map

$$\operatorname{gr} Z\colon (\operatorname{gr} \mathbb{F}G = \mathcal{A}(G)) \to (\operatorname{gr} \mathcal{A}(G) = \mathcal{A}(G))$$

is the identity. An equivalent way to phrase this is that the degree m piece of Z restricted to $\mathcal{I}^m$ is the projection onto $\mathcal{I}^m/\mathcal{I}^{m+1}$.

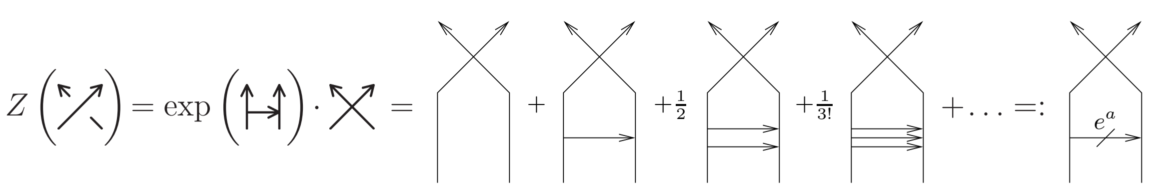
Exercise 2.13. Verify that for the groups PvB_n and PwB_n the m-th power of the augmentation ideal coincides with the span of all resolutions of m-singular v- or w-braids (by a resolution we mean the formal linear combination where each semivirtual crossing is replaced by the appropriate difference of a virtual and a regular crossing). Then check that the notion of expansion defined above is the same as that of Definition 2.11, restricted to pure braids.

Finally, note the functorial nature of the construction above. What we have described is a functor, called "projectivization" proj : *Groups* $\to$ *GradedAlgebras*, which assigns to each group G the graded algebra $\mathcal{A}(G)$. To each homomorphism $\phi\colon G \to H$, proj assigns the induced map $\operatorname{gr}\phi\colon (\mathcal{A}(G) = \operatorname{gr}\mathbb{F}G) \to (\mathcal{A}(H) = \operatorname{gr}\mathbb{F}H)$.

2.4. **Expansions for w-Braids.** The space $\mathcal{A}_n^w$ of arrow diagrams on n strands is an associative algebra in an obvious manner: If the permutations underlying two arrow diagrams are the identity permutations, we simply juxtapose the diagrams. Otherwise we "slide" arrows through permutations in the obvious manner — if τ is a permutation, we declare that $\tau a_{(\tau i)(\tau j)} = a_{ij}\tau$. Instead of seeking an expansion $wB_n \to \mathcal{A}_n^w$, we set the bar a little higher and seek a "homomorphic expansion":

Definition 2.14. A homomorphic expansion $Z\colon wB_n \to \mathcal{A}_n^w$ is an expansion that carries products in wB_n to products in $\mathcal{A}_n^w$.

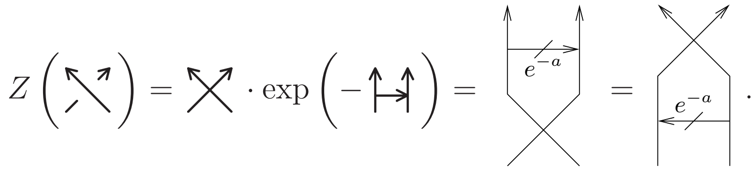
To find a homomorphic expansion, we just need to define it on the generators of wB_n and verify that it satisfies the relations defining wB_n and the universality condition. Following [BP, Section 5.3] and [AT, Section 8.1] we set $Z(\times) = \times$ (that is, a transposition in wB_n gets mapped to the same transposition in $\mathcal{A}_n^w$, adding no arrows) and $Z(\times) = \exp(\mapsto)\times$. This last formula is important so deserves to be magnified, explained and replaced by some new notation:



(15)

Thus the new notation $\xrightarrow{e^a}$ stands for an "exponential reservoir" of parallel arrows, much like $e^a = 1 + a + aa/2 + aaa/3! + \ldots$ is a "reservoir" of a's. With the obvious interpretation for $\xrightarrow{e^{-a}}$ (the $-$ sign indicates that the terms should have alternating signs, as in $e^{-a} =$

$1 - a + a^2/2 - a^3/3! + \ldots$), the second Reidemeister move $= 1$ forces that we set



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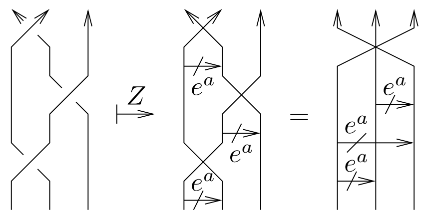
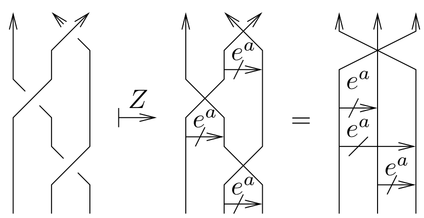
Theorem 2.15. *The above formulae define an invariant $Z\colon wB_n \to \mathcal{A}_n^w$ (that is, Z satisfies all the defining relations of wB_n). The resulting Z is a homomorphic expansion (that is, it satisfies the universality property of Definition 2.14).*

Proof. (Following [BP, AT]) For the invariance of Z, the only interesting relations to check are the Reidemeister 3 relation of (4) and the Overcrossings Commute relation of (10). For Reidemeister 3, we have



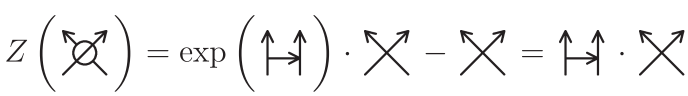
$$= e^{a_{12}}e^{a_{13}}e^{a_{23}}\tau \stackrel{1}{=} e^{a_{12}+a_{13}}e^{a_{23}}\tau \stackrel{2}{=} e^{a_{12}+a_{13}+a_{23}}\tau,$$

where τ is the permutation 321 and equality 1 holds because $[a_{12}, a_{13}] = 0$ by a Tails Commute (TC) relation and equality 2 holds because $[a_{12} + a_{13}, a_{23}] = 0$ by a $\overrightarrow{4T}$ relation. Likewise, again using TC and $\overrightarrow{4T}$,



$$= e^{a_{23}}e^{a_{13}}e^{a_{12}}\tau = e^{a_{23}}e^{a_{13}+a_{12}}\tau = e^{a_{23}+a_{13}+a_{12}}\tau,$$

and so Reidemeister 3 holds. An even simpler proof using just the Tails Commute relation shows that the Overcrossings Commute relation also holds. Finally, since Z is homomorphic, it is enough to check the universality property at degree 1, where it is very easy:



$+$ (terms of degree > 1),

and a similar computation manages the case. $\square$

Remark 2.16. Note that the main ingredient of the above proof was to show that $R := Z(\sigma_{12}) = e^{a_{12}}$ satisfies the famed Yang-Baxter equation,

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12},$$

where R^{ij} means "place R on strands i and j".

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2.5. **Some Further Comments.**

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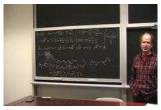


starts

2.5.1. *Compatibility with Braid Operations.* As with any new gadget, we would like to know how compatible the expansion Z of the previous section is with the gadgets we already have; namely, with various operations that are available on w-braids and with the action of w-braids on the free group F_n (Section 2.2.3).

2.5.1.1. Z is Compatible with Braid Inversion. Let θ denote both the "braid inversion" operation $\theta\colon wB_n \to wB_n$ defined by $B \mapsto B^{-1}$ and the "antipode" anti-automorphism $\theta\colon \mathcal{A}^w_n \to \mathcal{A}^w_n$ defined by mapping permutations to their inverses and arrows to their negatives (that is, $a_{ij} \mapsto -a_{ij}$). Then the diagram on the right commutes.

$$\begin{tikzcd} wB_n \arrow[r, "\theta"] \arrow[d, "Z"'] \arrow[dr, phantom, "\circlearrowleft"] & wB_n \arrow[d, "Z"] \\ \mathcal{A}^w_n \arrow[r, "\theta"'] & \mathcal{A}^w_n \end{tikzcd}$$

2.5.1.2. Braid Cloning and the Group-Like Property. Let Δ denote both the "braid cloning" operation $\Delta\colon wB_n \to wB_n \times wB_n$ defined by $B \mapsto (B, B)$ and the "co-product" algebra morphism $\Delta\colon \mathcal{A}^w_n \to \mathcal{A}^w_n \otimes \mathcal{A}^w_n$ defined by cloning permutations (that is, $\tau \mapsto \tau \otimes \tau$) and by treating arrows as primitives (that is, $a_{ij} \mapsto a_{ij} \otimes 1 + 1 \otimes a_{ij}$). Then the diagram on the right commutes. In formulae, this is $\Delta(Z(B)) = Z(B) \otimes Z(B)$, which is the statement "$Z(B)$ is group-like".

$$\begin{tikzcd} wB_n \arrow[r, "\Delta"] \arrow[d, "Z"'] \arrow[dr, phantom, "\circlearrowleft"] & wB_n \times wB_n \arrow[d, "Z \times Z"] \\ \mathcal{A}^w_n \arrow[r, "\Delta"'] & \mathcal{A}^w_n \otimes \mathcal{A}^w_n \end{tikzcd}$$

2.5.1.3. Strand Insertions. Let $\iota\colon wB_n \to wB_{n+1}$ be an operation of "inert strand insertion". Given $B \in wB_n$, the resulting $\iota B \in wB_{n+1}$ will be B with one strand S added at some location chosen in advance — to the left of all existing strands, or to the right, or starting from between the 3rd and the 4th strand of B and ending between the 6th and the 7th strand of B; when adding S, add it "inert", so that all crossings on it are virtual (this is well defined). There is a corresponding inert strand addition operation $\iota\colon \mathcal{A}^w_n \to \mathcal{A}^w_{n+1}$, obtained by adding a strand at the same location as for the original ι and adding no arrows. It is easy to check that Z is compatible with ι; namely, that the diagram on the right is commutative.

$$\begin{tikzcd} wB_n \arrow[r, "\iota"] \arrow[d, "Z"'] \arrow[dr, phantom, "\circlearrowleft"] & wB_{n+1} \arrow[d, "Z"] \\ \mathcal{A}^w_n \arrow[r, "\iota"'] & \mathcal{A}^w_{n+1} \end{tikzcd}$$

2.5.1.4. Strand Deletions. Given k between 1 and n, let $d_k\colon wB_n \to wB_{n-1}$ the operation of "removing the kth strand". This operation induces a homonymous operation $d_k\colon \mathcal{A}^w_n \to \mathcal{A}^w_{n-1}$: if $D \in \mathcal{A}^w_n$ is an arrow diagram, $d_k D$ is D with its kth strand removed if no arrows in D start or end on the kth strand, and it is 0 otherwise. It is easy to check that Z is compatible with d_k; namely, that the diagram on the right is commutative.[14]

$$\begin{tikzcd} wB_n \arrow[r, "d_k"] \arrow[d, "Z"'] \arrow[dr, phantom, "\circlearrowleft"] & wB_{n-1} \arrow[d, "Z"] \\ \mathcal{A}^w_n \arrow[r, "d_k"'] & \mathcal{A}^w_{n-1} \end{tikzcd}$$

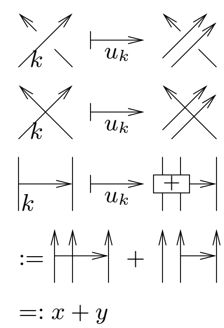
[14] Section 4.2, "$d_k\colon wB_n \to wB_{n-1}$" is an algebraic structure made of two spaces (wB_n and wB_{n-1}), two binary operations (braid composition in wB_n and in wB_{n-1}), and one unary operation, d_k. After projectivization we get the algebraic structure $d_k\colon \mathcal{A}^w_n \to \mathcal{A}^w_{n-1}$ with d_k as described above, and an alternative way of stating our assertion is to say that Z is a morphism of algebraic structures. A similar remark applies (sometimes in the negative form) to the other operations discussed in this section.

2.5.1.5. Compatibility with the action on F_n. Let FA_n denote the (degree-completed) free associative (but not commutative) algebra on generators $x_1, \dots, x_n$. Then there is an "expansion" $Z\colon F_n \to FA_n$ defined by $\xi_i \mapsto e^{x_i}$ (see [Lin] and the related "Magnus Expansion" of [MKS]). Also, there is a right action of $\mathcal{A}_n^w$ on FA_n defined on generators by $x_i\tau = x_{\tau i}$ for $\tau \in S_n$ and by $x_j a_{ij} = [x_i, x_j]$ and $x_k a_{ij} = 0$ for $k \neq j$ and extended by the Leibniz rule to the rest of FA_n and then multiplicatively to the rest of $\mathcal{A}_n^w$.

$$\begin{tikzcd} F_n \arrow[r, phantom, "\circlearrowleft"] \arrow[d, "Z"'] & wB_n \arrow[d, "Z"] \arrow[dl, phantom, "\circlearrowright"] \\ FA_n \arrow[r, phantom, "\circlearrowleft"] & \mathcal{A}_n^w \end{tikzcd}$$

Exercise 2.17. Using the language of Section 4.2, verify that $FA_n = \operatorname{proj} F_n$ and that when the actions involved are regarded as instances of the algebraic structure "one monoid acting on another", we have that $\left(FA_n \circlearrowleft \mathcal{A}_n^w\right) = \operatorname{proj}\left(F_n \circlearrowleft wB_n\right)$. Finally, use the definition of the action in (14) and the commutative diagrams of paragraphs 2.5.1.1, 2.5.1.2 and 2.5.1.3 to show that the diagram of paragraph 2.5.1.5 is also commutative.

2.5.1.6. Unzipping a Strand. Given k between 1 and n, let $u_k\colon wB_n \to wB_{n+1}$ the operation of "unzipping the kth strand", briefly defined on the right[15]. The induced operation $u_k\colon \mathcal{A}_n^w \to \mathcal{A}_{n+1}^w$ is also shown on the right — if an arrow starts (or ends) on the strand being doubled, it is replaced by a sum of two arrows that start (or end) on either of the two "daughter strands" (and this is performed for each arrow independently; so if there are t arrows touching the kth strands in a diagram D, then $u_k D$ will be a sum of 2^t diagrams).



In some sense, this whole paper as well as the work of Kashiwara and Vergne [KV] and Alekseev and Torossian [AT] is about coming to grips with the fact that Z is **not** compatible with u_k (that the diagram on the right is **not** commutative). Indeed, let $x := a_{13}$ and $y := a_{23}$ be as on the right, and let s be the permutation 21 and τ the permutation 231. Then $d_1 Z(\times) = d_1(e^{a_{12}} s) = e^{x+y}\tau$ while $Z(d_1 \times) = e^y e^x \tau$. So the failure of d_1 and Z to commute is the ill-behaviour of the exponential function when its arguments are not commuting, which is measured by the BCH formula, central to both [KV] and [AT].

$$\begin{tikzcd} wB_n \arrow[r, "u_k"] \arrow[d, "Z"'] & wB_{n+1} \arrow[d, "Z"] \\ \mathcal{A}_n^w \arrow[r, "u_k"'] \arrow[ur, phantom, "\not\circlearrowright"] & \mathcal{A}_{n+1}^w \end{tikzcd}$$

2.5.2. *Power and Injectivity.* The following theorem is due to Berceanu and Papadima [BP, Theorem 5.4]; a variant of this theorem are also true for ordinary braids [BN2, Ko, HM], and can be proven by similar means:

Theorem 2.18. *$Z\colon wB_n \to \mathcal{A}_n^w$ is injective. In other words, finite type invariants separate w-braids.*

Proof. Follows immediately from the faithfulness of the action $F_n \circlearrowleft wB_n$, from the compatibility of Z with this action, and from the injectivity of $Z\colon F_n \to FA_n$ (the latter is well known, see e.g. [MKS, Lin]). Indeed if B_1 and B_2 are w-braids and $Z(B_1) = Z(B_2)$, then $Z(\xi)Z(B_1) = Z(\xi)Z(B_2)$ for any $\xi \in F_n$, therefore $\forall \xi\, Z(\xi \,/\!\!/\, B_1) = Z(\xi \,/\!\!/\, B_2)$, therefore $\forall \xi\, \xi \,/\!\!/\, B_1 = \xi \,/\!\!/\, B_2$, therefore $B_1 = B_2$.

[15]Unzipping a knotted zipper turns a single band into two parallel ones. This operation is also known as "strand doubling", but for compatibility with operations that will be introduced later, we prefer "unzipping".

Remark 2.19. Apart from the obvious, that $\mathcal{A}^w_n$ can be computed degree by degree in exponential time, we do not know a simple formula for the dimension of the degree m piece of $\mathcal{A}^w_n$ or a natural basis of that space. This compares unfavourably with the situation for ordinary braids (see e.g. [BN5]). Also compare with Problem 2.9 and with Remark 2.10.

2.5.3. *Uniqueness.* There is certainly not a unique expansion for w-braids — if Z_1 is an expansion and and P is any degree-increasing linear map $\mathcal{A}^w \to \mathcal{A}^w$ (a "pollution" map), then $Z_2 := (I + P) \circ Z_1$ is also an expansion, where $I\colon \mathcal{A}^w \to \mathcal{A}^w$ is the identity. But that's all, and if we require a bit more, even that freedom disappears.

Proposition 2.20. *If $Z_{1,2}\colon wB_n \to \mathcal{A}^w_n$ are expansions then there exists a degree-increasing linear map $P\colon \mathcal{A}^w \to \mathcal{A}^w$ so that $Z_2 := (I + P) \circ Z_1$.*

Proof. (Sketch). Let $\widehat{wB_n}$ be the unipotent completion of wB_n. That is, let $\mathbb{Q}wB_n$ be the algebra of formal linear combinations of w-braids, let $\mathcal{I}$ be the ideal in $\mathbb{Q}wB_n$ be the ideal generated by $\semivirtualover = \overcrossing - \virtualcrossing$ and by $\semivirtualunder = \virtualcrossing - \undercrossing$, and set

$$\widehat{wB_n} := \varprojlim_{m\to\infty} \mathbb{Q}wB_n \,/\mathcal{I}^m \ .$$

$\widehat{wB_n}$ is filtered with $\mathcal{F}_m\widehat{wB_n} := \varprojlim_{m'>m} \mathcal{I}^m \,/\mathcal{I}^{m'}$. An "expansion" can be re-interpreted as an "isomorphism of $\widehat{wB_n}$ and $\mathcal{A}^w_n$ as filtered vector spaces". Always, any two isomorphisms differ by an automorphism of the target space, and that's the essence of $I + P$. $\square$

Proposition 2.21. *If $Z_{1,2}\colon wB_n \to \mathcal{A}^w_n$ are homomorphic expansions that commute with braid cloning (paragraph 2.5.1.2) and with strand insertion (paragraph 2.5.1.3), then $Z_1 = Z_2$.*

Proof. (Sketch). A homomorphic expansion that commutes with strand insertions is determined by its values on the generators $\overcrossing$, $\undercrossing$ and $\virtualcrossing$ of wB_2. Commutativity with braid cloning implies that these values must be (up to permuting the strands) group like, that is, the exponentials of primitives. But the only primitives are a_{12} and a_{21}, and one may easily verify that there is only one way to arrange these so that Z will respect $\virtualcrossing^2 = \overcrossing \cdot \undercrossing = 1$ and $\semivirtualover \mapsto \rightarrowdiagram +$ (higher degree terms). $\square$

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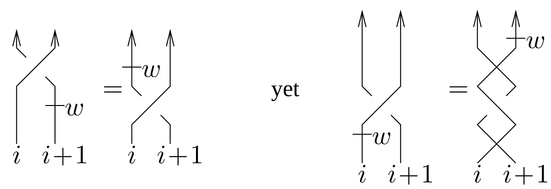
2.5.4. *The group of non-horizontal flying rings.* Let Y_n denote the space of all placements of n numbered disjoint oriented unlinked geometric circles in $\mathbb{R}^3$. Such a placement is determined by the centres in $\mathbb{R}^3$ of the circles, the radii, and a unit normal vector for each circle pointing in the positive direction, so $\dim Y_n = 3n + n + 3n = 7n$. $S_n \ltimes \mathbb{Z}_2^n$ acts on Y_n by permuting the circles and mapping each circle to its image in either an orientation-preserving or an orientation-reversing way. Let $\tilde{Y}_n$ denote the quotient $Y_n/S_n \ltimes \mathbb{Z}_2^n$. The fundamental group $\pi_1(\tilde{Y}_n)$ can be thought of as the "group of flippable flying rings". Without loss of generality, we can assume that the basepoint is chosen to be a horizontal placement. We want to study the relationship of this group to wB_n.

Theorem 2.22. *$\pi_1(\tilde{Y}_n)$ is a $\mathbb{Z}_2^n$-extension of wB_n, generated by s_i, σ_i ($1 \le i \le n-1$), and w_i ("flips"), for $1 \le i \le n$; with the relations as above, and in addition:*

$$w_i^2 = 1; \qquad w_iw_j = w_jw_i; \qquad w_js_i = s_iw_j \quad \text{when } i \neq j, j+1;$$
$$w_is_i = s_iw_{i+1}; \qquad w_{i+1}s_i = s_iw_i;$$

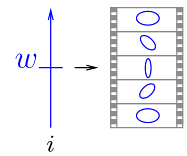
$$w_j\sigma_i = \sigma_i w_j \quad \text{if } j \neq i, i+1; \quad w_{i+1}\sigma_i = \sigma_i w_i; \quad \text{yet} \quad w_i\sigma_i = s_i\sigma_i^{-1}s_i w_{i+1}.$$

The two most interesting flip relations in pictures:



(16)

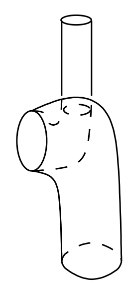
Instead of a proof, we provide some heuristics. Since each circle starts out in a horizontal position and returns to a horizontal position, there is an integer number of "flips" they do in between, these are the generators w_i, as shown on the right.

$w_i =$ 

The first relation says that a double flip is homotopic to doing nothing. Technically, there are two different directions of flips, and they are the same via this (non-obvious but true) relation. The rest of the first line is obvious: flips of different rings commute, and if two rings fly around each other while another one flips, the order of these events can be switched by homotopy. The second line says that if two rings trade places with no interaction while one flips, the order of these events can be switched as well. However, we have to re-number the flip to conform to the strand labelling convention.

The only subtle point is how flips interact with crossings. First of all, if one ring flies through another while a third one flips, the order clearly does not matter. If a ring flies through another and also flips, the order can be switched. However, if ring A flips and then ring B flies through it, this is homotopic to ring B flying through ring A from the other direction and then ring A flipping. In other words, commuting σ_i with w_i changes the "sign of the crossing" in the sense of Exercise 2.7. This gives the last, and the only truly non-commutative flip relation.

To explain why the flip is denoted by w, let us consider the alternative description by ribbon tubes. A flipping ring traces a so called wen[16] in $\mathbb{R}^4$. A wen is a Klein bottle cut along a meridian circle, as shown. The wen is embedded in $\mathbb{R}^4$.



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Finally, note that $\pi_1 Y_n$ is exactly the pure w-braid group PwB_n: since each ring has to return to its original position and orientation, each does an even number of flips. The flips (or wens) can all be moved to the bottoms of the braid diagram strands (to the bottoms of the tubes, to the beginning of words), at a possible cost, as specified by Equation (16). Once together, they pairwise cancel each other. As a result, this group can be thought of as not containing wens at all.

2.5.5. *The Relationship with u-Braids.* For the sake of ignoring strand permutations, we restrict our attention to pure braids.

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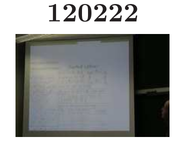

starts

By Section 2.3.2, for any expansion $Z^u\colon PuB_n \to \mathcal{A}_n^u$ (where PuB_n is the "usual" braid group and $\mathcal{A}_n^u$ is the algebra of horizontal chord diagrams on n strands), there is a square of maps as shown on the right. Here Z^w is the expansion constructed in Section 2.4, the left vertical map a is the composition of the inclusion and projection maps $PuB_n \to PvB_n \to PwB_n$. The map α is the induced map by the functoriality of projectivization, as noted after Exercise

$$\begin{tikzcd} PuB \arrow[r, dotted, "Z^u"] \arrow[d, "a"] & \mathcal{A}^u \arrow[d, "\alpha"] \\ PwB \arrow[r, "Z^w"] & \mathcal{A}^w \end{tikzcd}$$

[16]The term wen was coined by Kanenobu and Shima in [KS]

2.13. The reader can verify that α maps each chord to the sum of its two possible directed versions.

Note that this square is *not* commutative for any choice of Z^u even in degree 2: the image of a crossing under Z^w is outside the image of α.

$$\begin{tikzcd} PuB_n \arrow[r, "Z^u_c"] \arrow[d, "a"] & \mathcal{A}^u_n \arrow[d, "\alpha"] \\ PwB_n \arrow[r, "Z^w"] & \mathcal{A}^w_n \end{tikzcd}$$

More specifically, for any choice c of a "parenthesization" of n points, the KZ-construction / Kontsevich integral (see for example [BN3]) returns an expansion Z^u_c of u-braids. As we shall see in Proposition 6.15, for any choice of c, the two compositions $\alpha \circ Z^u_c$ and $Z^w \circ a$ are "conjugate in a bigger space": there is a map i from $\mathcal{A}^w$ to a larger space of "non-horizontal arrow diagrams", and in this space the images of the above composites are conjugate. However, we are not certain that i is an injection, and whether the conjugation leaves the i-image of $\mathcal{A}^w$ invariant, and so we do not know if the two compositions differ merely by an outer automorphism of $\mathcal{A}^w$.

3. w-Knots

Section Summary. In 3.1 we define v-knots and w-knots (long v-knots and long w-knots, to be precise) and discuss the map $v \to w$. In 3.2 we determine the space of "chord diagrams" for w-knots to be the space $\mathcal{A}^w(\uparrow)$ of arrow diagrams modulo $\overrightarrow{4T}$ and TC relations and in 3.3 we compute some relevant dimensions. In 3.5 we show that $\mathcal{A}^w(\uparrow)$ can be re-interpreted as a space of trivalent graphs modulo STU- and IHX-like relations, and is therefore related to Lie algebras (Sec. 3.6). This allows us to completely determine $\mathcal{A}^w(\uparrow)$. With no difficulty at all in 3.4 we construct a universal finite type invariant for w-knots. With a bit of further difficulty we show in Sec. 3.7 that it is essentially equal to the Alexander polynomial.

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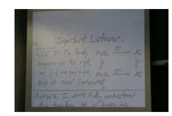

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Knots are the wrong objects for study in knot theory, v-knots are the wrong objects for study in the theory of v-knotted objects and w-knots are the wrong objects for study in the theory of w-knotted objects. Studying uvw-knots on their own is the parallel of studying cakes and pastries as they come out of the bakery — we sure want to make them our own, but the theory of desserts is more about the ingredients and how they are put together than about the end products. In algebraic knot theory this reflects through the fact that knots are not finitely generated in any sense (hence they must be made of some more basic ingredients), and through the fact that there are very few operations defined on knots (connected sums and satellite operations being the main exceptions), and thus most interesting properties of knots are transcendental, or non-algebraic, when viewed from within the algebra of knots and operations on knots [BN8].

The right objects for study in knot theory, or v-knot theory or w-knot theory, are thus the ingredients that make up knots and that permit a richer algebraic structure. These are braids, studied in the previous section, and even more so tangles and tangled graphs, studied in the following sections. Yet tradition has its place and the sweets are tempting, and we feel compelled to introduce some of the tools we will use in the deeper and healthier study of w-tangles and w-tangled foams in the limited but tasty arena of the baked goods of knot theory, the knots themselves.

3.1. v-Knots and w-Knots.

v-Knots may be understood either as knots drawn on surfaces modulo the addition or removal of empty handles [Ka2, Kup] or as "Gauss diagrams" (Remark 3.4), or simply "unembedded but wired together" crossings modulo the Reidemeister moves ([Ka2, Rou] and Section 4.4). But right now we forgo the topological and the abstract and give only the "planar" (and somewhat less philosophically satisfying) definition of v-knots.

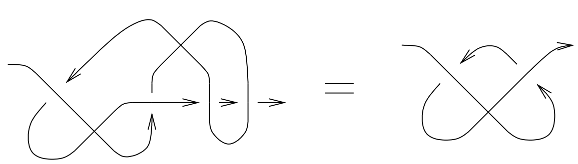

Figure 5. A long v-knot diagram with 2 virtual crossings, 2 positive crossings and 2 negative crossings. A positive-negative pair can easily be cancelled using R2, and then a virtual crossing can be cancelled using VR1, and it seems that the rest cannot be simplified any further.

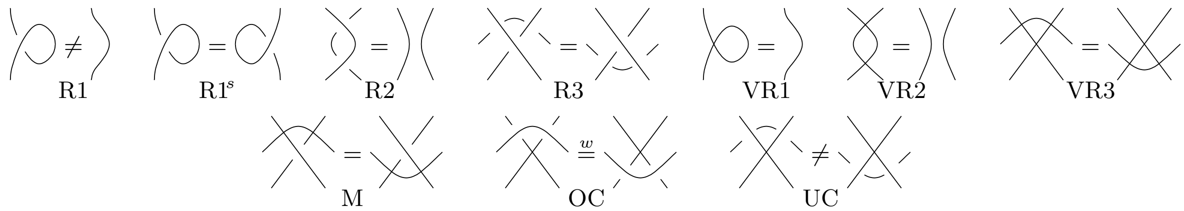

Figure 6. The relations defining v-knots and w-knots, along with two relations that are *not* imposed.

Definition 3.1. A "long v-knot diagram" is an arc smoothly drawn in the plane from $-\infty$ to $+\infty$, with finitely many self-intersections, divided into "virtual crossings" ⤫ and over- and under-crossings, ⤱ and ⤲, and regarded up to planar isotopy. A picture is worth more than a more formal definition, and one appears in Figure 5. A "long v-knot" is an equivalence class of long v-knot diagrams, modulo the equivalence generated by the Reidemeister 1^s, 2 and 3 moves (R1^s, R2 and R3)[17], the virtual Reidemeister 1 through 3 moves (VR1, VR2, VR3), and by the mixed relations (M); all these are shown in Figure 6. Finally, "long w-knots" are obtained from long v-knots by also dividing by the Overcrossings Commute (OC) relations, also shown in Figure 6. Note that we never mod out by the Reidemeister 1 (R1) move nor by the Undercrossings Commute relation (UC).

Definition and Warning 3.2. A "circular v-knot" is like a long v-knot, except parametrized by a circle rather than by a long line. Unlike the case of ordinary knots, circular v-knots are **not** equivalent to long v-knots. The same applies to w-knots.

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Definition and Warning 3.3. Long v-knots form a monoid using the concatenation operation #. Unlike the case of ordinary knots, the resulting monoid is **not** Abelian. The same applies to w-knots.

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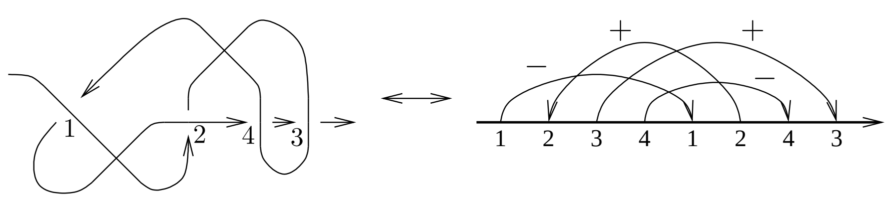
Remark 3.4. A "Gauss diagram" is a straight "skeleton line" along with signed directed chords (signed "arrows") marked along it (more at [Ka2, GPV]). Gauss diagrams are in an obvious bijection with long v-knot diagrams; the skeleton line of a Gauss diagram corresponds to the parameter space of the v-knot, and the arrows correspond to the crossings, with each arrow heading from the upper strand to the lower strand, marked by the sign of the relevant crossing:



One may also describe the relations in Figure 6 as well as circular v-knots and other types of v-knots (as we will encounter later) in terms of Gauss diagrams with varying skeletons.

[17] R1^s is the "spun" version of R1 — kinks can be spun around, but not removed outright. See Figure 6.

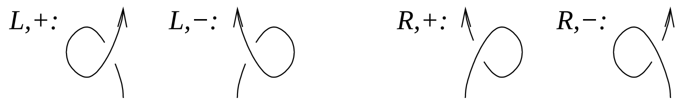


Figure 7. The positive and negative under-then-over kinks (left), and the positive and negative over-then-under kinks (right). In each pair the negative kink is the #-inverse of the positive kink.

Remark 3.5. Since we do not mod out by R1, it is perhaps more appropriate to call our class of v/w-knots "framed long v/w-knots", but since we care more about framed v/w-knots than about unframed ones, we reserve the unqualified name for the framed case, and when we do wish to mod out by R1 we will explicitly write "unframed long v/w-knots".

Recall that in the case of "usual knots", or u-knots, dropping the R1 relation altogether also results in a $\mathbb{Z}^2$-extension of unframed knot theory, where the two factors of $\mathbb{Z}$ are framing and rotation number. If one wants to talk about "true" framed knots, one mods out by the spun Reidemeister 1 relation (R1^s of Figure 6), which preserves the blackboard framing but does not preserve the rotation number. We take the analogous approach here, including the R1^s relation but not R1 also in the v and w cases.

This said, note that the monoid of long v-knots is just a central extension by $\mathbb{Z}$ of the monoid of unframed long v-knots, and so studying the framed case is not very different from studying the unframed case. Indeed the four "kinks" of Figure 7 generate a central $\mathbb{Z}$ within long v-knots, and it is not hard to show that the sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \{\text{long v-knots}\} \longrightarrow \{\text{unframed long v-knots}\} \longrightarrow 1 \tag{17}$$

is split and exact. The same can be said for w-knots.

Exercise 3.6. Show that a splitting of the sequence (17) is given by the "self-linking" invariant $sl\colon \{\text{long v-knots}\} \to \mathbb{Z}$ defined by

$$sl(K) := \sum_{\substack{\text{crossings}\\ x \text{ in } K}} \operatorname{sign} x,$$

where K is a v-knot diagram, and the sign of a crossing x is defined so as to agree with the signs in Figure 7.

Remark 3.7. w-Knots are strictly weaker than v-knots — a notorious example is the Kishino knot (e.g. [Dye]) which is non-trivial as a v-knot yet both it and its mirror are trivial as w-knots. Yet ordinary knots inject even into w-knots, as the Wirtinger presentation makes sense for w-knots and therefore w-knots have a "fundamental quandle" which generalizes the fundamental quandle of ordinary knots [Ka2], and as the fundamental quandle of ordinary knots separates ordinary knots [Joy].

3.1.1. *A topological construction of Satoh's tubing map.* Following Satoh [Sa] and using the same constructions as in Section 2.2.2, we can map w-knots to ("long") ribbon tubes in $\mathbb{R}^4$ (and the relations in Figure 6 still hold). It is natural to expect that this "tubing" map is an isomorphism; in other words, that the theory of w-knots provides a "Reidemeister framework" for long ribbon tubes in $\mathbb{R}^4$ — that every long ribbon tube is in the image of

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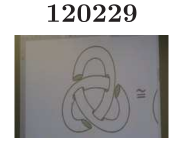


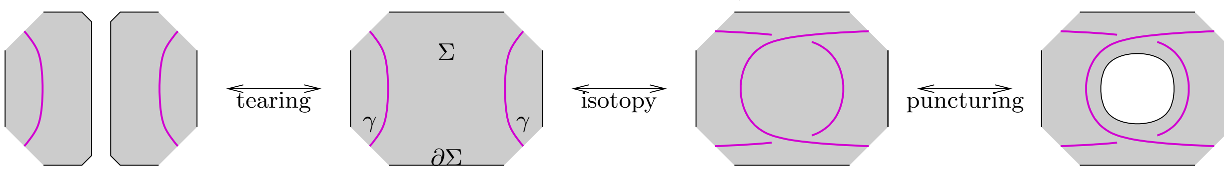
has more pictures, less formalism

this map and that two "w-knot diagrams" represent the same long ribbon tube iff they differ by a sequence of moves as in Figure 6. This remains unproven.

Let $\delta\colon \{\text{v-knots}\} \to \{\text{Ribbon tori in } \mathbb{R}^4\}$ denote the tubing map described in Section 2.2.2. In Satoh's [Sa] δ is called "Tube". It is worthwhile to give a completely "topological" definition of δ. To do this we must start with a topological interpretation of v-knots.

The standard topological interpretation of v-knots (e.g. [Kup]) is that they are oriented framed knots drawn[18] on an oriented surface Σ, modulo "stabilization", which is the addition and/or removal of empty handles (handles that do not intersect with the knot). We prefer an equivalent, yet even more bare-bones approach. For us, a virtual knot is an oriented framed knot γ drawn on a "virtual surface Σ for γ". More precisely, Σ is an oriented surface that may have a boundary, γ is drawn on Σ, and the pair (Σ, γ) is taken modulo the following relations:

- Isotopies of γ on Σ (meaning, in $\Sigma \times [-\epsilon, \epsilon]$).
- Tearing and puncturing parts of Σ away from γ:



(We call Σ a "virtual surface" because tearing and puncturing imply that we only care about it in the immediate vicinity of γ).

We can now define[19] a map δ, defined on v-knots and taking values in ribbon tori in $\mathbb{R}^4$: given (Σ, γ), embed Σ arbitrarily in $\mathbb{R}^3_{xzt} \subset \mathbb{R}^4$. Note that the unit normal bundle of Σ in $\mathbb{R}^4$ is a trivial circle bundle and it has a distinguished trivialization, constructed using its positive-y-direction section and the orientation that gives each fibre a linking number $+1$ with the base Σ. We say that a normal vector to Σ in $\mathbb{R}^4$ is "near unit" if its norm is between $1-\epsilon$ and $1+\epsilon$. The near-unit normal bundle of Σ has as fibre an annulus that can be identified with $[-\epsilon, \epsilon] \times S^1$ (identifying the radial direction $[1-\epsilon, 1+\epsilon]$ with $[-\epsilon, \epsilon]$ in an orientation-preserving manner), and hence the near-unit normal bundle of Σ defines an embedding of $\Sigma \times [-\epsilon, \epsilon] \times S^1$ into $\mathbb{R}^4$. On the other hand, γ is embedded in $\Sigma \times [-\epsilon, \epsilon]$ so $\gamma \times S^1$ is embedded in $\Sigma \times [-\epsilon, \epsilon] \times S^1$, and we can let $\delta(\gamma)$ be the composition

$$\gamma \times S^1 \hookrightarrow \Sigma \times [-\epsilon, \epsilon] \times S^1 \hookrightarrow \mathbb{R}^4,$$

which is a torus in $\mathbb{R}^4$, oriented using the given orientation of γ and the standard orientation of S^1.

A framing of a knot (or a v-knot) γ can be thought of as a "nearby companion" to γ. Applying the above procedure to a knot and a nearby companion simultaneously, we find that δ takes framed v-knots to framed ribbon tori in $\mathbb{R}^4$, where a framing of a tube in $\mathbb{R}^4$ is a continuous up-to-homotopy choice of unit normal vector at every point of the tube. Note that from the perspective of flying rings as in Section 2.2.1 a framing is a "companion ring" to a flying ring. In the framing of $\delta(\gamma)$ the companion ring is never linked with the main ring, but can fly parallel inside, outside, above or below it and change these positions, as shown below.

[18]Here and below, "drawn on Σ" means "embedded in $\Sigma \times [-\epsilon, \epsilon]$".

[19]Following a private discussion with Dylan Thurston.

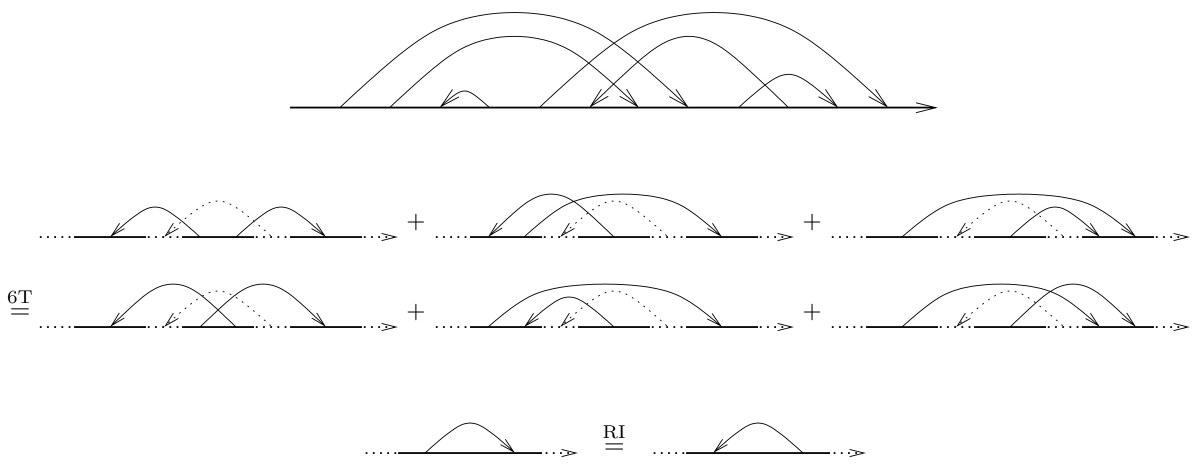


Figure 8. An arrow diagram of degree 6, a 6T relation, and an RI relation.

We leave it to the reader to verify that $\delta(\gamma)$ is ribbon, that it is independent of the choices made within its construction, that it is invariant under isotopies of γ and under tearing and puncturing of Σ, that it is also invariant under the "overcrossings commute" relation of Figure 6 and hence the true domain of δ is w-knots, and that it is equivalent to Satoh's tubing map.

3.2. **Finite Type Invariants of v-Knots and w-Knots.** Much as for v-braids and w-braids (Section 2.3) and much as for ordinary knots (e.g. [BN1]) we define finite type invariants for v-knots and for w-knots using an alternation scheme with $\otimes \to \nearrow\!\!\!\!\nwarrow - \nwarrow\!\!\!\!\nearrow$ and $\otimes \to \nwarrow\!\!\!\!\nearrow - \nearrow\!\!\!\!\nwarrow$. That is, we extend any Abelian-group-valued invariant of v- or w-knots to v- or w-knots also containing "semi-virtual crossings" like $\otimes$ and $\otimes$ using the above assignments, and we declare an invariant to be "of type m" if it vanishes on v- or w-knots with more than m semi-virtuals. As for v- and w-braids and as for ordinary knots, such invariants have an "mth derivative", their "weight system", which is a linear functional on the space $\mathcal{A}^{sv}(\uparrow)$ (for v-knots) or $\mathcal{A}^{sw}(\uparrow)$ (for w-knots). We turn to the definitions of these spaces, following [GPV, BHLR]:

Definition 3.8. An "arrow diagram" is a chord diagram along a long line (called "the skeleton"), in which the chords are oriented (hence "arrows"). An example is in Figure 8. Let $\mathcal{D}^v(\uparrow)$ be the space of formal linear combinations of arrow diagrams. Let $\mathcal{A}^v(\uparrow)$ be $\mathcal{D}^v(\uparrow)$ modulo all "6T relations". Here a 6T relation is any (signed) combination of arrow diagrams obtained from the diagrams in Figure 3 by placing the 3 vertical strands there along a long line in any order, and possibly adding some further arrows in between. An example is in Figure 8. Let $\mathcal{A}^{sv}(\uparrow)$ be the further quotient of $\mathcal{A}^v(\uparrow)$ by the RI relation, where the RI (for Rotation number Independence) relation asserts that an isolated arrow pointing to the right equals an isolated arrow pointing to the left, as shown in Figure 8[20].

[20] The XII relation of [BHLR] follows from RI and need not be imposed.

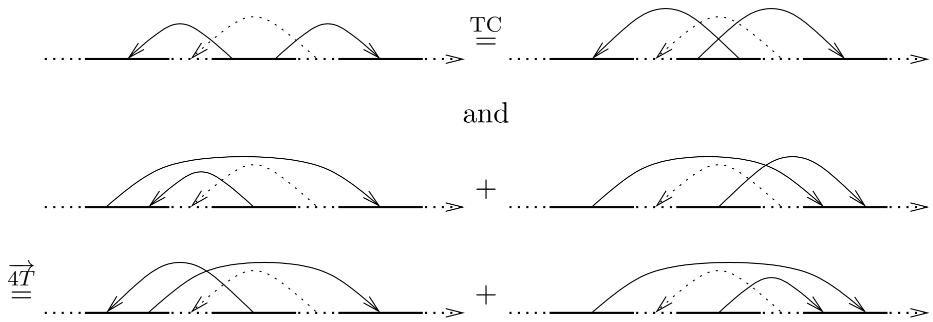


Figure 9. The TC and the $\overrightarrow{4T}$ relations for knots.

Let $\mathcal{A}^w(\uparrow)$ be the further quotient of $\mathcal{A}^v(\uparrow)$ by the "Tails Commute" (TC) relation, first displayed in Figure 4 and reproduced for the case of a long-line skeleton in Figure 9. Likewise, let $\mathcal{A}^{sw}(\uparrow) := \mathcal{A}^{sv}(\uparrow)/TC = \mathcal{A}^w(\uparrow)/RI$. Alternatively, noting that given TC two of the terms in 6T drop out, $\mathcal{A}^w(\uparrow)$ is the space of formal linear combinations of arrow diagrams modulo TC and $\overrightarrow{4T}$ relations, displayed in Figures 4 and 9. Likewise, $\mathcal{A}^{sw} = \mathcal{D}^v/TC, \overrightarrow{4T}, RI$. Finally, grade $\mathcal{D}^v(\uparrow)$ and all of its quotients by declaring that the degree of an arrow diagram is the number of arrows in it.

As an example, the spaces $\mathcal{A}^{v,sv,w,sw}(\uparrow)$ restricted to degrees up to 2 are studied in detail in Section 7.2.

In the same manner as in the theory of finite type invariants of ordinary knots (see especially [BN1, Section 3], the spaces $\mathcal{A}^{-}(\uparrow)$ carry much algebraic structure. The obvious juxtaposition product makes them into graded algebras. The product of two finite type invariants is a finite type invariant (whose type is the sum of the types of the factors); this induces a product on weight systems, and therefore a co-product Δ on arrow diagrams. In brief (and much the same as in the usual finite type story), the co-product ΔD of an arrow diagram D is the sum of all ways of dividing the arrows in D between a "left co-factor" and a "right co-factor". In summary,

Proposition 3.9. $\mathcal{A}^v(\uparrow)$*,* $\mathcal{A}^{sv}(\uparrow)$*,* $\mathcal{A}^w(\uparrow)$*, and* $\mathcal{A}^{sw}(\uparrow)$ *are co-commutative graded bi-algebras.*

By the Milnor-Moore theorem [MM] we find that $\mathcal{A}^{v,sv,w,sw}(\uparrow)$ are the universal enveloping algebras of their Lie algebras of primitive elements. Denote these (graded) Lie algebras by $\mathcal{P}^{v,sv,w,sw}(\uparrow)$, respectively.

When we grow up we'd like to understand $\mathcal{A}^v(\uparrow)$ and $\mathcal{A}^{sv}(\uparrow)$. At the moment we know only very little about these spaces beyond the generalities of Proposition 3.9. In the next section some dimensions of low degree parts of $\mathcal{A}^{v,sv}(\uparrow)$ are displayed. Also, given a finite dimensional Lie bialgebra and a finite dimensional representation thereof, we know how to construct linear functionals on $\mathcal{A}^v(\uparrow)$ (one in each degree) [Hav, Leu] (but not on $\mathcal{A}^{sv}(\uparrow)$). But we don't even know which degree m linear functionals on $\mathcal{A}^{sv}(\uparrow)$ are the weight systems of degree m invariants of v-knots (that is, we have not solved the "Fundamental Problem" [BS] for v-knots).

As we shall see below, the situation is much brighter for $\mathcal{A}^{w,sw}(\uparrow)$.

3.3. **Some Dimensions.** The table below lists what we could find about $\mathcal{A}^v$ and $\mathcal{A}^w$ by crude brute force computations in low degrees. We list degrees 0 through 7. The spaces we study are $\mathcal{A}^-(\uparrow)$, $\mathcal{A}^{s-}(\uparrow)$, $\mathcal{A}^{r-}(\uparrow)$ which is $\mathcal{A}^-(\uparrow)$ moded out by "isolated" arrows [21], $\mathcal{P}^-(\uparrow)$ which is the space of primitives in $\mathcal{A}^-(\uparrow)$, and $\mathcal{A}^-(\bigcirc)$, $\mathcal{A}^{s-}(\bigcirc)$, and $\mathcal{A}^{r-}(\bigcirc)$, which are the same as $\mathcal{A}^-(\uparrow)$, $\mathcal{A}^{s-}(\uparrow)$, and $\mathcal{A}^{r-}(\uparrow)$ except with closed knots (knots with a circle skeleton) replacing long knots. Each of these spaces we study in three variants: the "v" and the "w" variants, as well as the usual knots "u" variant which is here just for comparison. We also include a row "$\dim \mathcal{G}_m \mathcal{L}ie^-(\uparrow)$" for the dimensions of "Lie-algebraic weight systems". Those are explained in the u and v cases in [BN1, Hav, Leu], and in the w case in Section 3.6.

<table>
<tr><th rowspan="2">m</th><th rowspan="2"></th><th colspan="3">See Section 7.2</th><th rowspan="2">3</th><th rowspan="2">4</th><th rowspan="2">5</th><th rowspan="2">6</th><th rowspan="2">7</th><th rowspan="2">Comments</th></tr>
<tr><th>0</th><th>1</th><th>2</th></tr>
<tr><td rowspan="2">$\dim \mathcal{G}_m \mathcal{A}^-(\uparrow)$</td><td>$u \mid v$</td><td>1 | 1</td><td>1 | 2</td><td>2 | 7</td><td>3 | 27</td><td>6 | 139</td><td>10 | 813</td><td>19 |?</td><td>33 |?</td><td>1 | 2</td></tr>
<tr><td>w</td><td>1</td><td>2</td><td>4</td><td>7</td><td>12</td><td>19</td><td>30</td><td>45</td><td>3, 4, 5</td></tr>
<tr><td rowspan="2">$\dim \mathcal{G}_m \mathcal{L}ie^-(\uparrow)$</td><td>$u \mid v$</td><td>1 | 1</td><td>1 | 2</td><td>2 | 7</td><td>3 | 27</td><td>6 | ≥128</td><td>10 |?</td><td>19 |?</td><td>33 |?</td><td>1 | 6</td></tr>
<tr><td>w</td><td>1</td><td>2</td><td>4</td><td>7</td><td>12</td><td>19</td><td>30</td><td>45</td><td>5</td></tr>
<tr><td rowspan="2">$\dim \mathcal{G}_m \mathcal{A}^{s-}(\uparrow)$</td><td>$u \mid v$</td><td>– | 1</td><td>– | 1</td><td>– | 3</td><td>– | 10</td><td>– | 52</td><td>– | 298</td><td>– |?</td><td>– |?</td><td>7 | 2</td></tr>
<tr><td>w</td><td>1</td><td>1</td><td>2</td><td>3</td><td>5</td><td>7</td><td>11</td><td>15</td><td>3, 8</td></tr>
<tr><td rowspan="2">$\dim \mathcal{G}_m \mathcal{A}^{r-}(\uparrow)$</td><td>$u \mid v$</td><td>1 | 1</td><td>0 | 0</td><td>1 | 2</td><td>1 | 7</td><td>3 | 42</td><td>4 | 246</td><td>9 |?</td><td>14 |?</td><td>1 | 9</td></tr>
<tr><td>w</td><td>1</td><td>0</td><td>1</td><td>1</td><td>2</td><td>2</td><td>4</td><td>4</td><td>3, 10</td></tr>
<tr><td rowspan="2">$\dim \mathcal{G}_m \mathcal{P}^-(\uparrow)$</td><td>$u \mid v$</td><td>0 | 0</td><td>1 | 2</td><td>1 | 4</td><td>1 | 15</td><td>2 | 82</td><td>3 | 502</td><td>5 |?</td><td>8 |?</td><td>1 | 11</td></tr>
<tr><td>w</td><td>0</td><td>2</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>3</td></tr>
<tr><td rowspan="2">$\dim \mathcal{G}_m \mathcal{A}^-(\bigcirc)$</td><td>$u \mid v$</td><td>1 | 1</td><td>1 | 1</td><td>2 | 2</td><td>3 | 5</td><td>6 | 19</td><td>10 | 77</td><td>19 |?</td><td>33 |?</td><td>1 | 12</td></tr>
<tr><td>w</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>3</td></tr>
<tr><td rowspan="2">$\dim \mathcal{G}_m \mathcal{A}^{s-}(\bigcirc)$</td><td>$u \mid v$</td><td>– | 1</td><td>– | 1</td><td>– | 1</td><td>– | 2</td><td>– | 6</td><td>– | 23</td><td>– |?</td><td>– |?</td><td>7 | 2</td></tr>
<tr><td>w</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>3</td></tr>
<tr><td rowspan="2">$\dim \mathcal{G}_m \mathcal{A}^{r-}(\bigcirc)$</td><td>$u \mid v$</td><td>1 | 1</td><td>0 | 0</td><td>1 | 0</td><td>1 | 1</td><td>3 | 4</td><td>4 | 17</td><td>9 |?</td><td>14 |?</td><td>1 | 12</td></tr>
<tr><td>w</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>3</td></tr>
</table>

Comments 3.10.

(1) Much more is known computationally on the u-knots case. See especially [BN1, BN4, Kn, AS].
(2) These dimensions were computed by Louis Leung and DBN using a program available at [BND2, "Dimensions"].
(3) As we shall see in Section 3.5, the spaces associated with w-knots are understood to all degrees.
(4) To degree 4, these numbers were also verified by [BND2, "Dimensions"].
(5) The next few numbers in these sequences are 67, 97, 139, 195, 272.
(6) These dimensions were computed by Louis Leung and DBN using a program available at [BND2, "Arrow Diagrams and $gl(N)$"]. Note the match with the row above.
(7) There is no "s" quotient in the "u" case.
(8) The next few numbers in this sequence are 22, 30, 42, 56, 77.
(9) These numbers were computed by [BND2, "Dimensions"]. Contrary to the $\mathcal{A}^u$ case, $\mathcal{A}^{rv}$ is *not* the quotient of $\mathcal{A}^v$ by the ideal generated by degree 1 elements, and

[21] That is, $\mathcal{A}^{r-}(\uparrow)$ is $\mathcal{A}^-(\uparrow)$ modulo "Framing Independence" (FI) relations [BN1], with the isolated arrow taken with either orientation. It is the space related to finite type invariants of unframed knots, on which the first Reidemeister move is also imposed, in the same way as $\mathcal{A}^-(\uparrow)$ is related to framed knots.

therefore the dimensions of the graded pieces of these two spaces cannot be deduced from each other using the Milnor-Moore theorem.

(10) The next few numbers in this sequence are 7,8,12,14,21.

(11) These dimensions were deduced from the dimensions of $\mathcal{G}_m\mathcal{A}^v(\uparrow)$ using the Milnor-Moore theorem.

(12) Computed by [BND2, "Dimensions"]. Contrary to the $\mathcal{A}^u$ case, $\mathcal{A}^v(\bigcirc)$, $\mathcal{A}^{sv}(\bigcirc)$, and $\mathcal{A}^{rv}(\bigcirc)$ are *not* isomorphic to $\mathcal{A}^v(\uparrow)$, $\mathcal{A}^{sv}(\uparrow)$, and $\mathcal{A}^{rv}(\uparrow)$ and separate computations are required.

3.4. **Expansions for w-Knots.** The notion of "an expansion" (or "a universal finite type invariant") for w-knots (or v-knots) is defined in complete analogy with the parallel notion for ordinary knots (e.g. [BN1]), except replacing double points ($\times$) with semi-virtual crossings ($\times$ and $\times$) and replacing chord diagrams by arrow diagrams. Alternatively, it is the same as an expansion for w-braids (Definition 2.11), with the obvious replacement of w-braids by w-knots. Just as in the cases of ordinary knots and/or w-braids, the existence of an expansion $Z\colon \{\text{w-knots}\} \to \mathcal{A}^{sw}(\uparrow)$ is equivalent to the statement "every weight system integrates", i.e., "every degree m linear functional on $\mathcal{A}^{sw}(\uparrow)$ is the mth derivative of a type m invariant of long w-knots".

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Theorem 3.11. *There exists an expansion $Z\colon \{w\text{-}knots\} \to \mathcal{A}^{sw}(\uparrow)$.*

Proof. It is best to define Z by an example, and it is best to display the example only as a picture:

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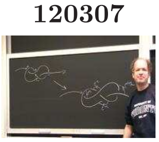

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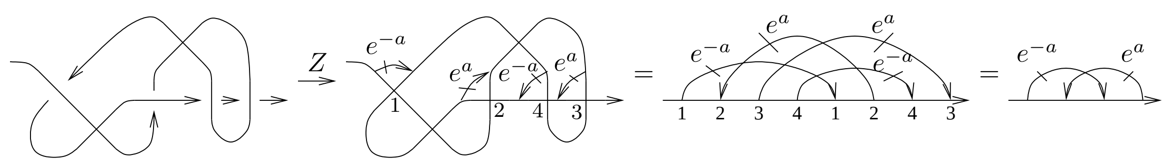
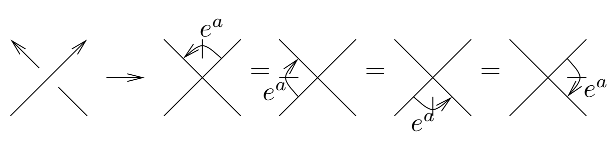


It is clear how to define $Z(K)$ in the general case — for every crossing in K place an exponential reservoir of arrows (compare with (15)) next to that crossing, with the arrows heading from the upper strand to the lower strand, taking positive reservoirs (e^a, with a symbolizing the arrow) for positive crossings and negative reservoirs (e^{-a}) for negative crossings, and then tug the skeleton until it looks like a straight line. Note that the Tails Commute relation in $\mathcal{A}^{sw}$ is used to show that all reasonable ways of placing an arrow reservoir at a crossing (with its heading and sign fixed) are equivalent:



The same proof that shows the invariance of Z in the braids case (Theorem 2.15) works here as well[22], and the same argument as in the braids case shows the universality of Z. ☐

Remark 3.12. Using the language of Gauss diagrams (Remark 3.4) the definition of Z is even simpler. Simply map every positive arrow in a Gauss diagram to a positive (e^a) reservoir,

[22]A tiny bit of extra care is required for invariance under R1^s: it easily follows from RI.

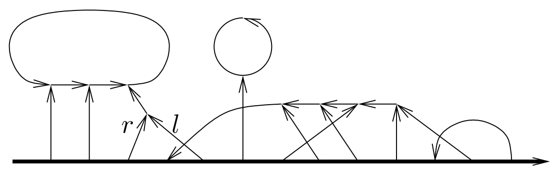
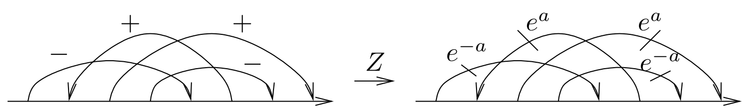


Figure 10. A degree 11 w-Jacobi diagram on a long line skeleton. It has a skeleton line at the bottom, 13 vertices along the skeleton (of which 2 are incoming and 11 are outgoing), 9 internal vertices (with only one explicitly marked with "left" (l) and "right" (r)) and one bubble. The five quadrivalent vertices that seem to appear in the diagram are just projection artifacts and graph-theoretically, they don't exist.

and every negative one to a negative (e^{-a}) reservoir:



An expansion (a universal finite type invariant) is as interesting as its target space, for it is just a tool that takes linear functionals on the target space to finite type invariants on its domain space. The purpose of the next section is to find out how interesting are our present target space, $\mathcal{A}^{sw}(\uparrow)$, and its "parent", $\mathcal{A}^{w}(\uparrow)$.

3.5. **Jacobi Diagrams, Trees and Wheels.** In studying $\mathcal{A}^{w}(\uparrow)$ we again follow the model set by ordinary knots. Compare the following definitions and theorem with [BN1, Section 3].

Definition 3.13. A "w-Jacobi diagram on a long line skeleton"[23] is a connected graph made of the following ingredients:

- A "long" oriented "skeleton" line. We usually draw the skeleton line a bit thicker for emphasis.
- Other directed edges, usually called "arrows".
- Trivalent "skeleton vertices" in which an arrow starts or ends on the skeleton line.
- Trivalent "internal vertices" in which two arrows end and one arrow begins. The internal vertices are "oriented" — of the two arrows that end in an internal vertices, one is marked as "left" and the other is marked as "right". In reality when a diagram is drawn in the plane, we almost never mark "left" and "right", but instead assume the "left" and "right" inherited from the plane, as seen from the outgoing arrow from the given vertex.

Note that we allow multiple arrows connecting the same two vertices (though at most two are possible, given connectedness and trivalence) and we allow "bubbles" — arrows that begin and end in the same vertex. Note that for the purpose of determining equality of diagrams the skeleton line is distinguished. The "degree" of a w-Jacobi diagram is half the number of trivalent vertices in it, including both internal and skeleton vertices. An example of a w-Jacobi diagram is in Figure 10.

[23]What a mouthful! We usually short this to "w-Jacobi diagram", or sometimes "arrow diagram" or just "diagram".

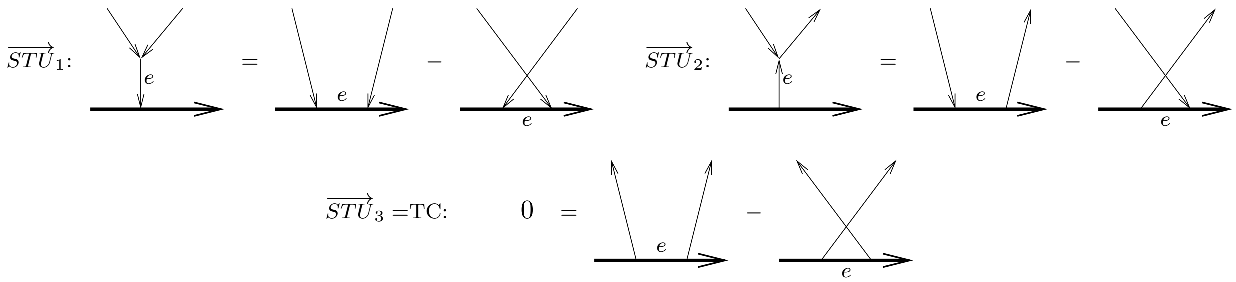


Figure 11. The $\overrightarrow{STU}_{1,2}$ and TC relations with their "central edges" marked e.

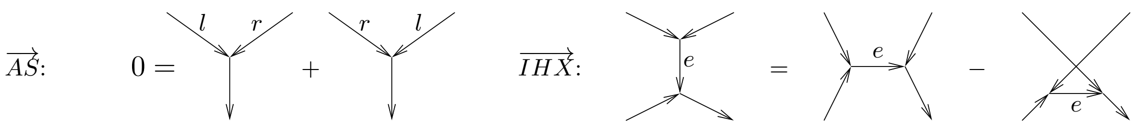


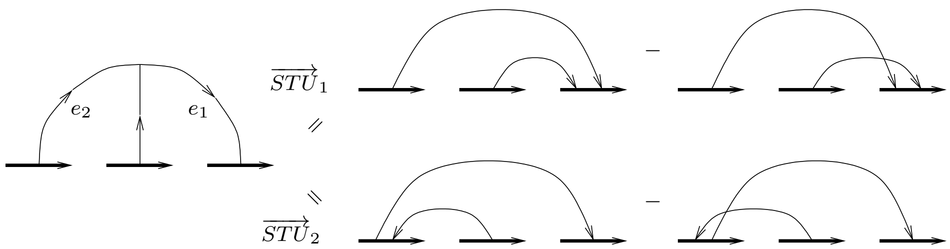
Figure 12. The $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations.

Definition 3.14. Let $\mathcal{D}^{wt}(\uparrow)$ be the graded vector space of formal linear combinations of w-Jacobi diagrams on a long line skeleton, and let $\mathcal{A}^{wt}(\uparrow)$ be $\mathcal{D}^{wt}(\uparrow)$ modulo the $\overrightarrow{STU}_1$, $\overrightarrow{STU}_2$, and TC relations of Figure 11. Note that that each diagram appearing in each $\overrightarrow{STU}$ relation has a "central edge" e which can serve as an "identifying name" for that $\overrightarrow{STU}$. Thus given a diagram D with a marked edge e which is either on the skeleton or which contacts the skeleton, there is an unambiguous $\overrightarrow{STU}$ relation "around" or "along" the edge e.

We like to call the following theorem "the bracket-rise theorem", for it justifies the introduction of internal vertices, and as should be clear from the $\overrightarrow{STU}$ relations and as will become even clearer in Section 3.6, internal vertices can be viewed as "brackets". Two other bracket-rise theorems are Theorem 6 of [BN1] and Ohtsuki's theorem, Theorem 4.9 of [Po].

Theorem 3.15 (bracket-rise)**.** *The obvious inclusion $\iota\colon \mathcal{D}^v(\uparrow) \to \mathcal{D}^{wt}(\uparrow)$ of arrow diagrams (Definition 3.8) into w-Jacobi diagrams descends to the quotient $\mathcal{A}^w(\uparrow)$ and induces an isomorphism $\bar{\iota}\colon \mathcal{A}^w(\uparrow) \xrightarrow{\sim} \mathcal{A}^{wt}(\uparrow)$. Furthermore, the $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations of Figure 12 hold in $\mathcal{A}^{wt}(\uparrow)$.*

Proof. The proof, joint with D. Thurston, is modelled after the proof of Theorem 6 of [BN1]. To show that ι descends to $\mathcal{A}^w(\uparrow)$ we just need to show that in $\mathcal{A}^{wt}(\uparrow)$, $\overrightarrow{4T}$ follows from $\overrightarrow{STU}_{1,2}$. Indeed, applying $\overrightarrow{STU}_1$ along the edge e_1 and $\overrightarrow{STU}_2$ along e_2 in the picture below, we get the two sides of $\overrightarrow{4T}$:



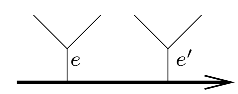
(18)

The fact that $\bar{\iota}$ is surjective is obvious; indeed, for diagrams in $\mathcal{A}^{wt}(\uparrow)$ that have no internal vertices there is nothing to show, for they are really in $\mathcal{A}^{w}(\uparrow)$. Further, by repeated use of $\overrightarrow{STU}_{1,2}$ relations, all internal vertices in any diagram in $\mathcal{A}^{wt}(\uparrow)$ can be removed (remember that the diagrams in $\mathcal{A}^{wt}(\uparrow)$ are always connected, and in particular, if they have an internal vertex they must have an internal vertex connected by an edge to the skeleton, and the latter vertex can be removed first).

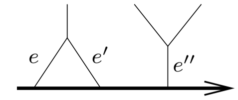
To complete the proof that $\bar{\iota}$ is an isomorphism it is enough to show that the "elimination of internal vertices" procedure of the last paragraph is well defined — that its output is independent of the order in which $\overrightarrow{STU}_{1,2}$ relations are applied in order to eliminate internal vertices. Indeed, this done, the elimination map would by definition satisfy the $\overrightarrow{STU}_{1,2}$ relations and thus descend to a well defined inverse for $\bar{\iota}$.

On diagrams with just one internal vertex, Equation (18) shows that all ways of eliminating that vertex are equivalent modulo $\overrightarrow{4T}$ relations, and hence the elimination map is well defined on such diagrams.

Now assume that we have shown that the elimination map is well defined on all diagrams with at most 7 internal vertices, and let D be a diagram with 8 internal vertices[24]. Let e and e' be edges in D that connect the skeleton of D to an internal vertex. We need to show that any elimination process that begins with eliminating e yields the same answer, modulo $\overrightarrow{4T}$, as any elimination process that begins with eliminating e'. There are several cases to consider.



Case I. e and e' connect the skeleton to *different* internal vertices of D. In this case, after eliminating e we get a signed sum of two diagrams with exactly 7 internal vertices, and since the elimination process is well defined on such diagrams, we may as well continue by eliminating e' in each of those, getting a signed sum of 4 diagrams with 6 internal vertices each. On the other hand, if we start by eliminating e' we can continue by eliminating e, and we get the *same* signed sum of 4 diagrams with 6 internal vertices.



Case II. e and e' are connected to the same internal vertex v of D, yet some other edge e'' exists in D that connects the skeleton of D to some other internal vertex v' in D. In that case, use the previous case and the transitivity of equality: (elimination starting with e)=(elimination starting with e'')=(elimination starting with e').

[24]"7" here is a symbol that denotes an arbitrary natural number greater than 1 and "8" denotes $7+1$.

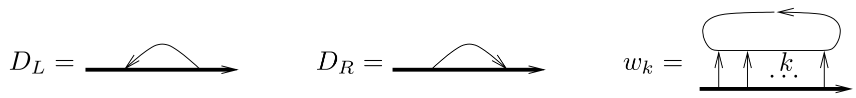
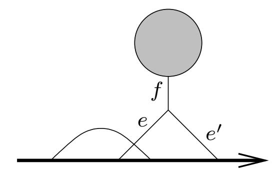
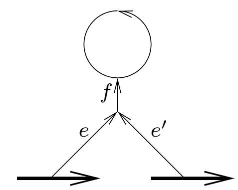


Figure 13. The left-arrow diagram D_L, the right-arrow diagram D_R and the k-wheel w_k.



Case III. Case III is what remains if neither Case I nor Case II hold. In that case, D must have a schematic form as on the right, with the "blob" not connected to the skeleton other than via e or e', yet further arrows may exist outside of the blob. Let f denote the edge connecting the blob to e and e'. The "two in one out" rule for vertices implies that any part of a diagram must have an excess of incoming edges over outgoing edges, equal to the total number of vertices in that diagram part. Applying this principle to the blob, we find that it must contain exactly one vertex, and that f and therefore e and e' must all be oriented upwards.



We leave it to the reader to verify that in this case the two ways of applying the elimination procedure, e and then f or e' and then f, yield the same answer modulo $\overrightarrow{4T}$ (in fact, that answer is 0).

We also leave it to the reader to verify that $\overrightarrow{STU}_1$ implies $\overrightarrow{AS}$ and $\overrightarrow{IHX}$. Algebraically, these are restatements of the anti-symmetry of the bracket and of Jacobi's identity: if $[x,y] := xy - yx$, then $0 = [x,y] + [y,x]$ and $[x,[y,z]] = [[x,y],z] - [[x,z],y]$. $\square$

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Note that $\mathcal{A}^{wt}(\uparrow)$ inherits algebraic structure from $\mathcal{A}^w(\uparrow)$: it is an algebra by concatenation of diagrams, and a co-algebra with $\Delta(D)$, for $D \in \mathcal{D}^{wt}(\uparrow)$, being the sum of all ways of dividing D between a "left co-factor" and a "right co-factor" so that connected components of $D - S$ are kept intact, where S is the skeleton line of D (compare with [BN1, Definition 3.7]).

As $\mathcal{A}^w(\uparrow)$ and $\mathcal{A}^{wt}(\uparrow)$ are canonically isomorphic, from this point on we will not keep the distinction between the two spaces.

One may add the RI relation to the definition of $\mathcal{A}^{wt}(\uparrow)$ to get a space $\mathcal{A}^{swt}(\uparrow)$, or the FI relation to get $\mathcal{A}^{rwt}(\uparrow)$. The statement and proof of the bracket rise theorem adapt with no difficulty, and we find that $\mathcal{A}^{sw}(\uparrow) \cong \mathcal{A}^{swt}(\uparrow)$ and $\mathcal{A}^{rw}(\uparrow) \cong \mathcal{A}^{rwt}(\uparrow)$.

Theorem 3.16. *The bi-algebra $\mathcal{A}^w(\uparrow)$ is the bi-algebra of polynomials in the diagrams D_L, D_R and w_k (for $k \geq 1$) shown in Figure 13, where $\deg D_L = \deg D_R = 1$ and $\deg w_k = k$, subject to the one relation $w_1 = D_L - D_R$. Thus $\mathcal{A}^w(\uparrow)$ has two generators in degree 1 and one generator in every degree greater than 1, as stated in Section 3.3.*

Proof. (sketch). Readers familiar with the diagrammatic PBW theorem [BN1, Theorem 8] will note that it has an obvious analogue for the $\mathcal{A}^w(\uparrow)$ case, and that the proof in [BN1] carries through almost verbatim. Namely, the space $\mathcal{A}^w(\uparrow)$ is isomorphic to a space $\mathcal{B}^w$ of "unitrivalent diagrams" with symmetrized univalent ends modulo $\overrightarrow{AS}$ and $\overrightarrow{IHX}$. Given the "two in one out" rule for arrow diagrams in $\mathcal{A}^w(\uparrow)$ (and hence in $\mathcal{B}^w$) the connected components of diagrams in $\mathcal{B}^w$ can only be trees or wheels. Trees vanish if they have more than one leaf, as their leafs are symmetric while their internal vertices are anti-symmetric, so $\mathcal{B}^w$ is generated by wheels (which become the w_k's in $\mathcal{A}^w(\uparrow)$) and by the one-leaf-one-root tree, which is simply a single arrow, and which becomes the average of D_L and D_R. The relation $w_1 = D_L - D_R$ is then easily verified using $\overrightarrow{STU}_2$.

One may also argue directly, without using sophisticated tools. In short, let D be a diagram in $\mathcal{A}^w(\uparrow)$ and S is its skeleton. Then $D - S$ may have several connected components, whose "legs" are intermingled along S. Using the $\overrightarrow{STU}$ relations these legs can be sorted (at a cost of diagrams with fewer connected components, which could have been treated earlier in an inductive proof). At the end of the sorting procedure one can see that the only diagrams that remain are our declared generators. It remains to show that our generators are linearly independent (apart for the relation $w_1 = D_L - D_R$). For the generators in degree 1, simply write everything out explicitly in the spirit of Section 7.2.2. In higher degrees there is only one primitive diagram in each degree, so it is enough to show that $w_k \neq 0$ for every k. This can be done "by hand", but it is more easily done using Lie algebraic tools in Section 3.6. □

Exercise 3.17. Show that the bi-algebra $\mathcal{A}^{rw}(\uparrow)$ (see Section 3.3) is the bi-algebra of polynomials in the wheel diagrams w_k ($k \geq 2$), and that $\mathcal{A}^{sw}(\uparrow)$ is the bi-algebra of polynomials in the same wheel diagrams and an additional generator $D_A := D_L = D_R$.

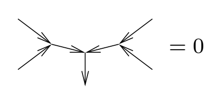
Theorem 3.18. *In $\mathcal{A}^w(\bigcirc)$ all wheels vanish and hence the bi-algebra $\mathcal{A}^w(\bigcirc)$ is the bi-algebra of polynomials in a single variable $D_L = D_R$.*

Proof. This is Lemma 2.7 of [Na]. In short, a wheel in $\mathcal{A}^w(\bigcirc)$ can be reduced using $\overrightarrow{STU}_2$ to a difference of trees. One of these trees has two adjoining leafs and hence is 0 by TC and $\overrightarrow{AS}$. In the other two of the leafs can be commuted "around the circle" using TC until they are adjoining and hence vanish by TC and $\overrightarrow{AS}$. A picture is worth a thousand words, but sometimes it takes up more space. □

Exercise 3.19. Show that $\mathcal{A}^{sw}(\bigcirc) \cong \mathcal{A}^w(\bigcirc)$ yet $\mathcal{A}^{rw}(\bigcirc)$ vanishes except in degree 0.

The following two exercises may help the reader to develop a better "feel" for $\mathcal{A}^w(\uparrow)$ and will be needed, within the discussion of the Alexander polynomial (especially within Definition 3.32).

Exercise 3.20. Show that the "Commutators Commute" (CC) relation, shown on the right, holds in $\mathcal{A}^w(\uparrow)$. (Interpreted in Lie algebras as in the next section, this relation becomes $[[x, y], [z, w]] = 0$, and hence the name "Commutators Commute"). Note that the proof of CC depends on the skeleton having a single component; later, when we will work with $\mathcal{A}^w$-spaces with more complicated skeleta, the CC relation will not hold.



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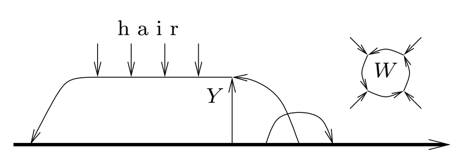
Exercise 3.21. Show that "detached wheels" and "hairy Y's" make sense in $\mathcal{A}^w(\uparrow)$. As on the right, a detached wheel is a wheel with a number of spokes, and a hairy Y is a combinatorial Y shape with further "hair" on its trunk (its outgoing arrow). It is specified where the trunk and the leafs of the Y connect to the skeleton, but it is not specified where the spokes of the wheel and where the hair on the Y connect to the skeleton. The content of the exercise is to show that modulo the relations of $\mathcal{A}^w(\uparrow)$, it is not necessary to specify this further information: all ways of connecting the



spokes and the hair to the skeleton are equivalent. Like the previous exercise, this result depends on the skeleton having a single component.

Remark 3.22. In the case of classical knots and classical chord diagrams, Jacobi diagrams have a topological interpretation using the Goussarov-Habiro calculus of claspers [Gou2, Hab]. In the w case a similar such calculus was developed by Watanabe in [Wa]. Various related results are at [HKS, HS].

3.6. **The Relation with Lie Algebras.** The theory of finite type invariants of knots is related to the theory of metrized Lie algebras via the space $\mathcal{A}$ of chord diagrams, as explained in [BN1, Theorem 4, Exercise 5.1]. In a similar manner the theory of finite type invariants of w-knots is related to arbitrary finite-dimensional Lie algebras (or equivalently, to doubles of co-commutative Lie bialgebra) via the space $\mathcal{A}^w(\uparrow)$ of arrow diagrams.

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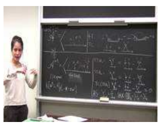

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3.6.1. *Preliminaries.* Given a finite dimensional Lie algebra $\mathfrak{g}$ let $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$ be the semi-direct product of the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ with $\mathfrak{g}$, with $\mathfrak{g}^*$ taken as an Abelian algebra and with $\mathfrak{g}$ acting on $\mathfrak{g}^*$ by the usual coadjoint action. In formulae,

$$I\mathfrak{g} = \{(\varphi, x)\colon\ \varphi \in \mathfrak{g}^*,\, x \in \mathfrak{g}\},$$
$$[(\varphi_1, x_1), (\varphi_2, x_2)] = (x_1\varphi_2 - x_2\varphi_1, [x_1, x_2]).$$

In the case where $\mathfrak{g}$ is the algebra $so(3)$ of infinitesimal symmetries of $\mathbb{R}^3$, its dual $\mathfrak{g}^*$ is itself $\mathbb{R}^3$ with the usual action of $so(3)$ on it, and $I\mathfrak{g}$ is the algebra $\mathbb{R}^3 \rtimes so(3)$ of infinitesimal affine isometries of $\mathbb{R}^3$. This is the Lie algebra of the Euclidean group of isometries of $\mathbb{R}^3$, which is often denoted $ISO(3)$. This explains our choice of the name $I\mathfrak{g}$.

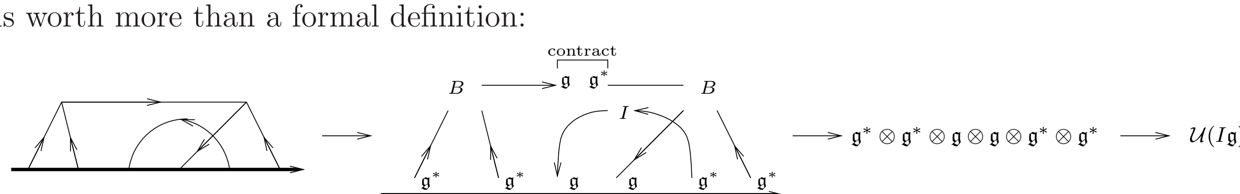
Note that if $\mathfrak{g}$ is a co-commutative Lie bialgebra then $I\mathfrak{g}$ is the "double" of $\mathfrak{g}$ [Dr1]. This is a significant observation, for it is a part of the relationship between this paper and the Etingof-Kazhdan theory of quantization of Lie bialgebras [EK]. Yet we will make no explicit use of this observation below.

In the construction that follows we are going to construct a map from $\mathcal{A}^w$ to $\mathcal{U}(I\mathfrak{g})$, the universal enveloping algebra of $I\mathfrak{g}$. Note that a map $\mathcal{A}^w \to \mathcal{U}(I\mathfrak{g})$ is "almost the same" as a map $\mathcal{A}^{sw} \to \mathcal{U}(I\mathfrak{g})$, in the following sense. There is an obvious quotient map $p\colon \mathcal{A}^w \to \mathcal{A}^{sw}$, and p has a one-sided inverse $F\colon \mathcal{A}^{sw} \to \mathcal{A}^w$ defined by

$$F(D) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} S_L^k(D) \cdot w_1^k.$$

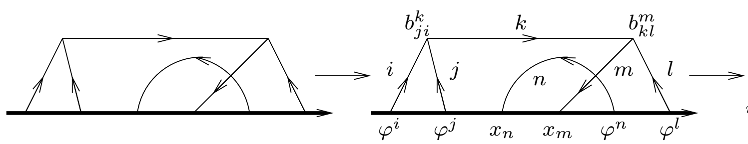
Here S_L denotes the map that sends an arrow diagram to the sum of all ways of deleting a left-going arrow, and w_1 denotes the 1-wheel, as shown in Figure 13. The reader can verify that F is well-defined, an algebra- and co-algebra homomorphism, and that $p \circ F = id_{\mathcal{A}^{sw}}$.

3.6.2. *The Construction.* Fixing a finite dimensional Lie algebra $\mathfrak{g}$ we construct a map $\mathcal{T}^w_{\mathfrak{g}}\colon \mathcal{A}^w \to \mathcal{U}(I\mathfrak{g})$ which assigns to every arrow diagram D an element of the universal enveloping algebra $\mathcal{U}(I\mathfrak{g})$. As is often the case in our subject, a picture of a typical example is worth more than a formal definition:



In short, we break up the diagram D into its constituent pieces and assign a copy of the structure constants tensor $B \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ to each internal vertex v of D (keeping an association between the tensor factors in $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ and the edges emanating from v, as dictated by the orientations of the edges and of the vertex v itself). We assign the identity tensor in $\mathfrak{g}^* \otimes \mathfrak{g}$ to every arrow in D that is not connected to an internal vertex, and contract any pair of factors connected by a fully internal arrow. The remaining tensor factors ($\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}^* \otimes \mathfrak{g}^*$ in our examples) are all along the skeleton and can thus be ordered by the skeleton. We then multiply these factors to get an output $\mathcal{T}^w_\mathfrak{g}(D)$ in $\mathcal{U}(I\mathfrak{g})$.

It is also useful to restate this construction given a choice of a basis. Let (x_j) be a basis of $\mathfrak{g}$ and let (φ^i) be the dual basis of $\mathfrak{g}^*$, so that $\varphi^i(x_j) = \delta^i_j$, and let b^k_{ij} denote the structure constants of $\mathfrak{g}$ in the chosen basis: $[x_i, x_j] = \sum b^k_{ij} x_k$. Mark every arrow in D with lower case Latin letter from within $\{i, j, k, \dots\}$[25]. Form a product P_D by taking one $b^\gamma_{\alpha\beta}$ factor for each internal vertex v of D using the letters marking the edges around v for α, β and γ and by taking one x_α or φ^β factor for each skeleton vertex of D, taken in the order that they appear along the skeleton, with the indices α and β dictated by the edge markings and with the choice between factors in $\mathfrak{g}$ and factors in $\mathfrak{g}^*$ dictated by the orientations of the edges. Finally let $\mathcal{T}^w_\mathfrak{g}(D)$ be the sum of P_D over the indices $i, j, k, \dots$ running from 1 to $\dim \mathfrak{g}$:



$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim \mathfrak{g}} b^k_{ij} b^m_{kl} \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g}) \qquad (19)$$

The following is easy to verify (compare with [BN1, Theorem 4, Exercise 5.1]):

Proposition 3.23. *The above two definitions of $T^w_\mathfrak{g}$ agree, are independent of the choices made within them, and respect all the relations defining $\mathcal{A}^w$.* $\square$

While we do not provide a proof of this proposition here, it is worthwhile to state the correspondence between the relations defining $\mathcal{A}^w$ and the Lie algebraic information in $\mathcal{U}(I\mathfrak{g})$: $\overrightarrow{AS}$ is the antisymmetry of the bracket of $\mathfrak{g}$, $\overrightarrow{IHX}$ is the Jacobi identity of $\mathfrak{g}$, $\overrightarrow{STU}_1$ and $\overrightarrow{STU}_2$ are the relations $[x_i, x_j] = x_i x_j - x_j x_i$ and $[\varphi^i, x_j] = \varphi^i x_j - x_j \varphi^i$ in $\mathcal{U}(I\mathfrak{g})$, TC is the fact that $\mathfrak{g}^\star$ is taken as an Abelian algebra, and $\overrightarrow{4T}$ is the fact that the identity tensor in $\mathfrak{g}^* \otimes \mathfrak{g}$ is $\mathfrak{g}$-invariant.

3.6.3. *Example: The 2 Dimensional Non-Abelian Lie Algebra.* Let $\mathfrak{g}$ be the Lie algebra with two generators $x_{1,2}$ satisfying $[x_1, x_2] = x_2$, so that the only non-vanishing structure constants b^k_{ij} of $\mathfrak{g}$ are $b^2_{12} = -b^2_{21} = 1$. Let $\varphi^i \in \mathfrak{g}^*$ be the dual basis of x_i; by an easy calculation, we find that in $I\mathfrak{g}$ the element φ^1 is central, while $[x_1, \varphi^2] = -\varphi^2$ and $[x_2, \varphi^2] = \varphi^1$. We calculate $\mathcal{T}^w_\mathfrak{g}(D_L)$, $\mathcal{T}^w_\mathfrak{g}(D_R)$ and $\mathcal{T}^w_\mathfrak{g}(w_k)$ using the "in basis" technique of Equation (19). The outputs of these calculations lie in $\mathcal{U}(I\mathfrak{g})$; we display these results in a PBW basis in which the elements of $\mathfrak{g}^*$ precede the elements of $\mathfrak{g}$:

[25] The supply of these can be made inexhaustible by the addition of numerical subscripts.

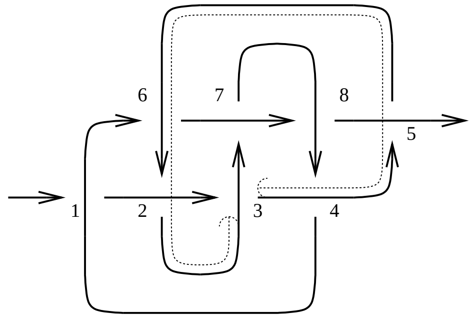
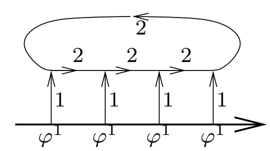


Figure 14. A long 8_{17}, with the span of crossing #3 marked. The projection is as in Brian Sanderson's garden. See [BND2]/`SandersonsGarden.html`.

$$\begin{aligned}
\mathcal{T}^w_{\mathfrak{g}}(D_L) &= x_1\varphi^1 + x_2\varphi^2 = \varphi^1 x_1 + \varphi^2 x_2 + [x_2, \varphi^2] = \varphi^1 x_1 + \varphi^2 x_2 + \varphi_1, \\
\mathcal{T}^w_{\mathfrak{g}}(D_R) &= \varphi^1 x_1 + \varphi^2 x_2, \\
\mathcal{T}^w_{\mathfrak{g}}(w_k) &= (\varphi^1)^k.
\end{aligned} \tag{20}$$



For the last assertion above, note that all non-vanishing structure constants b^k_{ij} in our case have $k = 2$, and therefore all indices corresponding to edges that exit an internal vertex must be set equal to 2. This forces the "hub" of w_k to be marked 2 and therefore the legs to be marked 1, and therefore w_k is mapped to $(\varphi^1)^k$.

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Note that the calculations in (20) are consistent with the relation $D_L - D_R = w_1$ of Theorem 3.16 and that they show that other than that relation, the generators of $\mathcal{A}^w$ are linearly independent.

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3.7. **The Alexander Polynomial.** Let K be a long w-knot, let $Z(K)$ be the invariant of Theorem 3.11. Theorem 3.27 below asserts that apart from self-linking, $Z(K)$ contains precisely the same information as the Alexander polynomial $A(K)$ of K (defined below). But we have to start with some definitions.

Definition 3.24. Enumerate the crossings of K from 1 to n in some arbitrary order. For $1 \leq j \leq n$, the "span" of crossing $\#i$ is the connected open interval along the line parametrizing K between the two times K "visits" crossing $\#i$ (see Figure 14). Form a matrix $T = T(K)$ with T_{ij} the indicator function of "the lower strand of crossing $\#j$ is within the span of crossing $\#i$" (so T_{ij} is 1 if for a given i, j the quoted statement is true, and 0 otherwise). Let s_i be the sign of crossing $\#i$ ($(-,-,-,-,+,+,+,+)$ for Figure 14), let d_i be $+1$ if K visits the "over" strand of crossing $\#i$ before visiting the "under" strand of that crossing, and let $d_i = -1$ otherwise ($(-,+,-,+,-,+,-,+)$ for Figure 14). Let $S = S(K)$ be the diagonal matrix with $S_{ii} = s_i d_i$, and for an indeterminate X, let X^{-S} denote the diagonal matrix with diagonal entries $X^{-s_i d_i}$. Finally, let $A(K)$ be the Laurent polynomial in $\mathbb{Z}[X, X^{-1}]$ given by

$$A(K)(X) := \det\left(I + T(I - X^{-S})\right). \tag{21}$$

Example 3.25. For the knot diagram in Figure 14,

$$T=\begin{pmatrix}0&1&1&1&1&0&1&0\\0&0&1&0&1&0&0&0\\0&1&0&0&1&0&0&0\\0&1&0&0&1&0&1&0\\0&1&0&1&0&1&1&1\\0&1&0&1&0&0&1&0\\0&0&0&1&0&1&0&0\\0&0&0&1&0&1&0&0\end{pmatrix},\quad S=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&-1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&-1&0&0&0&0\\0&0&0&0&-1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&-1&0\\0&0&0&0&0&0&0&1\end{pmatrix},\quad\text{and}\quad A=\begin{vmatrix}1&1-X&1-X^{-1}&1-X&1-X&0&1-X&0\\0&1&1-X^{-1}&0&1-X&0&0&0\\0&1-X&1&0&1-X&0&0&0\\0&1-X&0&1&1-X&0&1-X&0\\0&1-X&0&1-X&1&1-X^{-1}&1-X&1-X^{-1}\\0&1-X&0&1-X&0&1&1-X&0\\0&0&0&1-X&0&1-X^{-1}&1&0\\0&0&0&1-X&0&1-X^{-1}&0&1\end{vmatrix}.$$

The last determinant equals $-X^3+4X^2-8X+11-8X^{-1}+4X^{-2}-X^{-3}$, the Alexander polynomial of the knot 8_{17} (e.g. [Rol]).

Theorem 3.26. *(P. Lee,* [Lee2]*) For any (classical) knot K, $A(K)$ is equal to the normalized Alexander polynomial* [Rol] *of K.* □

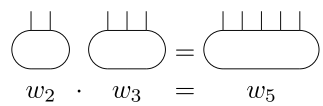
The Mathematica notebook [BND2, "wA"] verifies Theorem 3.26 for all prime knots with up to 11 crossings.

The following theorem asserts that $Z(K)$ can be computed from $A(K)$ (Equation (22)) and that modulo a certain additional relation and with the appropriate identifications in place, $Z(K)$ *is* $A(K)$ (Equation (23)).

Theorem 3.27. *(Proof in Section 3.8). Let x be an indeterminate, let sl be self-linking as in Exercise 3.6, let $D_A := D_L = D_R$ and w_k be as in Figure 13, and let $w\colon \mathbb{Q}[\![x]\!] \to \mathcal{A}^w$ be the linear map defined by $x^k \mapsto w_k$. Then for a w-knot K,*

$$Z(K) = \underbrace{\exp_{\mathcal{A}^{sw}}\left(sl(K)D_A\right)}_{sl\text{ coded in arrows}} \cdot \underbrace{\exp_{\mathcal{A}^{sw}}\left(-w\left(\log_{\mathbb{Q}[\![x]\!]} A(K)(e^x)\right)\right)}_{\text{main part: Alexander coded in wheels}}, \tag{22}$$

where the logarithm and inner exponentiation are computed by formal power series in $\mathbb{Q}[\![x]\!]$ and the outer exponentiations are likewise computed in $\mathcal{A}^{sw}$.



Let $\mathcal{A}^{\text{reduced}}$ be $\mathcal{A}^{sw}$ modulo the additional relations $D_A = 0$ and $w_k w_l = w_{k+l}$ for $k, l \neq 1$. The quotient $\mathcal{A}^{\text{reduced}}$ can be identified with vector space of (infinite) linear combinations of w_k's (with $k \neq 1$). Identifying the k-wheel w_k with x^k, we see that $\mathcal{A}^{\text{reduced}}$ is the space of power series in x having no linear terms. Note by inspecting (21) that $A(K)(e^x)$ never has a term linear in x, and that modulo $w_k w_l = w_{k+l}$, the exponential and the logarithm in (22) cancel each other out. Hence within $\mathcal{A}^{\text{reduced}}$,

$$Z(K) = A^{-1}(K)(e^x). \tag{23}$$

Remark 3.28. In [HKS] K. Habiro, T. Kanenobu, and A. Shima show that all coefficients of the Alexander polynomial are finite type invariants of w-knots, and in [HS] K. Habiro and A. Shima show that all finite type invariants of w-knots are polynomials in the coefficients of the Alexander polynomial. Thus Theorem 3.27 is merely an "explicit form" of these earlier results.

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3.8. **Proof of Theorem 3.27.** We start with a sketch. The proof of Theorem 3.27 can be divided in three parts: differentiation, bulk management, and computation.

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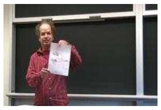

has further background on E, the differential of exp, and the BCH formula.

Differentiation. Both sides of our goal, Equation (22), are exponential in nature. When seeking to show an equality of exponentials it is often beneficial to compare their derivatives[26]. In our case the useful "derivatives" to use are the "Euler operator" E ("multiply every term by its degree", an analogue of $f \mapsto xf'$, defined in Section 3.8.1), and the "normalized Euler operator" $Z \mapsto \tilde{E}Z := Z^{-1}EZ$, which is a variant of the logarithmic derivative $f \mapsto x(\log f)' = xf'/f$. Since $\tilde{E}$ is one to one (Section 3.8.1) and since we know how to apply $\tilde{E}$ to the right hand side of Equation (22) (Section 3.8.1), it is enough to show that with $B := T(\exp(-xS) - I)$ and suppressing the fixed w-knot K from the notation,

$$EZ = Z \cdot \big(sl \cdot D_A - w\big[x \operatorname{tr}\big((I-B)^{-1}TS\exp(-xS)\big)\big]\big) \qquad \text{in } \mathcal{A}^{sw}. \tag{24}$$

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Bulk Management. Next we seek to understand the left hand side of (24). Z is made up of "quantities in bulk": arrows that come in exponential "reservoirs". As it turns out, EZ is made up of the same bulk quantities, but also allowing for a single non-bulk "red excitation" (compare with $Ee^x = x \cdot e^x$; the "bulk" e^x remains, and single "excited red" x gets created). We wish manipulate and simplify that red excitation. This is best done by introducing a certain module, IAM_K, the "Infinitesimal Alexander Module" of K (see Section 3.8.2). The elements of IAM_K can be thought of as names for "bulk objects with a red excitation", and hence there is an "interpretation map" $\iota\colon IAM_K \to \mathcal{A}^{sw}$, which maps every "name" into the object it represents. There are three special elements in IAM_K: an element λ, which is the name of EZ (that is, $\iota(\lambda) = EZ$), the element δ_A which is the name of $D_A \cdot Z$ (so $\iota(\delta_A) = D_A \cdot Z$), and an element ω_1 which is the name of a "detached" 1-wheel that is appended to Z. The latter can take a coefficient which is a power of x, with $\iota(x^k\omega_1) = w(x^{k+1}) \cdot Z = (Z \text{ times a } (k+1)\text{-wheel})$. Thus it is enough to show that in IAM_K,

$$\lambda = sl \cdot \delta_A - \operatorname{tr}\big((I-B)^{-1}TSX^{-S}\big)\,\omega_1, \quad \text{with} \quad X = e^x. \tag{25}$$

Indeed, applying ι to both sides of the above equation, we get Equation (24) back again.
Computation. Last, we show in Section 3.8.3 that (25) holds true. This is a computation that happens entirely in IAM_K and does not mention finite type invariants, expansions or arrow diagrams in any way.

3.8.1. *The Euler Operator.* Let A be a completed graded algebra with unit, in which all degrees are ≥ 0. Define a continuous linear operator $E\colon A \to A$ by setting $Ea = (\deg a)a$ for homogeneous $a \in A$. In the case $A = \mathbb{Q}[\![x]\!]$, we have $Ef = xf'$, the standard "Euler operator", and hence we adopt this name for E in general.

We say that $Z \in A$ is a "perturbation of the identity" if its degree 0 piece is 1. Such a Z is always invertible. For such a Z, set $\tilde{E}Z := Z^{-1} \cdot EZ$, and call the thus (partially) defined operator $\tilde{E}\colon A \to A$ the "normalized Euler operator". From this point on when we write $\tilde{E}Z$ for some $Z \in A$, we automatically assume that Z is a perturbation of the identity or that it is trivial to show that Z is a perturbation of the identity. Note that for $f \in \mathbb{Q}[\![x]\!]$, we have $\tilde{E}f = x(\log f)'$, so $\tilde{E}$ is a variant of the logarithmic derivative.

Claim 3.29. *$\tilde{E}$ is one to one.*

[26]Thanks, Dylan.

Proof. Assume $Z_1 \neq Z_2$ and let d be the smallest degree in which they differ. Then $d > 0$ and in degree d the difference $\tilde{E}Z_1 - \tilde{E}Z_2$ is d times the difference $Z_1 - Z_2$, and hence $\tilde{E}Z_1 \neq \tilde{E}Z_2$. □

Thus in order to prove our goal, Equation (22), it is enough to compute $\tilde{E}$ of both sides and to show the equality then. We start with the right hand side of (22); but first, we need some simple properties of E and $\tilde{E}$. The proofs of these properties are routine and hence they are omitted.

Proposition 3.30. *The following hold true:*

1. *E is a derivation: $E(fg) = (Ef)g + f(Eg)$.*
2. *If Z_1 commutes with Z_2, then $\tilde{E}(Z_1Z_2) = \tilde{E}Z_1 + \tilde{E}Z_2$.*
3. *If z commutes with Ez, then $Ee^z = e^z(Ez)$ and $\tilde{E}e^z = Ez$.*
4. *If $w\colon A \to \mathcal{A}$ is a morphism of graded algebras, then it commutes with E and $\tilde{E}$.* □

Let us denote the right hand side of (22) by $Z_1(K)$. Then by the above proposition, remembering (Theorem 3.16) that $\mathcal{A}^{sw}$ is commutative and that $\deg D_A = 1$, we have

$$\tilde{E}Z_1(K) = sl \cdot D_A - w(E \log A(K)(e^x)) = sl \cdot D_A - w\left(x\frac{d}{dx}\log A(K)(e^x)\right).$$

The rest is an exercise in matrices and differentiation. $A(K)$ is a determinant (21), and in general, $\frac{d}{dx}\log\det(M) = \operatorname{tr}\left(M^{-1}\frac{d}{dx}M\right)$. So with $B = T(e^{-xS} - I)$ (so $M = I - B$), we have

$$\tilde{E}Z_1(K) = sl \cdot D_A + w\left(x\operatorname{tr}\left((I-B)^{-1}\frac{d}{dx}B\right)\right) = sl \cdot D_A - w\left(x\operatorname{tr}\left((I-B)^{-1}TSe^{-xS}\right)\right),$$

as promised in Equation (24).

3.8.2. *The Infinitesimal Alexander Module.* Let K be a w-knot diagram. The Infinitesimal Alexander Module IAM_K of K is a certain module made from a certain space IAM_K^0 of pictures "annotating" K with "red excitations" modulo some pictorial relations that indicate how the red excitations can be moved around. The space IAM_K^0 in itself is made of three pieces, or "sectors". The "A sector" in which the excitations are red arrows, the "Y sector" in which the excitations are "red hairy Y-diagrams", and a rank 1 "W sector" for "red hairy wheels". There is an "interpretation map" $\iota\colon IAM_K^0 \to \mathcal{A}^w$ which descends to a well defined (and homonymous) $\iota\colon IAM_K \to \mathcal{A}^w$. Finally, there are some special elements λ and δ_A that live in the A sector of IAM_K^0 and ω_1 that lives in the W sector.

In principle, the description of IAM_K^0 and of IAM_K can be given independently of the interpretation map ι, and there are some good questions to ask about IAM_K (and the special elements in it) that are completely independent of the interpretation of the elements of IAM_K as "perturbed bulk quantities" within $\mathcal{A}^{sw}$. Yet IAM_K is a complicated object and we fear its definition will appear completely artificial without its interpretation. Hence below the two definitions will be woven together.

IAM_K and ι may equally well be described in terms of K or in terms of the Gauss diagram of K (Remark 3.4). For pictorial simplicity, we choose to use the latter; so let $G = G(K)$ be the Gauss diagram of K. It is best to read the following definition while at the same time studying Figure 15.

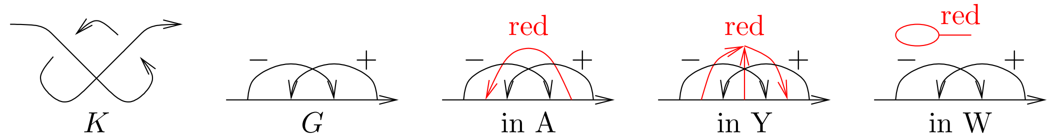

Figure 15. A sample w-knot K, it's Gauss diagram G, and one generator from each of the A, Y, and W sectors of IAM^0_K. Red parts are marked with the word "red".

Definition 3.31. Let R be the ring $\mathbb{Z}[X, X^{-1}]$ of Laurent polynomials in X, and let R_1 be the subring of polynomials that vanish at $X = 1$ (i.e., whose sum of coefficients is 0)[27]. Let IAM^0_K be the direct sum of the following three modules (which for the purpose of taking the direct sum, are all regarded as $\mathbb{Z}$-modules):

(1) The "A sector" is the free $\mathbb{Z}$-module generated by all diagrams made from G by the addition of a single unmarked "red excitation" arrow, whose endpoints are on the skeleton of G and are distinct from each other and from all other endpoints of arrows in G. Such diagrams are considered combinatorially — so two are equivalent iff they differ only by an orientation preserving diffeomorphism of the skeleton. Let us count: if K has n crossings, then G has n arrows and the skeleton of G get subdivided into $m := 2n+1$ arcs. An A sector diagram is specified by the choice of an arc for the tail of the red arrow and an arc for the head (m^2 choices), except if the head and the tail fall within the same arc, their relative ordering has to be specified as well (m further choices). So the rank of the A sector over $\mathbb{Z}$ is $m(m+1)$.

(2) The "Y sector" is the free R_1-module generated by all diagrams made from G by the addition of a single "red excitation" Y-shape single-vertex graph, with two incoming edges ("tails") and one outgoing ("head"), modulo anti-symmetry for the two incoming edges (again, considered combinatorially). Counting is more elaborate: when the three edges of the Y end in distinct arcs in the skeleton of G, we have $\frac{1}{2}m(m-1)(m-2)$ possibilities ($\frac{1}{2}$ for the antisymmetry). When the two tails of the Y lie on the same arc, we get 0 by anti-symmetry. The remaining possibility is to have the head and one tail on one arc (order matters!) and the other tail on another, at $2m(m-1)$ possibilities. So the rank of the Y sector over R_1 is $m(m-1)(\frac{1}{2}m+1)$.

(3) The "W sector" is the rank 1 free R-module with a single generator w_1. It is not necessary for w_1 to have a pictorial representation, yet one, involving a single "red" 1-wheel, is shown in Figure 15.

Definition 3.32. The "interpretation map" $\iota\colon IAM^0_K \to \mathcal{A}^w$ is defined by sending the arrows (marked $+$ or $-$) of a diagram in IAM^0_K to $e^{\pm a}$-exponential reservoirs of arrows, as in the definition of Z (see Remark 3.12). In addition, the red excitations of diagrams in IAM^0_K are interpreted as follows:

(1) In the A sector, the red arrow is simply mapped to itself, with the colour red suppressed.

(2) In the Y sector diagrams have red Y's and coefficients $f \in R_1$. Substitute $X = e^x$ in f, expand in powers of x, and interpret x^kY as a "hairy Y with $k-1$ hairs" as in

[27] R_1 is only very lightly needed, and only within Definition 3.32. In particular, all that we say about IAM_K that does not concern the interpretation map ι is equally valid with R replacing R_1.

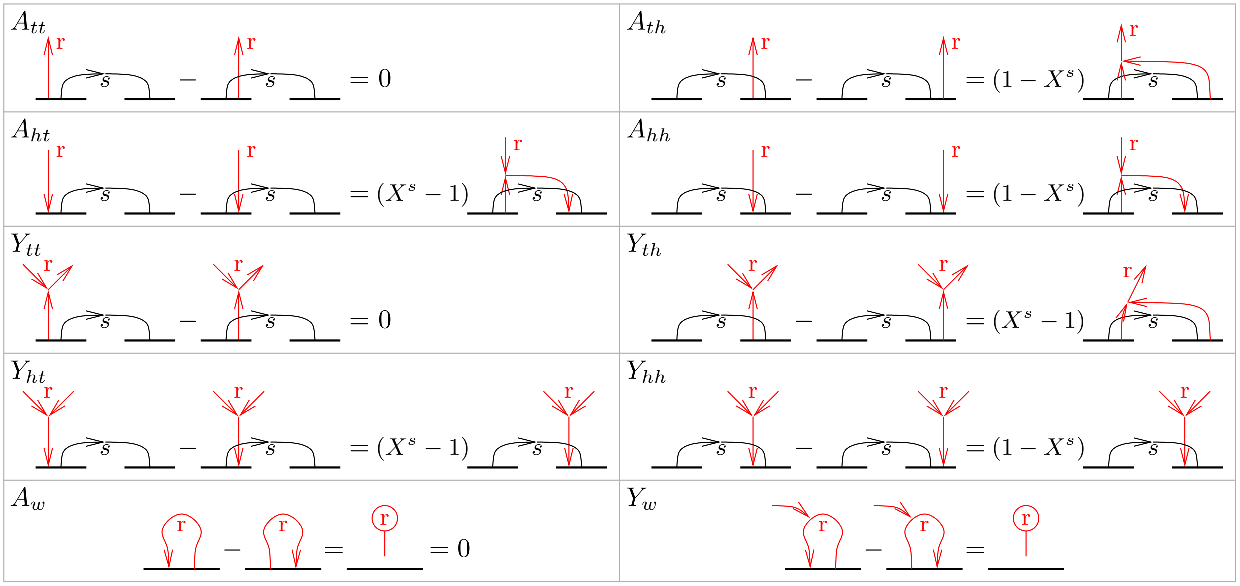

Figure 16. The relations $\mathcal{R}$ making IAM_K.

Exercise 3.21. Note that $f(1) = 0$, so only positive powers of x occur, so we never need to worry about "Y's with -1 hairs". This is the only point where the condition $f \in R_1$ (as opposed to $f \in R$) is needed.

(3) In the W sector treat the coefficients as above, but interpret $x^k w_1$ as a detached w_{k+1}. I.e., as a detached wheel with $k+1$ spokes, as in Exercise 3.21.

As stated above, IAM_K is the quotient of IAM_K^0 by some set of relations. The best way to think of this set of relations is as "everything that's obviously annihilated by ι". Here's the same thing, in a more formal language:

Definition 3.33. Let $IAM_K := IAM_K^0/\mathcal{R}$, where $\mathcal{R}$ is the linear span of the relations depicted in Figure 16. The top 8 relations are about moving a leg of the red excitation across an arrow head or an arrow tail in G. Since the red excitation may be either an arrow (A) or a Y, its leg in motion may be either a tail or a head, and it may be moving either past a tail or past a head, there are 8 relations of that type. The next relation corresponds to $D_L - D_R = w_1 = 0$. The last relation indicates the "price" (always a red w_1) of commuting a red head across a red tail. As per custom, in each case only the changing part of the diagrams involved is shown. Further, the red excitations are marked with the letter "r" and the sign of an arrow in G is marked s; so always $s \in \{\pm 1\}$. The relations in the left column may be multiplied by a scalar in $\mathbb{Z}$, while the relations in the right column may be multiplied by a scalar in R. Hence, for example, $x^0 w_1 = 0$ by A_w, yet $x^k w_1 \neq 0$ for $k > 0$.

Proposition 3.34. *The interpretation map ι indeed annihilates all the relations in $\mathcal{R}$.*

Proof. ιA_{tt} and ιY_{tt} follow immediately from "Tails Commute". The formal identity $e^{\operatorname{ad} b}(a) = e^b a e^{-b}$ implies $e^{\operatorname{ad} b}(a) e^b = e^b a$ and hence $a e^b - e^b a = (1 - e^{\operatorname{ad} b})(a) e^b$. With a interpreted as "red head", b as "black head", and $\operatorname{ad} b$ as "hair" (justified by the ι-meaning of hair and by the $\overrightarrow{STU}_1$ relation, Figure 11), the last equality becomes a proof of ιY_{hh}.

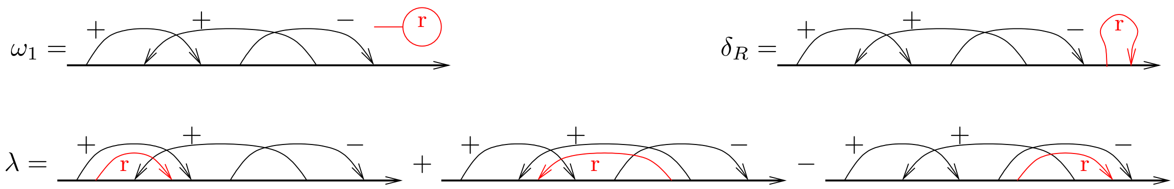

Figure 17. The special elements ω_1, δ_A, and λ in IAM_G, for a sample 3-arrow Gauss diagram G.

Further pushing that same equality, we get $ae^b - e^b a = \frac{1-e^{\operatorname{ad} b}}{\operatorname{ad} b}([b,a])$, where $\frac{1-e^{\operatorname{ad} b}}{\operatorname{ad} b}$ is first interpreted as a power series $\frac{1-e^y}{y}$ involving only non-negative powers of y, and then the substitution $y = \operatorname{ad} b$ is made. But that's ιA_{hh}, when one remembers that ι on the Y sector automatically contains a single "$\frac{1}{\text{hair}}$" factor. Similar arguments, though using $\overrightarrow{STU}_2$ instead of $\overrightarrow{STU}_1$, prove that Y_{ht}, Y_{th}, A_{ht}, and A_{th} are all in $\ker \iota$. Finally, ιA_w is RI, and ιY_w is a direct consequence of $\overrightarrow{STU}_2$. $\square$

Finally, we come to the special elements λ, δ_A, and ω_1.

Definition 3.35. Within IAM_G, let ω_1 be, as before, the generator of the W sector. Let δ_A be a "short" red arrow, as in the A_w relation (exercise: modulo $\mathcal{R}$, this is independent of the placement of the short arrows within G). Finally, let λ be the signed sum of exciting each of the (black) arrows in G in turn. The picture says all, and it is Figure 17.

Proposition 3.36. *In $\mathcal{A}^{sw}(\uparrow)$, the special elements of IAM_G are interpreted as follows: $\iota(\omega_1) = Zw_1$, $\iota(\delta_A) = ZD_A$, and most interesting, $\iota(\lambda) = EZ$. Therefore, Equation* (25) *(if true) implies Equation* (24) *and hence it implies our goal, Theorem 3.27.*

Proof. For the proof of this proposition, the only thing that isn't done yet and isn't trivial is the assertion $\iota(\lambda) = EZ$. But this assertion is a consequence of $Ee^{\pm a} = \pm ae^{\pm a}$ and of a Leibniz law for the derivation E, appropriately generalized to a context where Z can be thought of as a "product" of "arrow reservoirs". The details are left to the reader. $\square$

3.8.3. *The Computation of λ.* Naturally, our next task is to prove Equation (25). This is done entirely algebraically within the finite rank module IAM_G. To read this section one need not know about $\mathcal{A}^{sw}(\uparrow)$, or ι, or Z, but we do need to lay out some notation. Start by marking the arrows of G with a_1 through a_n in some order.

Let ϵ stand for the informal yet useful quantity "a little". Let λ_{ij} denote the difference $\lambda'_{ij} - \lambda''_{ij}$ of red excitations in the A sector of IAM_G, where λ'_{ij} is the diagram with a red arrow whose tail is ϵ to the right of the left end of a_i and whose head is $\frac{1}{2}\epsilon$ away from head of a_j in the direction of the tail of a_j, and where λ''_{ij} has a red arrow whose tail is ϵ to the left of the right end of a_i and whose head is as before, $\frac{1}{2}\epsilon$ away from head of a_j in the direction of the tail of a_j. Let $\Lambda = (\lambda_{ij})$ be the matrix whose entries are the λ_{ij}'s, as shown in Figure 18.

Similarly, let y_{ij} denote the element in the Y sector of IAM_G whose red Y has its head $\frac{1}{2}\epsilon$ away from head of a_j in the direction of the tail of a_j, its right tail (as seen from the head) ϵ to the left of the right end of a_i and its left tail ϵ to the right of the left end of a_i. Let $Y = (y_{ij})$ be the matrix whose entries are the y_{ij}'s, as shown in Figure 18.

Proposition 3.37. *With S and T as in Definition 3.24, and with $B = T(X^{-S} - I)$ and λ as above, the following identities between elements of IAM_G and matrices with entries in*

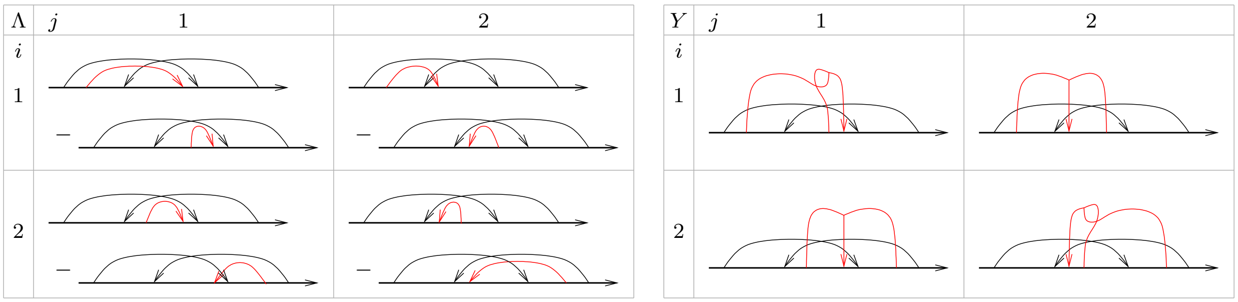

Figure 18. The matrices Λ and Y for a sample 2-arrow Gauss diagram (the signs on a_1 and a_2 are suppressed, and so are the r marks). The twists in y_{11} and y_{22} may be replaced by minus signs.

IAM_G hold true:

$$\begin{aligned} \lambda - sl\cdot D_A &= \operatorname{tr} S\Lambda & (26)\\ \Lambda &= -BY - TX^{-S}w_1 & (27)\\ Y &= BY + TX^{-S}w_1 & (28) \end{aligned}$$

Proof of Equation (25) *given Proposition 3.37.* The last of the equalities above implies that $Y = (I-B)^{-1}TX^{-S}w_1$. Thus

$$\begin{aligned} \lambda - sl\cdot D_A = \operatorname{tr} S\Lambda = -\operatorname{tr} S(BY + TX^{-S}w_1) &= -\operatorname{tr} S(B(I-B)^{-1}TX^{-S} + TX^{-S})w_1 \\ &= -\operatorname{tr}\left((I-B)^{-1}TSX^{-S}\right)w_1, \end{aligned}$$

and this is exactly Equation (25). □

Proof of Proposition 3.37. Equation (26) is trivial. The proofs of Equations (27) and (28) both have the same simple cores, that have to be supplemented by highly unpleasant tracking of signs and conventions and powers of X. Let us start from the cores.

To prove Equation (27) we wish to "compute" $\lambda_{ik} = \lambda'_{ik} - \lambda''_{ik}$. As λ'_{ik} and λ''_{ik} have their heads in the same place, we can compute their difference by gradually sliding the tail of λ'_{ik} from its original position near the left end of a_i towards the right end of a_i, where it would be cancelled by λ''_{ik}. As the tail slides we pick up a y_{jk} term each time it crosses a head of an a_j (relation A_{th}), we pick up a vanishing term each time it crosses a tail (relation A_{tt}), and we pick up a w_1 term if the tail needs to cross over its own head (relation A_w). Ignoring signs and $(X^{\pm 1}-1)$ factors, the sum of the y_{jk}-terms should be proportional to TY, for indeed, the matrix T has non-zero entries precisely when the head of an a_j falls within the span of an a_i. Unignoring these signs and factors, we get $-BY$ (recall that $B = T(X^{-S}-I)$ is just T with added $(X^{\pm 1}-1)$ factors). Similarly, a w_1 term arises in this process when a tail has to cross over its own head, that is, when the head of a_k is within the span of a_i. Thus the w_1 term should be proportional to Tw_1, and we claim it is $-TX^{-S}w_1$.

The core of the proof of Equation (28) is more or less the same. We wish to "compute" y_{ik} by sliding its left leg, starting near the left end of a_i, towards its right leg, which is stationary near the right end of a_i. When the two legs come together, we get 0 because of the anti-symmetry of Y excitations. Along the way we pick up further Y terms from the

Y_{th} relations, and sometimes a w_1 term from the Y_w relation. When all signs and $(X^{\pm 1}-1)$ factors are accounted for, we get Equation (28).

We leave it to the reader to complete the details in the above proofs. It is a major headache, and we would not have trusted ourselves had we not written a computer program to manipulate quantities in IAM_G by a brute force application of the relations in $\mathcal{R}$. Everything checks; see [BND2, "The Infinitesimal Alexander Module"]. □

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This concludes the proof of Theorem 3.27. □

Remark 3.38. We chose the name "Infinitesimal Alexander Module" as in our mind there is some similarity between IAM_K and the "Alexander Module" of K. Yet beyond the above, we did not embark on any serious study of IAM_K. In particular, we do not know if IAM_K in itself is an invariant of K (though we suspect it wouldn't be hard to show that it is), we do not know if IAM_K contains any further information beyond sl and the Alexander polynomial, and we do not know if there is any formal relationship between IAM_K and the Alexander module of K.

Remark 3.39. The logarithmic derivative of the Alexander polynomial also appears in Lescop's [Les1, Les2]. We don't know if its appearances there are related to its appearance here.

3.9. **The Relationship with u-Knots.** Unlike in the case of braids, there is a canonical universal finite type invariant of u-knots: the Kontsevich integral Z^u. So it makes sense to ask how it is related to the expansion Z^w.

$$\begin{tikzcd} \mathcal{K}^u(\uparrow) \arrow[r, "Z^u"] \arrow[d, "a"] & \mathcal{A}^u(\uparrow) \arrow[d, "\alpha"] \\ \mathcal{K}^w(\uparrow) \arrow[r, "Z^w"] & \mathcal{A}^w(\uparrow) \end{tikzcd}$$

We claim that the square on the left commutes, where $\mathcal{K}^u(\uparrow)$ stands for long u-knots (knottings of an oriented line), and similarly $\mathcal{K}^w(\uparrow)$ denotes long w-knots. As before, a is the composition of the maps u-knots $\to$ v-knots $\to$ w-knots, and α is the induced map on the projectivizations, mapping each chord to the sum of the two ways to direct it.

Recall that α kills everything but wheels and arrows. We are going to use the formula for the "wheel part" of the Kontsevich integral as stated in [Kr]. Let K be a 0-framed long knot, and let $A(K)$ denote the Alexander polynomial. Then by [Kr],

$$Z^u(K) = \exp_{\mathcal{A}^u}\left(-\frac{1}{2}\log A(K)(e^h)|_{h^{2n}\to w^u_{2n}}\right) + \text{ "loopy terms"},$$

where w^u_{2n} stands for the unoriented wheel with $2n$ spokes; and "loopy terms" means terms that contain diagrams with more than one loop, which are killed by α. Note that by the symmetry $A(z)=A(z^{-1})$ of the Alexander polynomial, $A(K)(e^h)$ contains only even powers of h, as suggested by the formula.

We need to understand how α acts on wheels. Due to the two-in-one-out rule, a wheel is zero unless all the "spokes" are oriented inward, and the cycle oriented in one direction. In other words, there are two ways to orient an unoriented wheel: clockwise or counterclockwise. Due to the anti-symmetry of chord vertices, we get that for odd wheels $\alpha(w^u_{2h+1})=0$ and for even wheels $\alpha(w^u_{2h})=2w^w_{2h}$. As a result,

$$\alpha Z^u(K) = \exp_{\mathcal{A}^w}\left(-\frac{1}{2}\log A(K)(e^h)|_{h^{2n}\to 2w_{2n}}\right) = \exp_{\mathcal{A}^w}\left(-\log A(K)(e^h)|_{h^{2n}\to w_{2n}}\right)$$

which agrees with the formula (22) of Theorem 3.27. Note that since K is 0-framed, the first part ("sl coded in arrows") of (22) is trivial.

4. Algebraic Structures, Projectivizations, Expansions, Circuit Algebras

Section Summary. In this section we define the "projectivization" (Sec. 4.2) of an arbitrary algebraic structure (4.1) and introduce the notions of "expansions" and "homomorphic expansions" (4.3) for such projectivizations. Everything is so general that practically anything is an example. The baby-example of quandles is built in into the section; the braid groups and w-braid groups appeared already in Section 2, yet our main goal is to set the language for the examples of w-tangles and w-tangled foams, which appear later in this paper. Both of these examples are types of "circuit algebras", and hence we end this section with a general discussion of circuit algebras (Sec. 4.4).

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4.1. **Algebraic Structures.** An "algebraic structure" $\mathcal{O}$ is some collection $(\mathcal{O}_\alpha)$ of sets of objects of different kinds, where the subscript α denotes the "kind" of the objects in $\mathcal{O}_\alpha$, along with some collection of "operations" ψ_β, where each ψ_β is an arbitrary map with domain some product $\mathcal{O}_{\alpha_1} \times \cdots \times \mathcal{O}_{\alpha_k}$ of sets of objects, and range a single set $\mathcal{O}_{\alpha_0}$ (so operations may be unary or binary or multinary, but they always return a value of some fixed kind). We also allow some named "constants" within some $\mathcal{O}_\alpha$'s (or equivalently, allow some 0-nary operations).[28] The operations may or may not be subject to axioms — an "axiom" is an identity asserting that some composition of operations is equal to some other composition of operations.

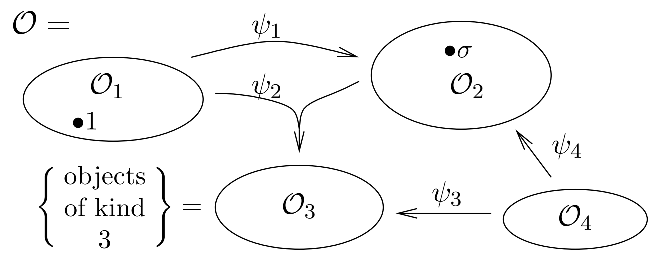

Figure 19. An algebraic structure $\mathcal{O}$ with 4 kinds of objects and one binary, 3 unary and two 0-nary operations (the constants 1 and σ).

Figure 19 illustrates the general notion of an algebraic structure. Here are a few specific examples:

- Groups: one kind of objects, one binary "multiplication", one unary "inverse", one constant "the identity", and some axioms.
- Group homomorphisms: Two kinds of objects, one for each group. 7 operations — 3 for each of the two groups and the homomorphism itself, going between the two groups. Many axioms.
- A group acting on a set, a group extension, a split group extension and many other examples from group theory.
- A quandle. It is worthwhile to quote the abstract of the paper that introduced the definition (Joyce, [Joy]):

  The two operations of conjugation in a group, $x \triangleright y = y^{-1}xy$ and $x \triangleright^{-1} y = yxy^{-1}$ satisfy certain identities. A set with two operations satisfying these

[28] alternatively define "algebraic structures" using the theory of "multicategories" [Lei]. Using this language, an algebraic structure is simply a functor from some "structure" multicategory $\mathcal{C}$ into the multicategory **Set** (or into **Vect**, if all $\mathcal{O}_i$ are vector spaces and all operations are multilinear). A "morphism" between two algebraic structures over the same multicategory $\mathcal{C}$ is a natural transformation between the two functors representing those structures.

identities is called a quandle. The Wirtinger presentation of the knot group involves only relations of the form $y^{-1}xy = z$ and so may be construed as presenting a quandle rather than a group. This quandle, called the knot quandle, is not only an invariant of the knot, but in fact a classifying invariant of the knot.

Also see Definition 4.2.

- Planar algebras as in [Jon] and circuit algebras as in Section 4.4.
- The algebra of knotted trivalent graphs as in [BN8, Da].
- Let $\varsigma\colon B \to S$ be an arbitrary homomorphism of groups (though our notation suggests what we have in mind — B may well be braids, and S may well be permutations). We can consider an algebraic structure $\mathcal{O}$ whose kinds are the elements of S, for which the objects of kind $s \in S$ are the elements of $\mathcal{O}_s := \varsigma^{-1}(s)$, and with the product in B defining operations $\mathcal{O}_{s_1} \times \mathcal{O}_{s_2} \to \mathcal{O}_{s_1 s_2}$.
- Clearly, many more examples appear throughout mathematics.

4.2. **Projectivization.** Any algebraic structure $\mathcal{O}$ has a projectivization. First extend $\mathcal{O}$ to allow formal linear combinations of objects of the same kind (extending the operations in a linear or multi-linear manner), then let $\mathcal{I}$, the "augmentation ideal", be the sub-structure made out of all such combinations in which the sum of coefficients is 0, then let $\mathcal{I}^m$ be the set of all outputs of algebraic expressions (that is, arbitrary compositions of the operations in $\mathcal{O}$) that have at least m inputs in $\mathcal{I}$ (and possibly, further inputs in $\mathcal{O}$), and finally, set

$$\operatorname{proj}\mathcal{O} := \bigoplus_{m\geq 0} \mathcal{I}^m/\mathcal{I}^{m+1}. \tag{29}$$

Clearly, with the operations inherited from $\mathcal{O}$, the projectivization $\operatorname{proj}\mathcal{O}$ is again algebraic structure with the same multi-graph of spaces and operations, but with new objects and with new operations that may or may not satisfy the axioms satisfied by the operations of $\mathcal{O}$. The main new feature in $\operatorname{proj}\mathcal{O}$ is that it is a "graded" structure; we denote the degree m piece $\mathcal{I}^m/\mathcal{I}^{m+1}$ of $\operatorname{proj}\mathcal{O}$ by $\operatorname{proj}_m\mathcal{O}$.

We believe that many of the most interesting graded structures that appear in mathematics are the result of this construction, and that many of the interesting graded equations that appear in mathematics arise when one tries to find "expansions", or "universal finite type invariants", which are also morphisms[29] $Z\colon \mathcal{O} \to \operatorname{proj}\mathcal{O}$ (see Section 4.3) or when one studies "automorphisms" of such expansions[30]. Indeed, the paper you are reading now is really the study of the projectivizations of various algebraic structures associated with w-knotted objects. We would like to believe that much of the theory of quantum groups (at "generic" $\hbar$) will eventually be shown to be a study of the projectivizations of various algebraic structures associated with v-knotted objects.

Thus we believe that the operation described in Equation (29) is truly fundamental and therefore worthy of a catchy name. So why "projectivization"? Well, it reminds us of graded spaces, but really, that's all. We simply found no better name. We're open to suggestions.

[29]Indeed, if $\mathcal{O}$ is finitely presented then finding such a morphism $Z\colon \mathcal{O} \to \operatorname{proj}\mathcal{O}$ amounts to finding its values on the generators of $\mathcal{O}$, subject to the relations of $\mathcal{O}$. Thus it is equivalent to solving a system of equations written in some graded spaces.

[30]The Drinfel'd graded Grothendieck-Teichmuller group GRT is an example of such an automorphism group. See [Dr3, BN6].

Let us end this section with two examples.

Proposition 4.1. *If G is a group,* $\operatorname{proj} G$ *is a graded associative algebra with unit.* $\square$

Definition 4.2. A quandle is a set Q with a binary operation $\uparrow\colon Q \times Q \to Q$ satisfying the following axioms:

(1) $\forall x \in Q$, $x{\uparrow}x = x$.
(2) For any fixed $y \in Q$, the map $x \mapsto x{\uparrow}y$ is invertible[31].
(3) Self-distributivity: $\forall x, y, x \in Q$, $(x{\uparrow}y){\uparrow}z = (x{\uparrow}z){\uparrow}(y{\uparrow}z)$.

We say that a quandle Q has a unit, or is unital, if there is a distinguished element $1 \in Q$ satisfying the further axiom:

(4) $\forall x \in Q$, $x{\uparrow}1 = x$ and $1{\uparrow}x = 1$.

If G is a group, it is also a (unital) quandle by setting $x{\uparrow}y := y^{-1}xy$, yet there are many quandles that do not arise from groups in this way.

Proposition 4.3. *If Q is a unital quandle,* $\operatorname{proj}_0 Q$ *is one-dimensional and* $\operatorname{proj}_{>0} Q$ *is a graded right Leibniz algebra*[32] *generated by* $\operatorname{proj}_1 Q$.

Proof. For any algebraic structure A with just one kind of objects, $\operatorname{proj}_0 A$ is one-dimensional, generated by the equivalence class $[x]$ of any single object x. In particular, $\operatorname{proj}_0 Q$ is one-dimensional and generated by $[1]$. Let $\mathcal{I} \subset \mathbb{Q}Q$ be the augmentation ideal of Q. For any $x \in Q$ set $\bar{x} := x - 1 \in \mathcal{I}$. Then $\mathcal{I}$ is generated by the $\bar{x}$'s, and therefore $\mathcal{I}^m$ is generated by expressions involving the operation $\uparrow$ applied to some m elements of $\bar{Q} := \{\bar{x}\colon x \in Q\}$ and possibly some further elements $y_i \in Q$. When regarded in $\mathcal{I}^m/\mathcal{I}^{m+1}$, any y_i in such a generating expression can be replaced by 1, for the difference would be the same expression with y_i replaced by $\bar{y}_i$, and this is now a member of $\mathcal{I}^{m+1}$. But for any element $z \in \mathcal{I}$ we have $z{\uparrow}1 = z$ and $1{\uparrow}z = 0$, so all the 1's can be eliminated from the expressions generating $\mathcal{I}^m$. Thus $\operatorname{proj}_{>0} Q$ is generated by $\bar{Q}$ and hence by $\operatorname{proj}_1 Q$.

Let $\Delta\colon \mathbb{Q}Q \to \mathbb{Q}Q \otimes \mathbb{Q}Q$ be the linear extension of the operation $x \mapsto x \otimes x$ defined on $x \in Q$, and extend $\uparrow$ to a binary operator $\uparrow_2\colon (\mathbb{Q}Q \otimes \mathbb{Q}Q) \otimes (\mathbb{Q}Q \otimes \mathbb{Q}Q) \to \mathbb{Q}Q \otimes \mathbb{Q}Q$ by using $\uparrow$ twice, to pair the first and third tensor factors and then to pair the second and the fourth tensor factors. With this language in place, the self-distributivity axiom becomes the following *linear* statement, which holds for every $x, y, z \in \mathbb{Q}Q$:

$$(x{\uparrow}y){\uparrow}z = {\uparrow} \circ {\uparrow_2}(x \otimes y \otimes \Delta z). \tag{30}$$

Clearly, we need to understand Δ better. By direct computation, if $x \in Q$ then $\Delta\bar{x} = \bar{x} \otimes 1 + 1 \otimes \bar{x} + \bar{x} \otimes \bar{x}$. We claim that in general, if z is a generating expression of $\mathcal{I}^m$ (that is, a formula made of m elements of $\bar{Q}$ and $m-1$ applications of $\uparrow$), then

$$\Delta z = z \otimes 1 + 1 \otimes z + \sum z_i' \otimes z_i'', \qquad \text{with} \qquad \sum z_i' \otimes z_i'' \in \sum_{\substack{m'+m''=m+1,\\ m',m''>0}} \mathcal{I}^{m'} \otimes \mathcal{I}^{m''}. \tag{31}$$

[31] This can alternatively be stated as "there exists a second binary operation $\uparrow^{-1}$ so that $\forall x$, $x = (x{\uparrow}y){\uparrow}^{-1}y = (x{\uparrow}^{-1}y){\uparrow}y$", so this axiom can still be phrased within the language of "algebraic structures". Yet note that below we do not use this axiom at all.

[32] A Leibniz algebra is a Lie algebra without anticommutativity, as defined by Loday in [Lod].

Indeed, for the generators of $\mathcal{I}^1$ this had just been shown, and if $z = z_1{\uparrow}z_2$ is a generator of $\mathcal{I}^m$, with z_1 and z_2 generators of $\mathcal{I}^{m_1}$ and $\mathcal{I}^{m_2}$ with $1 \leq m_1, m_2 < m$ and $m_1 + m_2 = m$, then (using $w{\uparrow}1 = w$ and $1{\uparrow}w = 0$ for $w \in \mathcal{I}$),

$$\begin{aligned}
\Delta z &= \Delta(z_1{\uparrow}z_2) = (\Delta z_1){\uparrow}_2(\Delta z_2) \\
&= (z_1 \otimes 1 + 1 \otimes z_1 + \sum z'_{1j} \otimes z''_{1j}){\uparrow}_2(z_2 \otimes 1 + 1 \otimes z_2 + \sum z'_{2k} \otimes z''_{2k}) \\
&= (z_1{\uparrow}z_2) \otimes 1 + 1 \otimes (z_1{\uparrow}z_2) \\
&\quad + \sum_j \left((z'_{1j}{\uparrow}z_2) \otimes z''_{1j} + z'_{1j} \otimes (z''_{1j}{\uparrow}z_2) + \sum_k (z'_{1j}{\uparrow}z'_{2k}) \otimes (z''_{1j}{\uparrow}z''_{2k}) \right),
\end{aligned}$$

and it is easy to see that the last line agrees with (31).

We can now combine Equations (30) and (31) to get that for any $x, y, z \in \mathbb{Q}Q$,

$$(x{\uparrow}y){\uparrow}z = (x{\uparrow}z){\uparrow}y + x{\uparrow}(y{\uparrow}z) + \sum (x{\uparrow}z'_i){\uparrow}(y{\uparrow}z''_i).$$

If $x \in \mathcal{I}^{m_1}$, $y \in \mathcal{I}^{m_2}$, and $z \in \mathcal{I}^{m_3}$, then by (31) the last term above is in $\mathcal{I}^{m_1+m_2+m_3+1}$, and so the above identity becomes the Jacobi identity $(x{\uparrow}y){\uparrow}z = (x{\uparrow}z){\uparrow}y + x{\uparrow}(y{\uparrow}z)$ in $\operatorname{proj}_{m_1+m_2+m_3} Q$.

Note that in the above proof neither axiom (1) nor axiom (2) of Definition 4.2 was used.

Exercise 4.4. Show that axiom (1) implies the antisymmetry of $\uparrow$ on $\mathcal{I}^1$.

4.3. **Expansions and Homomorphic Expansions.** We start with the definition. Given an algebraic structure $\mathcal{O}$ let fil $\mathcal{O}$ denote the filtered structure of linear combinations of objects in $\mathcal{O}$ (respecting kinds), filtered by the powers $(\mathcal{I}^m)$ of the augmentation ideal $\mathcal{I}$. Recall also that any graded space $G = \bigoplus_m G_m$ is automatically filtered, by $\left(\bigoplus_{n \geq m} G_n\right)_{m=0}^{\infty}$.

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Definition 4.5. An "expansion" Z for $\mathcal{O}$ is a map $Z\colon \mathcal{O} \to \operatorname{proj} \mathcal{O}$ that preserves the kinds of objects and whose linear extension (also called Z) to fil $\mathcal{O}$ respects the filtration of both sides, and for which $(\operatorname{gr} Z) : (\operatorname{gr} \operatorname{fil} \mathcal{O} = \operatorname{proj} \mathcal{O}) \to (\operatorname{gr} \operatorname{proj} \mathcal{O} = \operatorname{proj} \mathcal{O})$ is the identity map of $\operatorname{proj} \mathcal{O}$.

In practical terms, this is equivalent to saying that Z is a map $\mathcal{O} \to \operatorname{proj} \mathcal{O}$ whose restriction to $\mathcal{I}^m$ vanishes in degrees less than m (in $\operatorname{proj} \mathcal{O}$) and whose degree m piece is the projection $\mathcal{I}^m \to \mathcal{I}^m/\mathcal{I}^{m+1}$.

We come now to what is perhaps the most crucial definition in this paper.

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Definition 4.6. A "homomorphic expansion" is an expansion which also commutes with all the algebraic operations defined on the algebraic structure $\mathcal{O}$.

Why Bother with Homomorphic Expansions? Primarily, for two reasons:

- Often times $\operatorname{proj} \mathcal{O}$ is simpler to work with than $\mathcal{O}$; for one, it is graded and so it allows for finite "degree by degree" computations, whereas often times, such as in many topological examples, anything in $\mathcal{O}$ is inherently infinite. Thus it can be beneficial to translate questions about $\mathcal{O}$ to questions about $\operatorname{proj} \mathcal{O}$. A simplistic example would be, "is some element $a \in \mathcal{O}$ the square (relative to some fixed operation) of an element $b \in \mathcal{O}$?". Well, if Z is a homomorphic expansion and by a finite computation it can be shown that $Z(a)$ is not a square already in degree 7 in $\operatorname{proj} \mathcal{O}$, then we've given a conclusive negative answer to the example question. Some less simplistic and more relevant examples appear in [BN8].

- Often times $\operatorname{proj}\mathcal{O}$ is "finitely presented", meaning that it is generated by some finitely many elements $g_1, \dots, g_k \in \mathcal{O}$, subject to some relations $R_1 \dots R_n$ that can be written in terms of $g_1, \dots, g_k$ and the operations of $\mathcal{O}$. In this case, finding a homomorphic expansion Z is essentially equivalent to guessing the values of Z on $g_1, \dots, g_k$, in such a manner that these values $Z(g_1), \dots, Z(g_k)$ would satisfy the $\operatorname{proj}\mathcal{O}$ versions of the relations $R_1 \dots R_n$. So finding Z amounts to solving equations in graded spaces. It is often the case (as will be demonstrated in this paper; see also [BN3, BN6]) that these equations are very interesting for their own algebraic sake, and that viewing such equations as arising from an attempt to solve a problem about $\mathcal{O}$ sheds further light on their meaning.

In practise, often times the first difficulty in searching for an expansion (or a homomorphic expansion) $Z\colon \mathcal{O} \to \operatorname{proj}\mathcal{O}$ is that its would-be target space $\operatorname{proj}\mathcal{O}$ is hard to identify. It is typically easy to make a suggestion $\mathcal{A}$ for what $\operatorname{proj}\mathcal{O}$ could be. It is typically easy to come up with a reasonable generating set $\mathcal{D}_m$ for $\mathcal{I}^m$ (keep some knot theoretic examples in mind, or the case of quandles as in Proposition 4.3). It is a bit harder but not exceedingly difficult to discover some relations $\mathcal{R}$ satisfied by the elements of the image of $\mathcal{D}$ in $\mathcal{I}^m/\mathcal{I}^{m+1}$ (4T, $\overrightarrow{4T}$, and more in knot theory, the Jacobi relation in Proposition 4.3). Thus we set $\mathcal{A} := \mathcal{D}/\mathcal{R}$; but it is often very hard to be sure that we found everything that ought to go in $\mathcal{R}$; so perhaps our suggestion $\mathcal{A}$ is still too big? Finding 4T, or Jacobi in Proposition 4.3 was actually not *that* easy. Perhaps we missed some further relations that are hiding in $\operatorname{proj} Q$, for example?

The notion of an $\mathcal{A}$-expansion, defined below, solves two problems are once. Once we find an $\mathcal{A}$-expansion we know that we've identified $\operatorname{proj}\mathcal{O}$ correctly, and we automatically get what we really wanted, a $(\operatorname{proj}\mathcal{O})$-valued expansion.

Definition 4.7. A "candidate projectivization" for an algebraic structure $\mathcal{O}$ is a graded structure $\mathcal{A}$ with the same operations as $\mathcal{O}$ along with a homomorphic surjective graded map $\pi\colon \mathcal{A} \to \operatorname{proj}\mathcal{O}$. An "$\mathcal{A}$-expansion" is a kind and filtration respecting map $Z_\mathcal{A}\colon \mathcal{O} \to \mathcal{A}$ for which $(\operatorname{gr} Z_\mathcal{A}) \circ \pi\colon \mathcal{A} \to \mathcal{A}$ is the identity. There's no need to define "homomorphic $\mathcal{A}$-expansions".

$$\begin{tikzcd} & \mathcal{A} \arrow[d, shift right, "\pi"'] \\ \mathcal{O} \arrow[ur, "Z_\mathcal{A}"] \arrow[r, "Z"'] & \operatorname{proj}\mathcal{O} \arrow[u, shift right, "\operatorname{gr} Z_\mathcal{A}"'] \end{tikzcd}$$

Proposition 4.8. *If $\mathcal{A}$ is a candidate projectivization of $\mathcal{O}$ and $Z_\mathcal{A}\colon \mathcal{O} \to \mathcal{A}$ is a homomorphic $\mathcal{A}$-expansion, then $\pi : \mathcal{A} \to \operatorname{proj}\mathcal{O}$ is an isomorphism and $Z := \pi \circ Z_\mathcal{A}$ is a homomorphic expansion. (Often in this case, $\mathcal{A}$ is identified with $\operatorname{proj}\mathcal{O}$ and $Z_\mathcal{A}$ is identified with Z).*

Proof. π is surjective by birth. Since $(\operatorname{gr} Z_\mathcal{A}) \circ \pi$ is the identity, π it is also injective and hence it is an isomorphism. The rest is immediate. □

4.4. **Circuit Algebras.** "Circuit algebras" are so common and everyday, and they make such a useful language (definitely for the purposes of this paper, but also elsewhere), we find it hard to believe they haven't made it into the standard mathematical vocabulary[33]. People familiar with planar algebras [Jon] may note that circuit algebras are just the same

[33]Or have they, and we've been looking the wrong way?

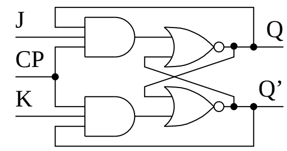

Figure 20. The J-K flip flop, a very basic memory cell, is an electronic circuit that can be realized using 9 components — two triple-input "and" gates, two standard "nor" gates, and 5 "junctions" in which 3 wires connect (many engineers would not consider the junctions to be real components, but we do). Note that the "crossing" in the middle of the figure is merely a projection artifact and does not indicate an electrical connection, and that electronically speaking, we need not specify how this crossing may be implemented in $\mathbb{R}^3$. The J-K flip flop has 5 external connections (labelled J, K, CP, Q, and Q') and hence in the circuit algebra of computer parts, it lives in C_5. In the directed circuit algebra of computer parts it would be in $C_{3,2}$ as it has 3 incoming wires (J, CP, and K) and two outgoing wires (Q and Q').

Figure 21. The circuit algebra product of 4 big black components and 1 small black component carried out using a green wiring diagram, is an even bigger component that has many golden connections (at bottom). When plugged into a yet bigger circuit, the CPU board of a laptop, our circuit functions as 4,294,967,296 binary memory cells.

as planar algebras, except with the planarity requirement dropped from the "connection diagrams" (and all colourings are dropped as well). For the rest, we'll start with an image and then move on to the dry definition.

Image 4.9. Electronic circuits are made of "components" that can be wired together in many ways. On a logical level, we only care to know which pin of which component is connected with which other pin of the same or other component. On a logical level, we don't really need to know how the wires between those pins are embedded in space (see Figures 20 and 21). "Printed Circuit Boards" (PCBs) are operators that make smaller components ("chips") into bigger ones ("circuits") — logically speaking, a PCB is simply a set of "wiring instructions", telling us which pins on which components are made to connect (and again, we never care precisely how the wires are routed provided they reach their intended destinations, and ever since the invention of multi-layered PCBs, all conceivable topologies for wiring are actually realizable). PCBs can be composed (think "plugging a graphics card onto a motherboard"); the result of a composition of PCBs, logically speaking, is simply a larger PCB which takes a larger number of components as inputs and outputs a larger circuit. Finally, it doesn't matter if several PCB are connected together and then the chips are placed on them, or if the chips are placed first and the PCBs are connected later; the resulting overall circuit remains the same.

We start process of drying (formalizing) this image by defining "wiring diagrams", the abstract analogs of printed circuit boards. Let $\mathbb{N}$ denote the set of natural numbers including 0, and for $n \in \mathbb{N}$ let $\underline{n}$ denote some fixed set with n elements, say $\{1, 2, \ldots, n\}$.

Definition 4.10. Let $k, n, n_1, \ldots, n_k \in \mathbb{N}$ be natural numbers. A "wiring diagram" D with inputs $\underline{n_1}, \ldots \underline{n_k}$ and outputs $\underline{n}$ is an unoriented compact 1-manifold whose boundary is $\underline{n} \amalg \underline{n_1} \amalg \cdots \amalg \underline{n_k}$, regarded up to homeomorphism. In strictly combinatorial terms, it is a pairing of the elements of the set $\underline{n} \amalg \underline{n_1} \amalg \cdots \amalg \underline{n_k}$ along with a single further natural number that counts closed circles. If $D_1; \ldots; D_m$ are wiring diagrams with inputs $\underline{n_{11}}, \ldots, \underline{n_{1k_1}}; \ldots; \underline{n_{m1}}, \ldots, \underline{n_{mk_m}}$ and outputs $\underline{n_1}; \ldots; \underline{n_m}$ and D is a wiring diagram with inputs $\underline{n_1}; \ldots; \underline{n_m}$ and outputs $\underline{n}$, there is an obvious "composition" $D(D_1, \ldots, D_m)$ (obtained by gluing the corresponding 1-manifolds, and also describable in completely combinatorial terms) which is a wiring diagram with inputs $(\underline{n_{ij}})_{1\leq i\leq k_j, 1\leq j\leq m}$ and outputs $\underline{n}$ (note that closed circles may be created in $D(D_1, \ldots, D_m)$ even if none existed in D and in $D_1; \ldots; D_m$).

A circuit algebra is an algebraic structure (in the sense of Section 4.2) whose operations are parametrized by wiring diagrams. Here's a formal definition:

Definition 4.11. A circuit algebra consists of the following data:

- For every natural number $n \geq 0$ a set (or a $\mathbb{Z}$-module) C_n "of circuits with n legs".
- For any wiring diagram D with inputs $\underline{n_1}, \ldots \underline{n_k}$ and outputs $\underline{n}$, an operation (denoted by the same letter) $D\colon C_{n_1} \times \cdots \times C_{n_k} \to C_n$ (or linear $D\colon C_{n_1} \otimes \cdots \otimes C_{n_k} \to C_n$ if we work with $\mathbb{Z}$-modules).

We insist that the obvious "identity" wiring diagrams with $\underline{n}$ inputs and $\underline{n}$ outputs act as the identity of C_n, and that the actions of wiring diagrams be compatible in the obvious sense with the composition operation on wiring diagrams.

A silly but useful example of a circuit algebra is the circuit algebra $\mathcal{S}$ of empty circuits, or in our context, of "skeletons". The circuits with n legs for $\mathcal{S}$ are wiring diagrams with n outputs and no inputs; namely, they are 1-manifolds with boundary $\underline{n}$ (so n must be even).

More generally one may pick some collection of "basic components" (perhaps some logic gates and junctions for electronic circuits as in Figure 20) and speak of the "free circuit algebra" generated by these components. Even more generally we can speak of circuit algebras given in terms of "generators and relations"; in the case of electronics, our relations may include the likes of De Morgan's law $\neg(p \vee q) = (\neg p) \wedge (\neg q)$ and the laws governing the placement of resistors in parallel or in series. We feel there is no need to present the details here, yet many examples of circuit algebras given in terms of generators and relations appear in this paper, starting with the next section. We will use the notation $C = \mathrm{CA}\langle G \mid R \rangle$ to denote the circuit algebra generated by a collection of elements G subject to some collection R of relations.

People familiar with electric circuits know very well that connectors sometimes come in "male" and "female" versions, and that you can't plug a USB cable into a headphone jack and expect your system to cooperate. Thus one may define "directed circuit algebras" in which the wiring diagrams are oriented, the circuit sets C_n get replaced by $C_{n_1 n_2}$ for "circuits with n_1 incoming wires and n_2 outgoing wires" and only orientation preserving connections are ever allowed. Likewise there is a "coloured" version of everything, in which the wires may be coloured by the elements of some given set X which may include among its members the elements "USB" and "audio" and in which connections are allowed only if the colour coding is respected. We will not give formal definitions of directed and/or coloured circuit algebras here, yet we will allow ourselves to freely use these notions. Likewise for the obvious analogues of the skeletons algebra $\mathcal{S}$ and for algebras given in terms of generators and relations.

Note that there is an obvious notion of "a morphism between two circuit algebras" and that circuit algebras (directed or not, coloured or not) form a category. We feel that a precise definition is not needed. Yet a lovely example is the "implementation morphism" of logic circuits in the style of Figure 20 into more basic circuits made of transistors and resistors.

Perhaps the prime mathematical example of a circuit algebra is tensor algebra. If t_1 is an element (a "circuit") in some tensor product of vector spaces and their duals, and t_2 is the same except in a possibly different tensor product of vector spaces and their duals, then once an appropriate pairing D (a "wiring diagram") of the relevant vector spaces is chosen, t_1 and t_2 can be contracted ("wired together") to make a new tensor $D(t_1, t_2)$. The pairing D must pair a vector space with its own dual, and so this circuit algebra is coloured by the set of vector spaces involved, and directed, by declaring (say) that some vector spaces are of one gender and their duals are of the other. We have in fact encountered this circuit algebra already, in Section 3.6.

Let G be a group. A G-graded algebra A is a collection $\{A_g \colon g \in G\}$ of vector spaces, along with products $A_g \otimes A_h \to A_{gh}$ that induce an overall structure of an algebra on $A := \bigoplus_{g \in G} A_g$. In a similar vein, we define the notion of an $\mathcal{S}$-graded circuit algebra:

Definition 4.12. An $\mathcal{S}$-graded circuit algebra, or a "circuit algebra with skeletons", is an algebraic structure C with spaces C_β, one for each element β of the circuit algebra of skeletons $\mathcal{S}$, along with composition operations $D_{\beta_1,\dots,\beta_k} \colon C_{\beta_1} \times \cdots \times C_{\beta_k} \to C_\beta$, defined whenever D is a wiring diagram and $\beta = D(\beta_1, \dots, \beta_k)$, so that with the obvious induced structure, $\coprod_\beta C_\beta$ is a circuit algebra. A similar definition can be made if/when the skeletons are taken to be directed or coloured.

Loosely speaking, a circuit algebra with skeletons is a circuit algebra in which every element T has a well-defined skeleton $\varsigma(T) \in \mathcal{S}$. Yet note that as an algebraic structure a circuit algebra with skeletons has more "spaces" than an ordinary circuit algebra, for its spaces are enumerated by skeleta and not merely by integers. The prime examples for circuit algebras with skeletons appear in the next section.

Section Summary. In Sec. 5.1 we introduce v-tangles and w-tangles, the obvious v- and w- counterparts of the standard knot-theoretic notion of "tangles", and briefly discuss their finite type invariants and their associated spaces of "arrow diagrams", $\mathcal{A}^v(\uparrow_n)$ and $\mathcal{A}^w(\uparrow_n)$. We then construct a homomorphic expansion Z, or a "well-behaved" universal finite type invariant for w-tangles. Once again, the only algebraic tool we need to use is $\exp(a) := \sum a^n/n!$, and indeed, Sec. 5.1 is but a routine extension of parts of Section 3. We break away in Sec. 5.2 and show that $\mathcal{A}^w(\uparrow_n) \cong \mathcal{U}(\mathfrak{a}_n \oplus \mathfrak{tder}_n \ltimes \mathfrak{tr}_n)$, where $\mathfrak{a}_n$ is an Abelian algebra of rank n and where $\mathfrak{tder}_n$ and $\mathfrak{tr}_n$, two of the primary spaces used by Alekseev and Torossian [AT], have simple descriptions in terms of words and free Lie algebras. We also show that some functionals studied in [AT], div and j, have a natural interpretation in our language. In 5.3 we discuss a subclass of w-tangles called "special" w-tangles, and relate them by similar means to Alekseev and Torossian's $\mathfrak{sder}_n$ and to "tree level" ordinary Vassiliev theory. Some conventions are described in Sec. 5.4 and the uniqueness of Z is studied in Sec.5.5.

5.1. **v-Tangles and w-Tangles.** With The (surprisingly pleasant) task of defining circuit algebras completed in Section 4.4, the definition of v-tangles and w-tangles is simple.

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Definition 5.1. The ($\mathcal{S}$-graded) circuit algebra vT of v-tangles is the $\mathcal{S}$-graded directed circuit algebra generated by two generators in $C_{2,2}$ called the "positive crossing" and the "negative crossing", modulo the usual R1^s, R2 and R3 moves as depicted in Figure 6 (these relations clearly make sense as circuit algebra relations between our two generators), with the obvious meaning for their skeleta. The circuit algebra wT of w-tangles is the same, except we also mod out by the OC relation of Figure 6 (note that each side in that relation involves only two generators, with the apparent third crossing being merely a projection artifact). In fewer words, $\mathit{vT} := \mathrm{CA}\langle\ ,\ \mid\ =\ ,\ =\ ,\ =\ \rangle$, and $\mathit{wT} := \mathit{vT}/\ =\ $.

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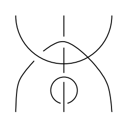
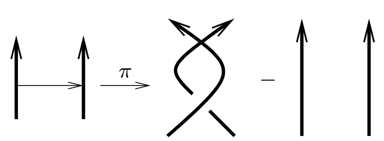
Remark 5.2. One may also define v-tangles and w-tangles using the language of planar algebras, except then another generator is required (the "virtual crossing") and also a few further relations (VR1–VR3, M), and some of the operations (non-planar wirings) become less elegant to define.



Our next task is to study the projectivizations proj vT and proj wT of vT and wT. Again, the language of circuit algebras makes it exceedingly simple.

Definition 5.3. The ($\mathcal{S}$-graded) circuit algebra $\mathcal{D}^v = \mathcal{D}^w$ of arrow diagrams is the graded and $\mathcal{S}$-graded directed circuit algebra generated by a single degree 1 generator a in $C_{2,2}$ called "the arrow" as shown on the right, with the obvious meaning for its skeleton. There are morphisms $\pi\colon \mathcal{D}^v \to \mathit{vT}$ and $\pi\colon \mathcal{D}^w \to \mathit{wT}$ defined by mapping the arrow to an overcrossing minus a no-crossing. (On the right some virtual crossings were added to make the skeleta match). Let $\mathcal{A}^v$ be $\mathcal{D}^v/6T$, let $\mathcal{A}^w := \mathcal{A}^v/TC = \mathcal{D}^w/(\overrightarrow{4T}, TC)$, and let $\mathcal{A}^{sv} := \mathcal{A}^v/RI$ and $\mathcal{A}^{sw} := \mathcal{A}^w/RI$ as usual, with RI, $6T$, $\overrightarrow{4T}$, and TC being the same relations as in Figures 8 and 9 (allowing skeleta parts that are not explicitly connected to really lie on separate skeleton components).



Proposition 5.4. *The maps π above induce surjections $\pi\colon \mathcal{A}^{sv} \to \operatorname{proj} v\mathcal{T}$ and $\pi\colon \mathcal{A}^{sw} \to \operatorname{proj} w\mathcal{T}$. Hence in the language of Definition 4.7, $\mathcal{A}^{sv}$ and $\mathcal{A}^{sw}$ are candidate projectivizations of $v\mathcal{T}$ and $w\mathcal{T}$.*

Proof. Proving that π is well-defined amounts to checking directly that the RI and 6T or RI, $\overrightarrow{4T}$ and TC relations are in the kernel of π. (Just like in the finite type theory of virtual knots and braids.) Thanks to the circuit algebra structure, it is enough to verify the surjectivity of π in degree 1. We leave this as an exercise for the reader. □

We do not know if $\mathcal{A}^{sv}$ is indeed the projectivizations of $v\mathcal{T}$ (also see [BHLR]). Yet in the w case, the picture is simple:

Theorem 5.5. *The assignment $\nearrow\!\!\!\!\nwarrow \mapsto e^a$ (with e^a denoting the exponential of a single arrow from the over strand to the under strand) extends to a well defined $Z\colon w\mathcal{T} \to \mathcal{A}^{sw}$. The resulting map Z is a homomorphic $\mathcal{A}^{sw}$-expansion, and in particular, $\mathcal{A}^{sw} \cong \operatorname{proj} w\mathcal{T}$ and Z is a homomorphic expansion.*

Proof. There is nothing new here. Z is satisfies the Reidemeister moves for the same reasons as in Theorem 2.15 and Theorem 3.11 and as there it also satisfies the universality property. The rest follows from Proposition 4.8. □

In a similar spirit to Definition 3.13, one may define a "w-Jacobi diagram" (often shorts to "arrow diagram") on an arbitrary skeleton. Denote the circuit algebra of formal linear combinations of arrow diagrams modulo $\overrightarrow{STU}_1$, $\overrightarrow{STU}_2$, and TC relations by $\mathcal{A}^{wt}$. We have the following bracket-rise theorem:

Theorem 5.6. *The obvious inclusion of diagrams induces a circuit algebra isomorphism $\mathcal{A}^w \cong \mathcal{A}^{wt}$. Furthermore, the $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations of Figure 12 hold in $\mathcal{A}^{wt}$. Similarly, $\mathcal{A}^{sw} \cong \mathcal{A}^{swt}$, with the expected definition for $\mathcal{A}^{swt}$.*

Proof. The proof of Theorem 3.15 can be repeated verbatim. Note that that proof does not make use of the connectivity of the skeleton. □

Given the above theorem, we no longer keep the distinction between $\mathcal{A}^w$ and $\mathcal{A}^{wt}$ and between $\mathcal{A}^{sw}$ and $\mathcal{A}^{swt}$.

Remark 5.7. Note that if T is an arbitrary w tangle, then the equality on the left side of the figure below always holds, while the one on the right generally doesn't:

(32)

The arrow diagram version of this statement is that if D is an arbitrary arrow diagram in $\mathcal{A}^w$, then the left side equality in the figure below always holds (we will sometimes refer to this as the "head-invariance" of arrow diagrams), while the right side equality ("tail-invariance") generally fails.

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shows a direct proof of (33)

(33)

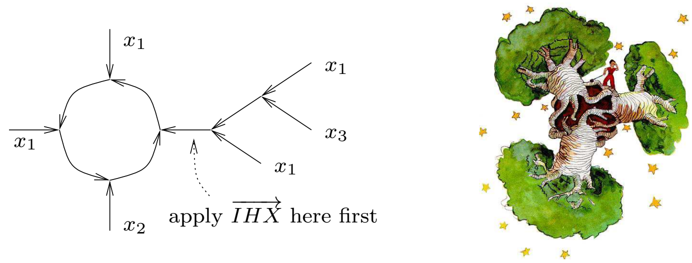

Figure 22. A wheel of trees can be reduced to a combination of wheels, and a wheel of trees with a Little Prince.

We leave it to the reader to ascertain that Equation (32) implies Equation (33). There is also a direct proof of Equation (33) which we also leave to the reader, though see an analogous statement and proof in [BN3, Lemma 3.4]. Finally note that a restricted version of tail-invariance does hold — see Section 5.3.

5.2. $\mathcal{A}^w(\uparrow_n)$ and the Alekseev-Torossian Spaces.

Definition 5.8. Let $\mathit{vT}(\uparrow_n)$ (likewise $\mathit{wT}(\uparrow_n)$) be the set of v-tangles (w-tangles) whose skeleton is the disjoint union of n directed lines. Likewise let $\mathcal{A}^v(\uparrow_n)$ be the part of $\mathcal{A}^v$ in which the skeleton is the disjoint union of n directed lines, with similar definitions for $\mathcal{A}^w(\uparrow_n)$, $\mathcal{A}^{sv}(\uparrow_n)$, and $\mathcal{A}^{sw}(\uparrow_n)$.

In the same manner as in the case of knots (Theorem 3.16), $\mathcal{A}^w(\uparrow_n)$ is a bi-algebra isomorphic (via a diagrammatic PBW theorem, applied independently on each component of the skeleton) to a space $\mathcal{B}^w_n$ of unitrivalent diagrams with symmetrized ends coloured with colours in some n-element set (say $\{x_1,\dots,x_n\}$), modulo $\overrightarrow{AS}$ and $\overrightarrow{IHX}$. Note that the RI relation becomes $w_1=0$, where w_1 denotes the 1-wheel of any colour.

The primitives $\mathcal{P}^w_n$ of $\mathcal{B}^w_n$ are the connected diagrams (and hence the primitives of $\mathcal{A}^w(\uparrow_n)$ are the diagrams that remain connected even when the skeleton is removed). Given the "two in one out" rule for internal vertices, the diagrams in $\mathcal{P}^w_n$ can only be trees or wheels ("wheels of trees" can be reduced to simple wheels by repeatedly using $\overrightarrow{IHX}$, as in Figure 22).

Thus as a vector space $\mathcal{P}^w_n$ is easy to identify. It is a direct sum $\mathcal{P}^w_n=\langle\text{trees}\rangle\oplus\langle\text{wheels}\rangle$. The wheels part is simply the graded vector space generated by all cyclic words in the letters $x_1,\dots,x_n$. Alekseev and Torossian [AT] denote the space of cyclic words by $\mathfrak{tr}_n$, and so shall we. The trees in $\mathcal{P}^w_n$ have leafs coloured $x_1,\dots,x_n$. Modulo $\overrightarrow{AS}$ and $\overrightarrow{IHX}$, they correspond to elements of the free Lie algebra $\mathfrak{lie}_n$ on the generators $x_1,\dots,x_n$. But the root of each such tree also carries a label in $\{x_1,\dots,x_n\}$, hence there are n types of such trees as separated by their roots, and so $\mathcal{P}^w_n$ is isomorphic to the direct sum $\mathfrak{tr}_n\oplus\bigoplus_{i=1}^n\mathfrak{lie}_n$. With $\mathcal{B}^{sw}_n$ and $\mathcal{P}^{sw}_n$ defined in the analogous manner, we can also conclude that $\mathcal{P}^{sw}_n\cong\mathfrak{tr}_n/(\deg 1)\oplus\bigoplus_{i=1}^n\mathfrak{lie}_n$.

By the Milnor-Moore theorem [MM], $\mathcal{A}^w(\uparrow_n)$ is isomorphic to the universal enveloping algebra $\mathcal{U}(\mathcal{P}^w_n)$, with $\mathcal{P}^w_n$ identified as the subspace $\mathcal{P}^w(\uparrow_n)$ of primitives of $\mathcal{A}^w(\uparrow_n)$ using the PBW symmetrization map $\chi\colon\mathcal{B}^w_n\to\mathcal{A}^w(\uparrow_n)$. Thus in order to understand $\mathcal{A}^w(\uparrow_n)$ as an associative algebra, it is enough to understand the Lie algebra structure induced on $\mathcal{P}^w_n$ via the commutator bracket of $\mathcal{A}^w(\uparrow_n)$.

We now wish to identify $\mathcal{P}^w(\uparrow_n)$ as the Lie algebra $\mathfrak{tr}_n \rtimes(\mathfrak{a}_n \oplus \mathfrak{tder}_n)$, which in itself is a combination of the Lie algebras $\mathfrak{a}_n$, $\mathfrak{tder}_n$ and $\mathfrak{tr}_n$ studied by Alekseev and Torossian [AT]. Here are the relevant definitions:

Definition 5.9. Let $\mathfrak{a}_n$ denote the vector space with basis $x_1, \dots, x_n$, also regarded as an Abelian Lie algebra of dimension n. As before, let $\mathfrak{lie}_n = \mathfrak{lie}(\mathfrak{a}_n)$ denote the free Lie algebra on n generators, now identified as the basis elements of $\mathfrak{a}_n$. Let $\mathfrak{der}_n = \mathfrak{der}(\mathfrak{lie}_n)$ be the (graded) Lie algebra of derivations acting on $\mathfrak{lie}_n$, and let

$$\mathfrak{tder}_n = \{D \in \mathfrak{der}_n \colon \forall i\ \exists a_i \text{ s.t. } D(x_i) = [x_i, a_i]\}$$

denote the subalgebra of "tangential derivations". A tangential derivation D is determined by the a_i's for which $D(x_i) = [x_i, a_i]$, and determines them up to the ambiguity $a_i \mapsto a_i + \alpha_i x_i$, where the α_i's are scalars. Thus as vector spaces, $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_{i=1}^n \mathfrak{lie}_n$.

Definition 5.10. Let $\mathrm{Ass}_n = \mathcal{U}(\mathfrak{lie}_n)$ be the free associative algebra "of words", and let Ass_n^+ be the degree > 0 part of Ass_n. As before, we let $\mathfrak{tr}_n = \mathrm{Ass}_n^+ / (x_{i_1} x_{i_2} \cdots x_{i_m} = x_{i_2} \cdots x_{i_m} x_{i_1})$ denote "cyclic words" or "(coloured) wheels". Ass_n, Ass_n^+, and $\mathfrak{tr}_n$ are $\mathfrak{tder}_n$-modules and there is an obvious equivariant "trace" $\mathrm{tr}\colon \mathrm{Ass}_n^+ \to \mathfrak{tr}_n$. **wClip 120510 ends**

Proposition 5.11. *There is a split short exact sequence of Lie algebras*

$$0 \longrightarrow \mathfrak{tr}_n \xrightarrow{\iota} \mathcal{P}^w(\uparrow_n) \xrightarrow{\pi} \mathfrak{a}_n \oplus \mathfrak{tder}_n \longrightarrow 0.$$

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Proof. The inclusion ι is defined the natural way: $\mathfrak{tr}_n$ is spanned by coloured "floating" wheels, and such a wheel is mapped into $\mathcal{P}^w(\uparrow_n)$ by attaching its ends to their assigned strands in arbitrary order. Note that this is well-defined: wheels have only tails, and tails commute.

As vector spaces, the statement is already proven: $\mathcal{P}^w(\uparrow_n)$ is generated by trees and wheels (with the all arrow endings fixed on n strands). When factoring out by the wheels, only trees remain. Trees have one head and many tails. All the tails commute with each other, and commuting a tail with a head on a strand costs a wheel (by $\overrightarrow{STU}$), thus in the quotient the head also commutes with the tails. Therefore, the quotient is the space of floating (coloured) trees, which we have previously identified with $\bigoplus_{i=1}^n \mathfrak{lie}_n \cong \mathfrak{a}_n \oplus \mathfrak{tder}_n$.

It remains to show that the maps ι and π are Lie algebra maps as well. For ι this is easy: the Lie algebra $\mathfrak{tr}_n$ is commutative, and is mapped to the commutative (due to TC) subalgebra of $\mathcal{P}^w(\uparrow_n)$ generated by wheels.

To show that π is a map of Lie algebras we give two proofs, first a "hands-on" one, then a "conceptual" one.

Hands-on argument. $\mathfrak{a}_n$ is the image of single arrows on one strand. These commute with everything in $\mathcal{P}^w(\uparrow_n)$, and so does $\mathfrak{a}_n$ in the direct sum $\mathfrak{a}_n \oplus \mathfrak{tder}_n$.

It remains to show that the bracket of $\mathfrak{tder}_n$ works the same way as commuting trees in $\mathcal{P}^w(\uparrow_n)$. Let D and D' be elements of $\mathfrak{tder}_n$ represented by $(a_1, \dots, a_n)$ and $(a'_1, \dots, a'_n)$, meaning that $D(x_i) = [x_i, a_i]$ and $D'(x_i) = [x_i, a'_i]$ for $i = 1, \dots, n$. Let us compute the commutator of these elements:

$$[D, D'](x_i) = (DD' - D'D)(x_i) = D[x_i, a'_i] - D'[x_i, a_i] = \\ = [[x_i, a_i], a'_i] + [x_i, Da'_i] - [[x_i, a'_i], a_i] - [x_i, D'a_i] = [x_i, Da'_i - D'a_i + [a_i, a'_i]].$$

Now let T and T' be two trees in $\mathcal{P}^w(\uparrow_n)/\,\mathfrak{tr}_n$, their heads on strands i and j, respectively (i may or may not equal j). Let us denote by a_i (resp. a'_j) the element in $\mathfrak{lie}_n$ given by forming the appropriate commutator of the colours of the tails of T's (resp. T'). In $\mathfrak{tder}_n$, let $D = \pi(T)$ and $D' = \pi(T')$. D and D' are determined by $(0,\ldots,a_i,\ldots,0)$, and $(0,\ldots,a'_j,\ldots 0)$, respectively. (In each case, the i-th or the j-th is the only non-zero component.) The commutator of these elements is given by $[D,D'](x_i) = [Da'_i - D'a_i + [a_i,a'_i], x_i]$, and $[D,D'](x_j) = [Da'_j - D'a_j + [a_j,a'_j], x_j]$. Note that unless $i = j$, $a_j = a'_i = 0$.

In $\mathcal{P}^w(\uparrow_n)/\,\mathfrak{tr}_n$, all tails commute, as well as a head of a tree with its own tails. Therefore, commuting two trees only incurs a cost when commuting a head of one tree over the tails of the other on the same strand, and the two heads over each other, if they are on the same strand.

If $i \neq j$, then commuting the head of T over the tails of T' by $\overrightarrow{STU}$ costs a sum of trees given by Da'_j, with heads on strand j, while moving the head of T' over the tails of T costs exactly $-D'a_i$, with heads on strand i, as needed.

If $i = j$, then everything happens on strand i, and the cost is $(Da'_i - D'a_i + [a_i,a'_i])$, where the last term happens when the two heads cross each other.

Conceptual argument. There is an action of $\mathcal{P}^w(\uparrow_n)$ on $\mathfrak{lie}_n$, as follows: introduce and extra strand on the right. An element L of $\mathfrak{lie}_n$ corresponds to a tree with its head on the extra strand. Its commutator with an element of $\mathcal{P}^w(\uparrow_n)$ (considered as an element of $\mathcal{P}^w(\uparrow_{n+1})$ by the obvious inclusion) is again a tree with head on strand $(n+1)$, defined to be the result of the action.

Since L has only tails on the first n strands, elements of $\mathfrak{tr}_n$, which also only have tails, act trivially. So do single (local) arrows on one strand ($\mathfrak{a}_n$). It remains to show that trees act as $\mathfrak{tder}_n$, and it is enough to check this on the generators of $\mathfrak{lie}_n$ (as the Leibniz rule is obviously satisfied). The generators of $\mathfrak{lie}_n$ are arrows pointing from one of the first n strands, say strand i, to strand $(n+1)$. A tree with head on strand i acts on this element, according $\overrightarrow{STU}$, by forming the commutator, which is exactly the action of $\mathfrak{tder}_n$.

To identify $\mathcal{P}^w(\uparrow_n)$ as the semidirect product $\mathfrak{tr}_n \rtimes (\mathfrak{a}_n \oplus \mathfrak{tder}_n)$, it remains to show that the short exact sequence of the Proposition splits. This is indeed the case, although not canonically. Two —of the many— splitting maps $u, l\colon \mathfrak{tder}_n \oplus \mathfrak{a}_n \to \mathcal{P}^w(\uparrow_n)$ are described as follows: $\mathfrak{tder}_n \oplus \mathfrak{a}_n$ is identified with $\bigoplus_{i=1}^n \mathfrak{lie}_n$, which in turn is identified with floating (coloured) trees. A map to $\mathcal{P}^w(\uparrow_n)$ can be given by specifying how to place the legs on their specified strands. A tree may have many tails but has only one head, and due to TC, only the positioning of the head matters. Let u (for *upper*) be the map placing the head of each tree above all its tails on the same strand, while l (for *lower*) places the head below all the tails. It is obvious that these are both Lie algebra maps and that $\pi \circ u$ and $\pi \circ l$ are both the identity of $\mathfrak{tder}_n \oplus \mathfrak{a}_n$. This makes $\mathcal{P}^w(\uparrow_n)$ a semidirect product. $\square$

Remark 5.12. Let $\mathfrak{tr}_n^s$ denote $\mathfrak{tr}_n$ mod out by its degree one part (one-wheels). Since the RI relation is in the kernel of π, there is a similar split exact sequence

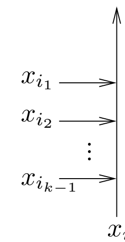
$$0 \to \mathfrak{tr}_n^s \xrightarrow{\bar{\iota}} \mathcal{P}^{sw} \xrightarrow{\bar{\pi}} \mathfrak{a}_n \oplus \mathfrak{tder}_n\,.$$

Definition 5.13. For any $D \in \mathfrak{tder}_n$, $(l-u)D$ is in the kernel of π, therefore is in the image of ι, so $\iota^{-1}(l-u)D$ makes sense. We call this element $\operatorname{div} D$.

Definition 5.14. In [AT] div is defined as follows: $\mathrm{div}(a_1, \dots, a_n) := \sum_{k=1}^n \mathrm{tr}((\partial_k a_k)x_k)$, where ∂_k picks out the words of a sum which end in x_k and deletes their last letter x_k, and deletes all other words (the ones which do not end in x_k).

Proposition 5.15. *The div of Definition 5.13 and the div of* [AT] *are the same.*

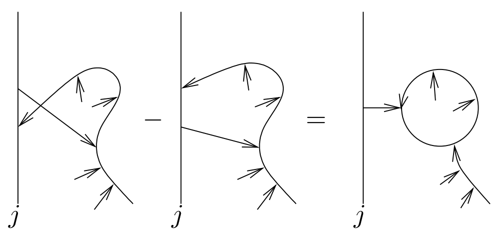
Proof. It is enough to verify the claim for the linear generators of $\mathfrak{tder}_n$, namely, elements of the form $(0, \dots, a_j, \dots, 0)$, where $a_j \in \mathfrak{lie}_n$ or equivalently, single (floating, coloured) trees, where the colour of the head is j. By the Jacobi identity, each a_j can be written in a form $a_j = [x_{i_1}, [x_{i_2}, [\dots, x_{i_k}]\dots]$. Equivalently, by $\overrightarrow{IHX}$, each tree has a standard "comb" form, as shown on the picture on the right.



For an associative word $Y = y_1 y_2 \dots y_l \in \mathrm{Ass}_n^+$, we introduce the notation $[Y] := [y_1, [y_2, [\dots, y_l]\dots]$. The div of [AT] picks out the words that end in x_j, forgets the rest, and considers these as cyclic words. Therefore, by interpreting the Lie brackets as commutators, one can easily check that for a_j written as above,

$$\mathrm{div}((0, \dots, a_j, \dots, 0)) = \sum_{\alpha\colon i_\alpha = x_j} -x_{i_1} \dots x_{i_{\alpha-1}} [x_{i_{\alpha+1}} \dots x_{i_k}] x_j. \tag{34}$$

In Definition 5.13, div of a tree is the difference between attaching its head on the appropriate strand (here, strand j) below all of its tails and above. As shown in the figure on the right, moving the head across each of the tails on strand j requires an $\overrightarrow{STU}$ relation, which "costs" a wheel (of trees, which is equivalent to a sum of honest wheels). Namely, the head gets connected to the tail in question. So div of the tree represented by a_j is given by



$$\textstyle\sum_{\alpha\colon x_{i_\alpha} = j} \text{“connect the head to the } \alpha \text{ leaf”}.$$

This in turn gets mapped to the formula above via the correspondence between wheels and cyclic words. $\square$

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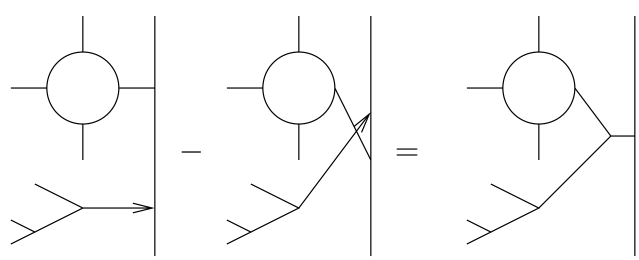
has extra material on the relationship of all this with differential operators

Remark 5.16. There is an action of $\mathfrak{tder}_n$ on $\mathfrak{tr}_n$ as follows. Represent a cyclic word $w \in \mathfrak{tr}_n$ as a wheel in $\mathcal{P}^w(\uparrow_n)$ via the map ι. Given an element $D \in \mathfrak{tder}_n$, $u(D)$, as defined above, is a tree in $\mathcal{P}^w(\uparrow_n)$ whose head is above all of its tails. We define $D \cdot w := \iota^{-1}(u(D)\iota(w) - \iota(w)u(D))$. Note that $u(D)\iota(w) - \iota(w)u(D)$ is in the image of ι, i.e., a linear combination of wheels, for the following reason. The wheel $\iota(w)$ has only tails. As we commute the tree $u(D)$ across the wheel, the head of the tree is commuted across tails of the wheel on the same strand. Each time this happens the cost, by the $\overrightarrow{STU}$ relation, is a wheel with the tree attached to it, as shown on the right, which in turn (by $\overrightarrow{IHX}$ relations, as Figure 22 shows) is a sum of wheels. Once the head of the tree has been moved to the top, the tails of the tree commute up for free by TC. Note that the alternative definition, $D \cdot w := \iota^{-1}(l(D)\iota(w) - \iota(w)l(D))$ is in fact equal to the definition above.



Definition 5.17. In [AT], the group TAut_n is defined as $\exp(\mathfrak{tder}_n)$. Note that $\mathfrak{tder}_n$ is positively graded, hence it integrates to a group. Note also that TAut_n is the group of "basis-conjugating" automorphisms of $\mathfrak{lie}_n$, i.e., for $g \in \mathrm{TAut}_n$, and any x_i, $i = 1, \ldots, n$ generator of $\mathfrak{lie}_n$, there exists an element $g_i \in \exp(\mathfrak{lie}_n)$ such that $g(x_i) = g_i^{-1} x_i g_i$.

The action of $\mathfrak{tder}_n$ on $\mathfrak{tr}_n$ lifts to an action of TAut_n on $\mathfrak{tr}_n$, by interpreting exponentials formally, in other words e^D acts as $\sum_{n=0}^{\infty} \frac{D^n}{n!}$. The lifted action is by conjugation: for $w \in \mathfrak{tr}_n$ and $e^D \in \mathrm{TAut}_n$, $e^D \cdot w = \iota^{-1}(e^{uD}\iota(w)e^{-uD})$.

Recall that in Section 5.1 of [AT] Alekseev and Torossian construct a map $j\colon \mathrm{TAut}_n \to \mathfrak{tr}_n$ which is characterized by two properties: the cocycle property

$$j(gh) = j(g) + g \cdot j(h), \tag{35}$$

where in the second term multiplication by g denotes the action described above; and the condition

$$\frac{d}{ds} j(\exp(sD))|_{s=0} = \mathrm{div}(D). \tag{36}$$

Now let us interpret j in our context.

Definition 5.18. The adjoint map $*\colon \mathcal{A}^w(\uparrow_n) \to \mathcal{A}^w(\uparrow_n)$ acts by "flipping over diagrams and negating arrow heads on the skeleton". In other words, for an arrow diagram D,

$$D^* := (-1)^{\#\{\text{tails on skeleton}\}} S(D),$$

where S denotes the map which switches the orientation of the skeleton strands (i.e. flips the diagram over), and multiplies by $(-1)^{\#\text{skeleton vertices}}$.

Proposition 5.19. *For $D \in \mathfrak{tder}_n$, define a map $J\colon \mathrm{TAut}_n \to \exp(\mathfrak{tr}_n)$ by $J(e^D) := e^{uD}(e^{uD})^*$. Then*

$$\exp(j(e^D)) = J(e^D).$$

Proof. Note that $(e^{uD})^* = e^{-lD}$, due to "Tails Commute" and the fact that a tree has only one head.

Let us check that $\log J$ satisfies properties (35) and (36). Namely, with $g = e^{D_1}$ and $h = e^{D_2}$, and using that $\mathfrak{tr}_n$ is commutative, we need to show that

$$J(e^{D_1}e^{D_2}) = J(e^{D_1})\big(e^{uD_1} \cdot J(e^{D_2})\big), \tag{37}$$

where $\cdot$ denotes the action of $\mathfrak{tder}_n$ on $\mathfrak{tr}_n$; and that

$$\frac{d}{ds} J(e^{sD})|_{s=0} = \mathrm{div}\, D. \tag{38}$$

Indeed, with $\mathrm{BCH}(D_1, D_2) = \log e^{D_1}e^{D_2}$ being the standard Baker–Campbell–Hausdorff formula,

$$\begin{aligned} J(e^{D_1}e^{D_2}) &= J(e^{\mathrm{BCH}(D_1,D_2)}) = e^{u(\mathrm{BCH}(D_1,D_2)}e^{-l(\mathrm{BCH}(D_1,D_2)} = e^{\mathrm{BCH}(uD_1,uD_2)}e^{-\mathrm{BCH}(lD_1,lD_2)} \\ &= e^{uD_1}e^{uD_2}e^{-lD_2}e^{-lD_1} = e^{uD_1}(e^{uD_2}e^{-lD_2})e^{-uD_1}e^{uD_1}e^{lD_1} = (e^{uD_1} \cdot J(D_2))J(D_1), \end{aligned}$$

as needed.

As for condition (36), a direct computation of the derivative yields

$$\frac{d}{ds} J(e^{sD})|_{s=0} = uD - lD = \mathrm{div}\, D,$$

as desired. □

5.3. **The Relationship with u-Tangles.** Let uT be the planar algebra of classical, or "*u*sual" tangles. There is a map $a\colon \mathit{uT} \to \mathit{wT}$ of u-tangles into w-tangles: algebraically, it is defined in the obvious way on the planar algebra generators of uT. (It can also be interpreted topologically as Satoh's tubing map, as in Section 3.1.1, where a u-tangle is a tangle drawn on a sphere. However, it is only conjectured that the circuit algebra presented here is a Reidemeister theory for "tangled ribbon tubes in $\mathbb{R}^4$".) The map a induces a corresponding map $\alpha\colon \mathcal{A}^u \to \mathcal{A}^{sw}$, which maps an ordinary Jacobi diagram (i.e., unoriented chords with internal trivalent vertices modulo the usual AS, IHX and STU relations) to the sum of all possible orientations of its chords (many of which are zero in $\mathcal{A}^{sw}$ due to the "two in one out" rule).

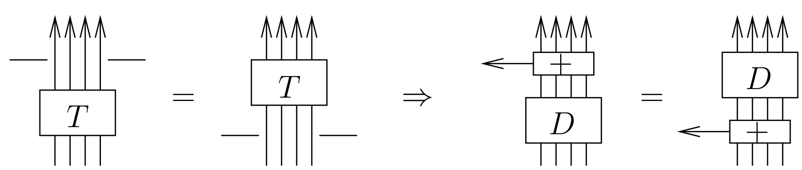
$$\begin{tikzcd} \mathit{uT} \arrow[r, dotted, "Z^u"] \arrow[d, "a"] & \mathcal{A}^u \arrow[d, "\alpha"] \\ \mathit{wT} \arrow[r, "Z^w"] & \mathcal{A}^{sw} \end{tikzcd}$$

It is tempting to ask whether the square on the left commutes. Unfortunately, this question hardly makes sense, as there is no canonical choice for the dotted line in it. Similarly to the braid case in Section 2.5.5, the definition of the Kontsevich integral for u-tangles typically depends on various choices of "parenthesizations". Choosing parenthesizations, this square becomes commutative up to some fixed corrections. The details are in Proposition 6.15.

Yet already at this point we can recover something from the existence of the map $a\colon \mathit{uT} \to \mathit{wT}$, namely an interpretation of the Alekseev-Torossian [AT] space of special derivations,

$$\mathfrak{sder}_n := \{D \in \mathfrak{tder}_n \colon D(\sum_{i=1}^n x_i) = 0\}.$$

Recall from Remark 5.7 that in general it is not possible to slide a strand under an arbitrary w-tangle. However, it is possible to slide strands freely under tangles *in the image of* a, and thus by reasoning similar to the reasoning in Remark 5.7, diagrams D in the image of α respect "tail-invariance":



(39)

Let $\mathcal{P}^u(\uparrow_n)$ denote the primitives of $\mathcal{A}^u(\uparrow_n)$, that is, Jacobi diagrams that remain connected when the skeleton is removed. Remember that $\mathcal{P}^w(\uparrow_n)$ stands for the primitives of $\mathcal{A}^w(\uparrow_n)$. Equation (39) readily implies that the image of the composition

$$\mathcal{P}^u(\uparrow_n) \xrightarrow{\alpha} \mathcal{P}^w(\uparrow_n) \xrightarrow{\pi} \mathfrak{a}_n \oplus \mathfrak{tder}_n$$

is contained in $\mathfrak{a}_n \oplus \mathfrak{sder}_n$. Even better is true.

Theorem 5.20. *The image of $\pi\alpha$ is precisely $\mathfrak{a}_n \oplus \mathfrak{sder}_n$.*

This theorem was first proven by Drinfel'd (Lemma after Proposition 6.1 in [Dr3]), but the proof we give here is due to Levine [Lev].

Proof. Let $\mathfrak{lie}_n^d$ denote the degree d piece of $\mathfrak{lie}_n$. Let V_n be the vector space with basis $x_1, x_2, \ldots, x_n$. Note that

$$V_n \otimes \mathfrak{lie}_n^d \cong \bigoplus_{i=1}^n \mathfrak{lie}_n^d \cong (\mathfrak{tder}_n \oplus \mathfrak{a}_n)^d,$$

where $\mathfrak{tder}_n$ is graded by the number of tails of a tree, and $\mathfrak{a}_n$ is contained in degree 1.

The bracket defines a map $\beta\colon V_n \otimes \mathfrak{lie}_n^d \to \mathfrak{lie}_n^{d+1}$: for $a_i \in \mathfrak{lie}_n^d$ where $i = 1, \dots, n$, the "tree" $D = (a_1, a_2, \dots, a_n) \in (\mathfrak{tder}_n \oplus \mathfrak{a}_n)^d$ is mapped to

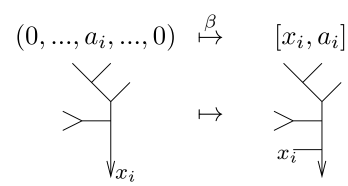
$$\beta(D) = \sum_{i=1}^{n} [x_i, a_i] = D\left(\sum_{i=1}^{n} x_i\right),$$

where the first equality is by the definition of tensor product and the bracket, and the second is by the definition of the action of $\mathfrak{tder}_n$ on $\mathfrak{lie}_n$.

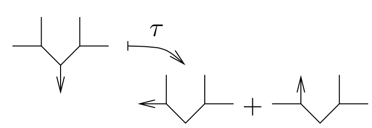
Since $\mathfrak{a}_n$ is contained in degree 1, by definition $\mathfrak{sder}_n^d = (\ker\beta)^d$ for $d \geq 2$. In degree 1, $\mathfrak{a}_n$ is obviously in the kernel, hence $(\ker\beta)^1 = \mathfrak{a}_n \oplus \mathfrak{sder}_n^1$. So overall, $\ker\beta = \mathfrak{a}_n \oplus \mathfrak{sder}_n$.

We want to study the image of the map $\mathcal{P}^u(\uparrow^n) \xrightarrow{\pi\alpha} \mathfrak{a}_n \oplus \mathfrak{tder}_n$. Under α, all connected Jacobi diagrams that are not trees or wheels go to zero, and under π so do all wheels. Furthermore, π maps trees that live on n strands to "floating" trees with univalent vertices coloured by the strand they used to end on. So for determining the image, we may replace $\mathcal{P}^u(\uparrow^n)$ by the space $\mathcal{T}_n$ of connected *un*oriented "floating trees" (uni-trivalent graphs), the ends (univalent vertices) of which are coloured by the $\{x_i\}_{i=1,..,n}$. We denote the degree d piece of $\mathcal{T}_n$, i.e., the space of trees with $d+1$ ends, by $\mathcal{T}_n^d$. Abusing notation, we shall denote the map induced by $\pi\alpha$ on $\mathcal{T}_n$ by $\alpha\colon \mathcal{T}_n \to \mathfrak{a}_n \oplus \mathfrak{tder}_n$. Since choosing a "head" determines the entire orientation of a tree by the two-in-one-out rule, α maps a tree in $\mathcal{T}_n^d$ to the sum of $d+1$ ways of choosing one of the ends to be the "head".

We want to show that $\ker\beta = \operatorname{im}\alpha$. This is equivalent to saying that $\bar\beta$ is injective, where $\bar\beta\colon V_n \otimes \mathfrak{lie}_n \,/\operatorname{im}\alpha \to \mathfrak{lie}_n$ is map induced by β on the quotient by $\operatorname{im}\alpha$.

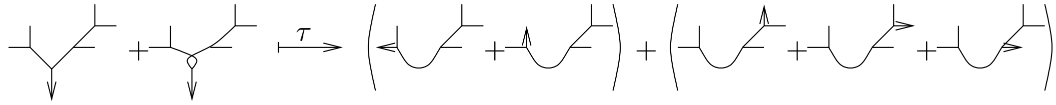


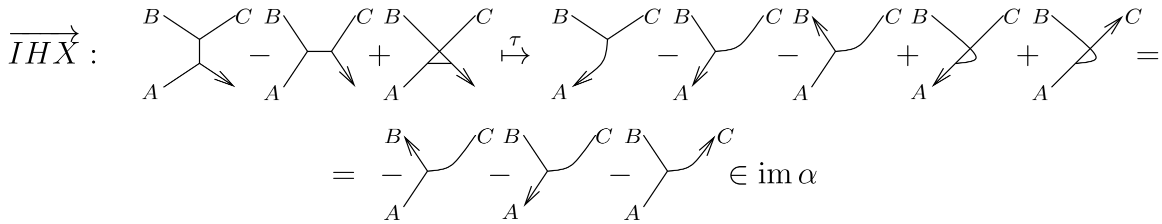
The degree d piece of $V_n \otimes \mathfrak{lie}_n$, in the pictorial description, is generated by floating trees with d tails and one head, all coloured by x_i, $i = 1, \dots, n$. This is mapped to $\mathfrak{lie}_n^{d+1}$, which is isomorphic to the space of floating trees with $d+1$ tails and one head, where only the tails are coloured by the x_i. The map β acts as shown on the picture on the right.



We show that $\bar\beta$ is injective by exhibiting a map $\tau\colon\ \mathfrak{lie}_n^{d+1} \to V_n \otimes \mathfrak{lie}_n^d \,/\operatorname{im}\alpha$ so that $\tau\bar\beta = I$. τ is defined as follows: given a tree with one head and $d+1$ tails τ acts by deleting the head and the arc connecting it to the rest of the tree and summing over all ways of choosing a new head from one of the tails on the left half of the tree relative to the original placement of the head (see the picture on the right). As long as we show that τ is well-defined, it follows from the definition and the pictorial description of β that $\tau\bar\beta = I$.

For well-definedness we need to check that the images of $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations under τ are in the image of α. This we do in the picture below. In both cases it is enough to check the case when the "head" of the relation is the head of the tree itself, as otherwise an $\overrightarrow{AS}$ or $\overrightarrow{IHX}$ relation in the domain is mapped to an $\overrightarrow{AS}$ or $\overrightarrow{IHX}$ relation, thus zero, in the image.

$\overrightarrow{AS}$:  $\in \operatorname{im}\alpha$

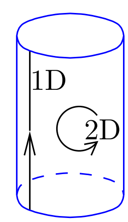


In the $\overrightarrow{IHX}$ picture, in higher degrees A, B and C may denote an entire tree. In this case, the arrow at A (for example) means the sum of all head choices from the tree A. $\square$

Comment 5.21. In view of the relation between the right half of Equation (39) and the special derivations $\mathfrak{sder}$, it makes sense to call w-tangles that satisfy the condition in the left half of Equation (39) "special". The a images of u-tangles are thus special. We do not know if the global version of Theorem 5.20 holds true. Namely, we do not know whether every special w-tangle is the a-image of a u-tangle.

5.4. **The local topology of w-tangles.** So far throughout this section we have presented w-tangles as a Reidemeister theory: a circuit algebra given by generators and relations. Note that Satoh's tubing map (see Sections 2.2.2 and 3.1.1) does extend to w-tangles in the obvious way, although it is not known whether it is an isomorphism between the circuit algebra described here and tangled tubes in $\mathbb{R}^4$. Nonetheless, this intuition explains the local relations (Reidemeister moves). The purpose of this subsection is to explain the local topology of crossings and understand orientations, signs and orientation reversals.

The tubes we consider are endowed with two orientations, we will call these the 1- and 2-dimensional orientations. The one dimensional orientation is the direction of the tube as a "strand" of the tangle. In other words, each tube has a "core"[34]: a distinguished line along the tube, which is oriented as a 1-dimensional manifold. Furthermore, the tube as a 2-dimensional surface is oriented as given by the tubing map. An example is shown on the right.

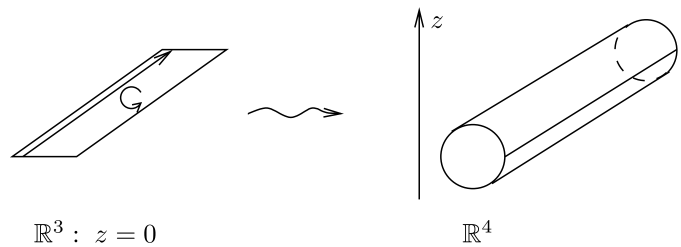


Note that a tube in $\mathbb{R}^4$ has a "filling": a solid (3-dimensional) cylinder embedded in $\mathbb{R}^4$, with boundary the tube, and the 2D orientation of the tube induces an orientation of its filling as a 3-dimensional manifold. A (non-virtual) crossing is when the core of one tube intersects the filling of another transversely. Due to the complementary dimensions, the intersection is a single point, and the 1D orientation of the core along with the 3D orientation of the filling it passes through determines an orientation of the ambient space. We say that the crossing is positive if this agrees with the standard orientation of $\mathbb{R}^4$, and negative otherwise. Hence, there are four types of crossings, given by whether the core of tube A intersects the filling of B or vice versa, and two possible signs in each case.

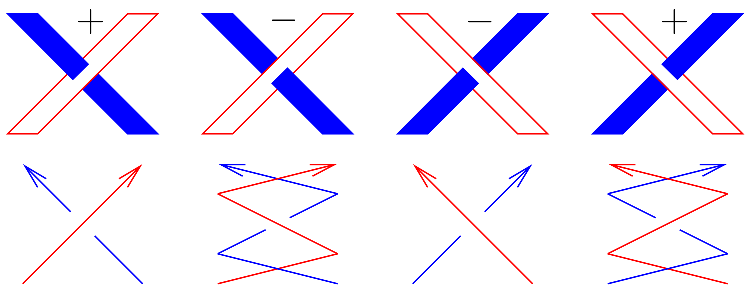
As discussed in Section 2.2, braided tubes in $\mathbb{R}^4$ can be thought of as movies of flying rings in $\mathbb{R}^3$, and in particular a crossing represents a ring flying through another ring. In this interpretation, the 1D orientation of the tube is given by time moving forward. The 2D and 1D orientations of the tube together induce an orientation of the flying ring which is a cross-section of the tube at each moment. Hence, saying "below" and "above" the ring makes sense, and as mentioned in Exercise 2.7 there are four types of crossings: ring A flies through ring B from below or from above; and ring B flies through ring A from below or from above. A crossing is positive if the inner ring comes from below, and negative otherwise.

[34]The core of Lord Voldemort's wand was made of a phoenix feather.

In Sections 2.2.2 and 3.1.1 we have discussed the tubing map from v- or w-diagrams of braids or knots to ribbon tubes in $\mathbb{R}^4$: the under-strand of a crossing is interpreted as a thinner tube (or a ring flying through another). This generalizes to tangles easily. We take the opportunity here to introduce another notation, to be called the "band notation", which is more suggestive of the 4D topology than the strand notation. We represent a tube in $\mathbb{R}^4$ by a picture of an oriented band in $\mathbb{R}^3$. By "oriented band" we mean that it has two orientations: a 1D direction (for example an orientation of one of the edges), and a 2D orientation as a surface. To interpret the 3D picture of a band as an tube in $\mathbb{R}^4$, we add an extra coordinate. Let us refer to the $\mathbb{R}^3$ coordinates as x, y and t, and to the extra coordinate as z. Think of $\mathbb{R}^3$ as being embedded in $\mathbb{R}^4$ as the hyperplane $z = 0$, and think of the band as being made of a thin double membrane. Push the membrane up and down in the z direction at each point as far as the distance of that point from the boundary of the band, as shown on the right. Furthermore, keep the 2D orientation of the top membrane (the one being pushed up), but reverse it on the bottom. This produces an oriented tube embedded in $\mathbb{R}^4$.



In band notation, the four possible crossings appear as follows, where underneath each crossing we indicate the corresponding strand picture, as mentioned in Exercise 2.7:



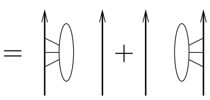
The signs for each type of crossing are shown in the figure above. Note that the sign of a crossing depends of the 2D orientation of the over-strand, as well as the 1D direction of the under-strand. Hence, switching only the direction (1D orientation) of a strand changes the sign of the crossing if and only if the strand of changing direction is the under strand. However, fully changing the orientation (both 1D and 2D) always switches the sign of the crossing. Note that switching the strand orientation in the strand notation corresponds to the total (both 1D and 2D) orientation switch.

5.5. **Good properties and uniqueness of the homomorphic expansion.** In much the same way as in Section 2.5.1, Z has a number of good properties with respect to various tangle operations: it is group-like; it commutes with adding an inert strand (note that this is a circuit algebra operation, hence it doesn't add anything beyond homomorphicity); and it commutes with deleting a strand and with strand orientation reversals. All but the last of these were explained in the context of braids and the explanations still hold. Orientation reversal $S_k\colon\, w\!T \to w\!T$ is the operation which reverses the orientation of the k-th component. Note that in the world of topology (via Satoh's tubing map) this means reversing both the 1D and the 2D orientations. The induced diagrammatic operation $S_k\colon \mathcal{A}^w(T) \to \mathcal{A}^w(S_k(T))$, where T denotes the skeleton of a given w-tangle, acts by multiplying each arrow diagram by (-1) raised to the power the number of arrow endings (both heads and tails) on the k-th strand, as well as reversing the strand orientation. Saying that "Z commutes with S_k" means that the appropriate square commutes.

The following theorem asserts that a well-behaved homomorphic expansion of w-tangles is unique:

Theorem 5.22. *The only homomorphic expansion satisfying the good properties described above is the Z defined in Section 5.1.*

Proof. We first prove the following claim: Assume, by contradiction, that Z' is a different homomorphic expansion of w-tangles with the good properties described above. Let $R' = Z'(\nearrow\!\!\!\!\nwarrow)$ and $R = Z(\nearrow\!\!\!\!\nwarrow)$, and denote by ρ the lowest degree homogeneous non-vanishing term of $R' - R$. (Note that R' determines Z', so if $Z' \neq Z$, then $R' \neq R$.) Suppose ρ is of degree k. Then we claim that $\rho = \alpha_1 w_k^1 + \alpha_2 w_k^2$ is a linear combination of w_k^1 and w_k^2, where w_k^i denotes a k-wheel living on strand i, as shown on the right.

$\rho =$ 

Before proving the claim, note that it leads to a contradiction. Let d_i denote the operation "delete strand i". Then up to degree k, we have $d_1(R') = \alpha_2 w_k^1$ and $d_2(R') = \alpha_1 w_k^2$, but Z' is compatible with strand deletions, so $\alpha_1 = \alpha_2 = 0$. Hence Z is unique, as stated.

On to the proof of the claim, note that Z' being an expansion determines the degree 1 term of R' (namely, the single arrow a^{12} from strand 1 to strand 2, with coefficient 1). So we can assume that $k \geq 2$. Note also that since both R' and R are group-like, ρ is primitive. Hence ρ is a linear combination of connected diagrams, namely trees and wheels.

Both R and R' satisfy the Reidemeister 3 relation:

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}, \qquad R'^{12}R'^{13}R'^{23} = R'^{23}R'^{13}R'^{12}$$

where the superscripts denote the strands on which R is placed (compare with Remark 2.16). We focus our attention on the degree $k+1$ part of the equation for R', and use that up to degree $k+1$. We can write $R' = R + \rho + \mu$, where μ denotes the degree $k+1$ homogeneous part of $R' - R$. Thus, up to degree $k+1$, we have

$$(R^{12}+\rho^{12}+\mu^{12})(R^{13}+\rho^{13}+\mu^{13})(R^{23}+\rho^{23}+\mu^{23}) = (R^{23}+\rho^{23}+\mu^{23})(R^{13}+\rho^{13}+\mu^{13})(R^{12}+\rho^{12}+\mu^{12}).$$

The homogeneous degree $k+1$ part of this equation is a sum of some terms which contain ρ and some which don't. The diligent reader can check that those which don't involve ρ cancel on both sides, either due to the fact that R satisfies the Reidemeister 3 relation, or by simple degree counting. Rearranging all the terms which do involve ρ to the left side, we get the following equation, where a^{ij} denotes an arrow pointing from strand i to strand j:

$$[a^{12}, \rho^{13}] + [\rho^{12}, a^{13}] + [a^{12}, \rho^{23}] + [\rho^{12}, a^{23}] + [a^{13}, \rho^{23}] + [\rho^{13}, a^{23}] = 0. \tag{40}$$

The third and fifth terms sum to $[a^{12}+a^{13}, \rho^{23}]$, which is zero due to the "head-invariance" of diagrams, as in Remark 5.7.

We treat the tree and wheel components of ρ separately. Let us first assume that ρ is a linear combination of trees. Recall that the space of trees on two strands is isomorphic to $\mathfrak{lie}_2 \oplus \mathfrak{lie}_2$, the first component given by trees whose head is on the first strand, and the second component by trees with their head on the second strand. Let $\rho = \rho_1 + \rho_2$, where ρ_i is the projection to the i-th component for $i = 1, 2$.

Note that due to TC, we have $[a^{12}, \rho_2^{13}] = [\rho_2^{12}, a^{13}] = [\rho_1^{12}, a^{23}] = 0$. So Equation (40) reduces to

$$[a^{12}, \rho_1^{13}] + [\rho_1^{12}, a^{13}] + [\rho_2^{12}, a^{23}] + [\rho_1^{13}, a^{23}] + [\rho_2^{13}, a^{23}] = 0$$

The left side of this equation lives in $\bigoplus_{i=1}^{3} \mathfrak{lie}_3$. Notice that only the first term lies in the second direct sum component, while the second, third and last terms live in the third one, and the fourth term lives in the first. This in particular means that the first term is itself zero. By $\overrightarrow{STU}$, this implies

$$0 = [a^{12}, \rho_1^{13}] = -[\rho_1, x_1]_2^{13},$$

where $[\rho_1, x_1]_2^{13}$ means the tree defined by the element $[\rho_1, x_1] \in \mathfrak{lie}_2$, with its tails on strands 1 and 3, and head on strand 2. Hence, $[\rho_1, x_1] = 0$, so ρ_1 is a multiple of x_1. The tree given by $\rho_1 = x_1$ is a degree 1 element, a possibility we have eliminated, so $\rho_1 = 0$.

Equation (40) is now reduced to

$$[\rho_2^{12}, a^{23}] + [\rho_2^{13}, a^{23}] = 0.$$

Both terms are words in $\mathfrak{lie}_3$, but notice that the first term does not involve the letter x_3. This means that if the second term involves x_3 at all, i.e., if ρ_2 has tails on the second strand, then both terms have to be zero individually. Assuming this and looking at the first term, ρ_2^{12} is a Lie word in x_1 and x_2, which does involve x_2 by assumption. We have $[\rho_2^{12}, a^{23}] = [x_2, \rho_2^{12}] = 0$, which implies ρ_2^{12} is a multiple of x_2, in other words, ρ is a single arrow on the second strand. This is ruled out by the assumption that $k \geq 2$.

On the other hand if the second term does not involve x_3 at all, then ρ_2 has no tails on the second strand, hence it is of degree 1, but again $k \geq 2$. We have proven that the "tree part" of ρ is zero.

So ρ is a linear combination of wheels. Wheels have only tails, so the first, second and fourth terms of (40) are zero due to the tails commute relation. What remains is $[\rho^{13}, a^{23}] = 0$. We assert that this is true if and only if each linear component of ρ has all of its tails on one strand.

To prove this, recall each wheel of ρ^{13} represents a cyclic word in letters x_1 and x_3. The map $r\colon \rho^{13} \mapsto [\rho^{13}, a^{23}]$ is a map $\mathfrak{tr}_2 \to \mathfrak{tr}_3$, which sends each cyclic word in letters x_1 and x_3 to the sum of all ways of substituting $[x_2, x_3]$ for one of the x_3's in the word. Note that if we expand the commutators, then all terms that have x_2 between two x_3's cancel. Hence all remaining terms will be cyclic words in x_1 and x_3 with a single occurrence of x_2 in between an x_1 and an x_3.

We construct an almost-inverse r' to r: for a cyclic word w in $\mathfrak{tr}_3$ with one occurrence of x_2, let r' be the map that deletes x_2 from w and maps it to the resulting word in $\mathfrak{tr}_2$ if x_2 is followed by x_3 in w, and maps it to 0 otherwise. On the rest of $\mathfrak{tr}_3$ the map r' may be defined to be 0.

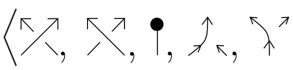
The composition $r'r$ takes a cyclic word in x_1 and x_3 to itself multiplied by the number of times a letter x_3 follows a letter x_1 in it. The kernel of this map can consist only of cyclic words that do not contain the sub-word x_3x_1, namely, these are the words of the form x_3^k or x_1^k. Such words are indeed in the kernel of r, so these make up exactly the kernel of r. This is exactly what needed to be proven: all wheels in ρ have all their tails on one strand.

This concludes the proof of the claim, and the proof of the theorem. $\square$

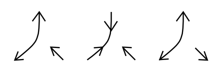
Section Summary. If you have come this far, you must have noticed the approximate Bolero spirit of this article. In every chapter a new instrument comes to play; the overall theme remains the same, but the composition is more and more intricate. In this chapter we add "foam vertices" to w-tangles (and a few lesser things as well) and ask the same questions we asked before; primarily, "is there a homomorphic expansion?". As we shall see, in the current context this question is equivalent to the Alekseev-Torossian [AT] version of the Kashiwara-Vergne [KV] problem and explains the relationship between these topics and Drinfel'd's theory of associators.

6.1. **The Circuit Algebra of w-Tangled Foams.** For reasons we will reluctantly acknowledge later in this section (see Comment 6.2), we will present the circuit algebra of w-tangled foams via its Reidemeister-style diagrammatic description (accompanied by a local topological interpretation) rather than as an entirely topological construct.

Definition 6.1. Let wTF^o (where o stands for "orientable", to be explained in Section 6.5) be the algebraic structure

$wTF^o = \mathrm{CA}\Big\langle$  $\Big|$ w-relations as in Section 6.1.2 $\Big|$ w-operations as in Section 6.1.3 $\Big\rangle$.

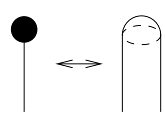
Hence wTF^o is the circuit algebra generated by the generators listed above and described below, modulo the relations described in Section 6.1.2, and augmented with several "auxiliary operations", which are a part of the algebraic structure of wTF^o but are not a part of its structure as a circuit algebra, as described in Section 6.1.3.



To be completely precise, we have to admit that wTF^o as a circuit algebra has more generators than shown above. The last two generators are "foam vertices", as will be explained shortly, and exist in all possible orientations of the three strands. Some examples are shown on the right. However, in Section 6.1.3 we will describe the operation "orientation switch" which allows switching the orientation of any given strand. In the algebraic structure which includes this extra operation in addition to the circuit algebra structure, the generators of the definition above are enough.

6.1.1. *The generators of wTF^o.* There is topological meaning to each of the generators of wTF^o: they each stand for a certain local feature of framed knotted ribbon tubes in $\mathbb{R}^4$. As in Section 5.4, the tubes are oriented as 2-dimensional surfaces, and also have a distinguished core with a 1-dimensional orientation (direction).

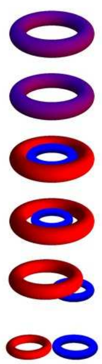
The crossings are as explained in Section 2.2.2 and Section 5.4: the under-strand denotes the ring flying through, or the "thin" tube. Remember that there really are four kinds of crossings, but in the circuit algebra the two not shown are obtained from the two that are shown by adding virtual crossings.



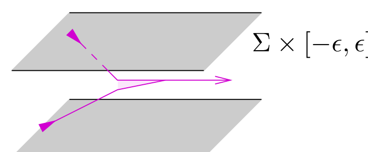
The bulleted end denotes a cap on the tube, or a flying ring that shrinks to a point, as in the figure on the right. In terms of Satoh's tubing map, the cap means that "the string is attached to the bottom of the thickened surface", as shown in the figure below. Recall from Section 3.1.1 that the tubing map is the composition

$$\gamma \times S^1 \hookrightarrow \Sigma \times [-\epsilon, \epsilon] \hookrightarrow \mathbb{R}^4.$$

Here γ is a trivalent tangle with "drawn on the virtual surface Σ", with caps ending on $\Sigma \times [-\epsilon, \epsilon]$. The first embedding above is the product of this "drawing" with an S^1, while the second arises from the unit normal bundle of Σ in $\mathbb{R}^4$. For each cap $(c, -\epsilon)$ the tube resulting from Satoh's map has a boundary component $\partial_c = (c, -\epsilon) \times S^1$. Follow the tubing map by gluing a disc to this boundary component to obtain the capped tube mentioned above.



The last two generators denote singular "foam vertices". As the notation suggests, a vertex can be thought of as "half of a crossing". To make this precise using the flying rings interpretation, the first singular vertex represents the movie shown on the left: the ring corresponding to the right strand approaches the ring represented by the left strand from below, flies inside it, and then the two rings fuse (as opposed to a crossing where the ring coming from the right would continue to fly out to above and to the left of the other one). The second vertex is the movie where a ring splits radially into a smaller and a larger ring, and the small one flies out to the right and below the big one.



The vertices can also be interpreted topologically via a natural extension of Satoh's tubing map. For the first generating vertex, imagine the broken right strand approaching the continuous left strand directly from below in a thickened surface, as shown.

The reader might object that there really are four types of vertices (as there are four types of crossings), and each of these can be viewed as a "fuse" or a "split" depending on the strand directions, as shown in Figure 23. However, looking at the fuse vertices for example, observe that the last two of these can be obtained from the first two by composing with virtual crossings, which always exist in a circuit algebra.

The sign of a vertex can be defined the same way as the sign of a crossing (see Section 5.4). We will sometimes refer to the first generator vertex as "the positive vertex" and to the second one as "the negative vertex". We use the band notation for vertices the same way we do for crossings: the fully coloured band stands for the thin (inner) ring.

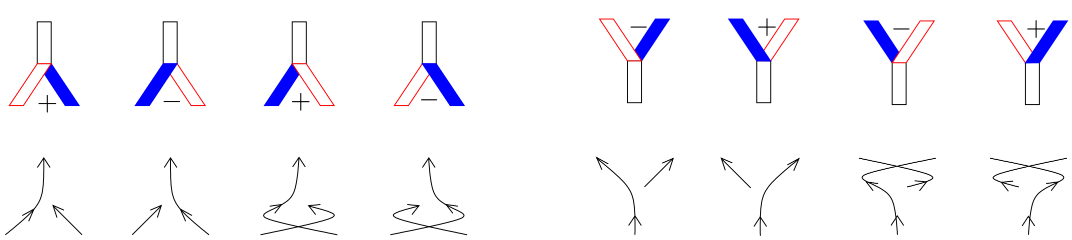


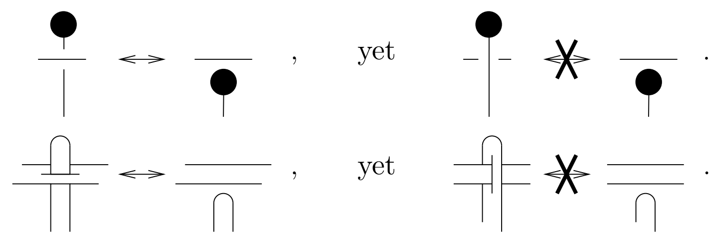
Figure 23. Vertex types in $w\mathit{TF}^o$.

6.1.2. *The relations of* $w\mathit{TF}^o$. In addition to the usual R1^s, R2, R3, and OC moves of Figure 6, we need more relations to describe the behaviour of the additional features.

Comment 6.2. As before, the relations have local topological explanations, and we conjecture that together they provide a Reidemeister theory for "w-tangled foams", that is, knotted ribbon tubes with foam vertices in $\mathbb{R}^4$. In this section we list the relations along with the topological reasoning behind them. However, for any rigorous purposes below, wTF^o is studied as a circuit algebra given by the declared generators and relations, regardless of their topological meaning.

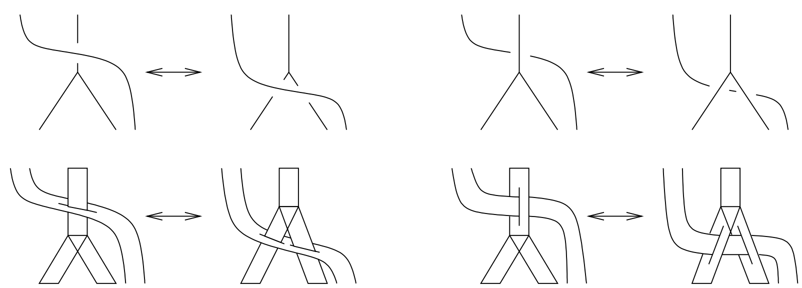
Recall that topologically, a cap represents a capped tube or equivalently, flying ring shrinking to a point. Hence, a cap on the thin (or under) strand can be "pulled out" from a crossing, but the same is not true for a cap on the thick (or over) strand, as shown below. This is the case for any orientation of the strands. We denote this relation by CP, for Cap Pull-out.

CP :



The Reidemeister 4 relations assert that a strand can be moved under or over a crossing, as shown in the picture below. The ambiguously drawn vertices in the picture denote a vertex of any kind (as described in Section 6.1.1), and the strands can be oriented arbitrarily. The local topological (tube or flying ring) interpretations can be read from the pictures below. These relations will be denoted R4.

$R4$:



6.1.3. *The auxiliary operations of* wTF^o. The circuit algebra wTF^o is equipped with several extra operations.

The first of these is the familiar orientation switch. We will, as mentioned in Section 5.4, distinguish between switching both the 2D and 1D orientations, or just the strand (1D) direction.

Topologically *orientation switch*, denoted S_e, is the switch of both orientations of the strand e. Diagrammatically (and this is the definition) S_e is the operation which reverses the orientation of a strand in a wTF^o diagram. The reader can check that when applying Satoh's tubing map, this amounts to reversing both the direction and the 2D orientation of the tube arising from the strand.

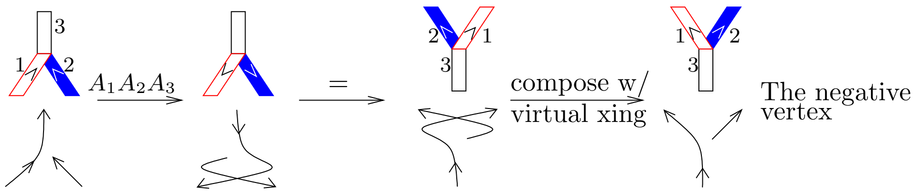
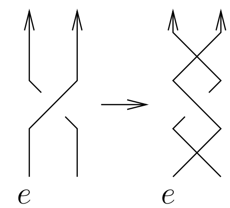


Figure 24. Switching strand orientations at a vertex. The adjoint operation only switches the tube direction, hence in the *band picture* only the arrows change. To express this vertex in terms of the negative generating vertex in strand notation, we use a virtual crossing (see Figure 23).



The operation which, in topology world, reverses a tube's direction but not its 2D orientation is called *"adjoint"*, and denoted by A_e. This is slightly more intricate to define rigorously in terms of diagrams. In addition to reversing the direction of the strand e of the $w\!T\!F^o$ diagram, A_e also locally changes each crossing of e *over* another strand by adding two virtual crossings, as shown on the right. We recommend for the reader to convince themselves that this indeed represents a direction switch in topology after reading Section 6.5.

Remark 6.3. As an example, let us observe how the negative generator vertex can be obtained from the positive generator vertex by adjoint operations and composition with virtual crossings, as shown in Figure 24. Note that also all other vertices can be obtained from the positive vertex via orientation switch and adjoint operations and composition by virtual crossings.

As a small exercise, it is worthwhile to convince ourselves of the effect of orientation switch operations on the *band picture*. For example, replace $A_1A_2A_3$ by $S_1S_2S_3$ in figure 24. In the strand diagram, this will only reverse the direction of the strands. The reader can check that in the band picture not only the arrows will reverse but also the blue band will switch to be on top of the red band.

Comment 6.4. Framings were discussed in Remark 3.5, but have not played a significant role so far, except to explain the lack of a Reidemeister 1 relation. Now we will need to discuss framings in order to provide a topological explanation for the unzip (tube doubling) operation.

In the local topological interpretation of $w\!T\!F^o$, strands represent ribbon-knotted tubes with foam vertices, which are also equipped with a framing, arising from the blackboard framing of the strand diagrams via Satoh's tubing map. Strand doubling is the operation of doubling a tube by "pushing it off itself slightly" in the framing direction, as shown in Figure 25.

Recall that ribbon knotted tubes have a "filling", with only "ribbon" self-intersections. When we double a tube, we want this ribbon property to be preserved. This is equivalent to saying that the ring obtained by pushing off any given girth of the tube in the framing direction is not linked with the original tube, which is indeed the case, as mentioned in Remark 3.5.

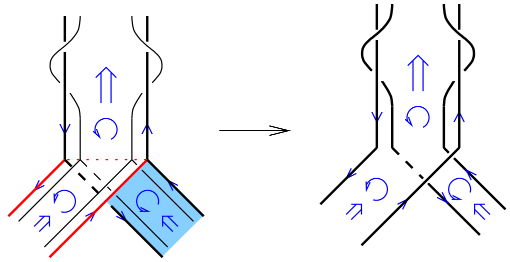
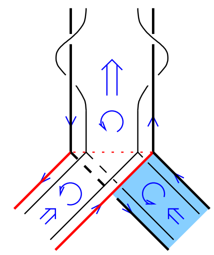


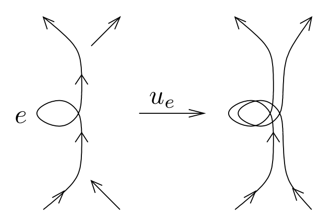
Figure 25. Unzipping a tube, in band notation with orientations and framing marked.

Framings arising from the blackboard framing of strand diagrams via Satoh's tubing map always match at the vertices, with the normal vectors pointing either directly towards or away from the centre of the singular ring. Note that the orientations of the three tubes may or may not match. An example of a vertex with the orientations and framings shown is on the right. Note that the framings on the two sides of each band are mirror images of each other, as they should be.

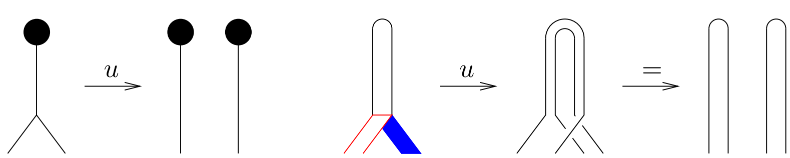


Unzip, or tube doubling is perhaps the most interesting of the auxiliary wTF^o operations. As mentioned above, topologically this means pushing the tube off itself slightly in the framing direction. At each of the vertices at the two ends of the doubled tube there are two tubes to be attached to the doubled tube. At each end, the normal vectors pointed either directly towards or away from the centre, so there is an "inside" and an "outside" ending ring. The two tubes to be attached also come as an "inside" and an "outside" one, which defines which one to attach to which. An example is shown in Figure 25. Unzip can only be done if the 1D and 2D orientations match at both ends.

To define unzip rigorously, we must talk only of strand diagrams. The natural definition is to let u_e double the strand e using the blackboard framing, and then attach the ends of the doubled strand to the connecting ones, as shown on the right. We restrict unzip to strands whose two ending vertices are of different signs. This is a somewhat artificial condition which we impose to get equations equivalent to the [AT] equations.



A related operation, *disk unzip*, is unzip done on a capped strand, pushing the tube off in the direction of the framing (in diagrammatic world, in the direction of the blackboard framing), as before. An example in the line and band notations (with the framing suppressed) is shown below.

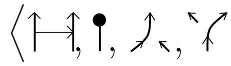


Finally, we allow the deletion of "long linear" strands, meaning strands that do not end in a vertex on either side.

The goal, as before, is to construct a homomorphic expansion for wTF^o. However, first we need to understand its target space, the projectivization proj wTF^o.

6.2. **The projectivization.** Mirroring the previous section, we describe the projectivization $\mathcal{A}^{sw}$ of wTF^o and its "full version" $\mathcal{A}^w$ as circuit algebras on certain generators modulo a number of relations. From now on we will write $\mathcal{A}^{(s)w}$ to mean "$\mathcal{A}^w$ and/or $\mathcal{A}^{sw}$".

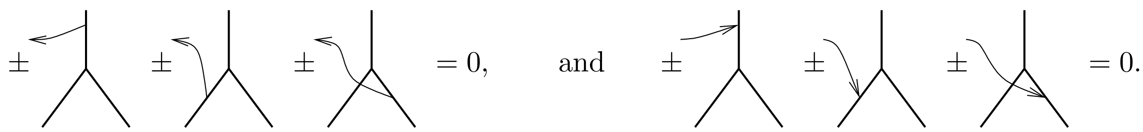
$$\mathcal{A}^{(s)w} = \mathrm{CA}\left\langle \text{generators} \;\middle|\; \begin{array}{c}\text{relations as in}\\ \text{Section 6.2.1}\end{array} \;\middle|\; \begin{array}{c}\text{operations as in}\\ \text{Section 6.2.2}\end{array} \right\rangle.$$



In other words, $\mathcal{A}^{(s)w}$ are the circuit algebras of arrow diagrams on trivalent (or foam) skeletons with caps. Note that all but the first of the generators are skeleton features (of degree 0), and that the single arrow is the only generator of degree 1. As for the generating vertices, the same remark applies as in Definition 6.1, that is, there are more vertices with all possible strand orientations needed to generate $\mathcal{A}^{(s)w}$ as circuit algebras.

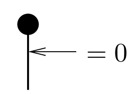
6.2.1. *The relations of* $\mathcal{A}^{(s)w}$. In addition to the usual $\overrightarrow{4T}$ and TC relations (see Section 2.3), as well as RI in the case of $\mathcal{A}^{sw} = \mathcal{A}^w/RI$, diagrams in $\mathcal{A}^{(s)w}$ satisfy the following additional relations:

Vertex invariance, denoted by VI, are relations arising the same way as $\overrightarrow{4T}$ does, but with the participation of a vertex as opposed to a crossing:



The other end of the arrow is in the same place throughout the relation, somewhere outside the picture shown. The signs are positive whenever the strand on which the arrow ends is directed towards the vertex, and negative when directed away. The ambiguously drawn vertex means any kind of vertex, but the same one throughout.

The CP relation (a cap can be pulled out from under a strand but not from over, Section 6.1.2) implies that arrow heads near a cap are zero, as shown on the right. Denote this relation also by CP. (Also note that a tail near a cap is not set to zero.)



As in the previous sections, and in particular in Definition 3.13, we define a "w-Jacobi diagram" (or just "arrow diagram") on a foam skeleton by allowing trivalent chord vertices. Denote the circuit algebra of formal linear combinations of arrow diagrams by $\mathcal{A}^{(s)wt}$. We have the following bracket-rise theorem:

Theorem 6.5. *The obvious inclusion of diagrams induces a circuit algebra isomorphism* $\mathcal{A}^{(s)w} \cong \mathcal{A}^{(s)wt}$. *Furthermore, the* $\overrightarrow{AS}$ *and* $\overrightarrow{IHX}$ *relations of Figure 12 hold in* $\mathcal{A}^{(s)wt}$.

Proof. Same as the proof of Theorem 3.15. □

As in Section 5.1, the primitive elements of $\mathcal{A}^{(s)w}$ are connected diagrams, namely trees and wheels. Before moving on to the auxiliary operations of $\mathcal{A}^{(s)w}$, let us make two useful observations:

Lemma 6.6. $\mathcal{A}^w(\uparrow)$, *the part of* $\mathcal{A}^w$ *with skeleton* $\uparrow$, *is isomorphic as a vector space to the completed polynomial algebra freely generated by wheels* w_k *with* $k \geq 1$. *Likewise* $\mathcal{A}^{sw}(\uparrow)$, *except here* $k \geq 2$.

Proof. Any arrow diagram with an arrow head at its top is zero by the Cap Pull-out (CP) relation. If D is an arrow diagram that has a head somewhere on the skeleton but not at the top, then one can use repeated $\overrightarrow{STU}$ relations to commute the head to the top at the cost of diagrams with one fewer skeleton head.

Iterating this procedure, we can get rid of all arrow heads, and hence write D as a linear combination of diagrams having no heads on the skeleton. All connected components of such diagrams are wheels.

To prove that there are no relations between wheels in $\mathcal{A}^{(s)w}(\overset{\bullet}{\uparrow})$, let $S_L\colon \mathcal{A}^{(s)w}(\uparrow_1) \to \mathcal{A}^{(s)w}(\uparrow_1)$ (resp. S_R) be the map that sends an arrow diagram to the sum of all ways of dropping one left (resp. right) arrow (on a vertical strand, left means down and right means up). Define

$$F := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} D_R^k (S_L + S_R)^k,$$

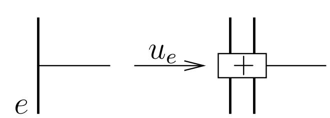
where D_R is the short right arrow as shown in Figure 13. We leave it as an exercise for the reader to check that F is a bi-algebra homomorphism that kills diagrams with an arrow head at the top (i.e., CP is in the kernel of F), and F is injective on wheels. This concludes the proof. $\square$

Lemma 6.7. *$\mathcal{A}^{(s)w}(Y) = \mathcal{A}^{(s)w}(\uparrow_2)$, where $\mathcal{A}^{(s)w}(Y)$ stands for the space of arrow diagrams whose skeleton is a Y-graph with any orientation of the strands, and as before $\mathcal{A}^{(s)w}(\uparrow_2)$ is the space of arrow diagrams on two strands.*

Proof. We can use the vertex invariance (VI) relation to push all arrow heads and tails from the "trunk" of the vertex to the other two strands. $\square$

6.2.2. *The auxiliary operations of $\mathcal{A}^{(s)w}$.* Recall from Section 5.4 that the orientation switch S_e (i.e. changing both the $1D$ and $2D$ orientations of a strand) always changes the sign of a crossing involving the strand e. Hence, letting S denote any foam (trivalent) skeleton, the induced arrow diagrammatic operation is a map $S_e\colon \mathcal{A}^{(s)w}(S) \to \mathcal{A}^{(s)w}(S_e(S))$ which acts by multiplying each arrow diagram by (-1) raised to the number of arrow endings on e (counting both heads and tails).

The adjoint operation A_e (i.e. switching only the strand direction), on the other hand, only changes the sign of a crossing when the strand being switched is the under- (or through) strand. (See section 5.4 for pictures and explanation.) Therefore, the arrow diagrammatic A_e acts by switching the direction of e and multiplying each arrow diagram by (-1) raised to the number of *arrow heads* on e. Note that in $\mathcal{A}^{(s)w}(\uparrow_n)$ taking the adjoint on every strand gives the adjoint map of Definition 5.18.



The arrow diagram operations induced by unzip and disc unzip (both to be denoted u_e, and interpreted appropriately according to whether the strand e is capped) are maps $u_e\colon \mathcal{A}^{(s)w}(S) \to \mathcal{A}^{(s)w}(u_e(S))$, where each arrow ending (head or tail) on e is mapped to a sum of two arrows, one ending on each of the new strands, as shown on the right. In other words, if in an arrow diagram D there are k arrow ends on e, then $u_e(D)$ is a sum of 2^k arrow diagrams.

The operation induced by deleting the long linear strand e is the map $d_e\colon \mathcal{A}^{(s)w}(S) \to \mathcal{A}^{(s)w}(d_e(S))$ which kills arrow diagrams with any arrow ending (head or tail) on e, and leaves all else unchanged, except with e removed.

6.3. **The homomorphic expansion.**

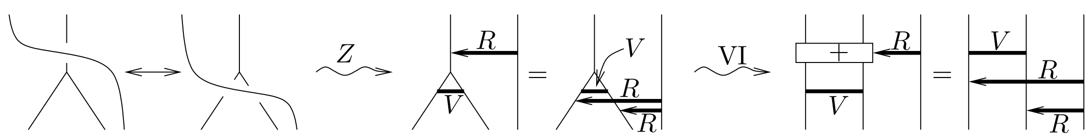
Theorem 6.8. *There exists a group-like*[35] *homomorphic expansion for $w\mathit{TF}^o$, i.e. a group-like expansion $Z\colon w\mathit{TF}^o \to \mathcal{A}^{sw}$ which is a map of circuit algebras and also intertwines the auxiliary operations of $w\mathit{TF}^o$ with their arrow diagrammatic counterparts.*

Since both $w\mathit{TF}^o$ and $\mathcal{A}^{sw}$ are circuit algebras defined by generators and relations, when looking for a suitable Z all we have to do is to find values for each of the generators of $w\mathit{TF}^o$ so that these satisfy (in $\mathcal{A}^{sw}$) the equations which arise from the relations in $w\mathit{TF}^o$ and the homomorphicity requirement. In this section we will derive these equations and show that they are equivalent to the Alekseev-Torossian version of the Kashiwara-Vergne equations [AT]. In [AET] Alekseev Enriquez and Torossian construct explicit solutions to these equations using associators. In a later paper we will interpret these results in our context of homomorphic expansions for w-tangled foams.

Let $R := Z(\diagup\!\!\!\!\diagdown) \in \mathcal{A}^{sw}(\uparrow_2)$. It follows from the Reidemeister 2 relation that $Z(\diagdown\!\!\!\!\diagup) = (R^{-1})^{21}$. As discussed in Sections 5.1 and 5.5, Reidemeister 3 with group-likeness and homomorphicity implies that $R = e^a$, where a is a single arrow pointing from the over to the under strand. Let $C := Z(\uparrow) \in \mathcal{A}^{sw}(\uparrow)$. By Lemma 6.6, we know that C is made up of wheels only. Finally, let $V = V^+ := Z(\curlywedge) \in \mathcal{A}^{sw}(\curlywedge) \cong \mathcal{A}^{sw}(\uparrow_2)$, and $V^- := Z(\curlyvee) \in \mathcal{A}^{sw}(\curlyvee) \cong \mathcal{A}^{sw}(\uparrow_2)$.

Before we translate each of the relations of Section 6.1.2 to equations let us slightly extend the notation used in Section 5.5. Recall that R^{23}, for instance, meant "R placed on strands 2 and 3". In this section we also need notation such as $R^{(23)1}$, which means "R with its first strand doubled, placed on strands 2, 3 and 1".

Now on to the relations, note that Reidemeister 2 and 3 and Overcrossings Commute have already been dealt with. Of the two Reidemeister 4 relations, the first one induces an equation that is automatically satisfied. Pictorially, the equation looks as follows:



In other words, we obtained the equation

$$V^{12}R^{3(12)} = R^{32}R^{31}V^{12}.$$

However, observe that by the "head-invariance" property of arrow diagrams (Remark 5.7) V^{12} and $R^{3(12)}$ commute on the left hand side. Hence the left hand side equals $R^{3(12)}V^{12} = R^{32}R^{31}V^{12}$. Also, $R^{3(12)} = e^{a^{31}+a^{32}} = e^{a^{32}}e^{a^{31}} = R^{32}R^{31}$, where the second step is due to the fact that a^{31} and a^{32} commute. Therefore, the equation is true independently of the choice of V.

We have no such luck with the second Reidemeister 4 relation, which, in the same manner as in the paragraph above, translates to the equation

$$V^{12}R^{(12)3} = R^{23}R^{13}V^{12}. \tag{41}$$

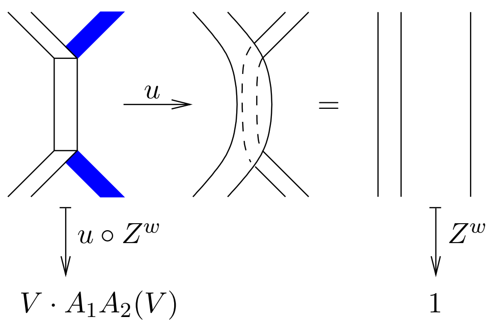
[35]The formal definition of the group-like property is along the lines of 2.5.1.2. In practise, it means that the Z-values of the vertices, crossings, and cap (denoted V, R and C below) are exponentials of linear combinations of connected diagrams.

There is no "tail invariance" of arrow diagrams, so V and R do not commute on the left hand side; also, $R^{(12)3} \neq R^{23}R^{13}$. As a result, this equation puts a genuine restriction on the choice of V.

The Cap Pull-out (CP) relation translates to the equation $R^{12}C^2 = C^2$. This is true independently of the choice of C: by head-invariance, $R^{12}C^2 = C^2R^{12}$. Now R^{12} is just below the cap on strand 2, and the cap "kills heads", in other words, every term of R^{12} with an arrow head at the top of strand 2 is zero. Hence, the only surviving term of R^{12} is 1 (the empty diagram), which makes the equation true.

The homomorphicity of the orientation switch operation was used to prove the uniqueness of R in Theorem 5.22. The homomorphicity of the adjoint leads to the equation $V_- = A_1A_2(V)$ (see Figure 24), eliminating V_- as an unknown. Note that we also silently assumed these homomorphicity properties when we did not introduce 32 different values of the vertex depending on the strand orientations.

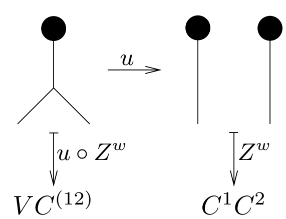
Homomorphicity of the (annular) unzip operation leads to an equation for V, which we are going to refer to as "unitarity". This is illustrated in the figure below. Recall that A_1 and A_2 denote the adjoint (direction switch) operation on strand 1 and 2, respectively.



Reading off the equation, we have

$$V \cdot A_1A_2(V) = 1. \tag{42}$$

Homomorphicity of the disk unzip leads to an equation for C which we will refer to as the "cap equation". The translation from homomorphicity to equation is shown in the figure on the right. C, as we introduced before, denotes the Z-value of the cap. Hence, the cap equation reads



$$V^{12}C^{(12)} = C^1C^2 \qquad \text{in} \quad \mathcal{A}^{sw}(\uparrow_2) \tag{43}$$

The homomorphicity of deleting long strands does not lead to an equation on its own, however it was used to prove the uniqueness of R (Theorem 5.22).

To summarize, we have reduced the problem of finding a homomorphic expansion Z to finding the Z-values of the (positive) vertex and the cap, denoted V and C, subject to three equations: the "hard Reidemeister 4" equation (41); "unitarity of V" equation (42); and the "cap equation" (43).

6.4. **The equivalence with the Alekseev-Torossian equations.** First let us recall Alekseev and Torossian's formulation of the generalized Kashiwara-Vergne problem (see [AT, Section 5.3]):

Generalized KV problem: Find an element $F \in \mathrm{TAut}_2$ with the properties

$$F(x+y) = \log(e^x e^y), \text{ and } j(F) \in \operatorname{im}(\tilde{\delta}). \tag{44}$$

Here $\tilde{\delta}\colon \mathfrak{tr}_1 \to \mathfrak{tr}_2$ is defined by $(\tilde{\delta}a)(x,y) = a(x) + a(y) - a(\log(e^x e^y))$, where elements of $\mathfrak{tr}_2$ are cyclic words in the letters x and y. (See [AT], Equation (8)). Note that an element of $\mathfrak{tr}_1$ is a polynomial with no constant term in one variable. In other words, the second condition says that there exists $a \in \mathfrak{tr}_1$ such that $jF = a(x) + a(y) - a(\log(e^x e^y))$.

Theorem 6.9. *Theorem 6.8, namely the existence of a group-like homomorphic expansion for wTF^o, is equivalent to the generalized Kashiwara-Vergne problem.*

Proof. We have reduced the problem of finding a homomorphic expansion to finding group-like solutions V and C to the hard Reidemeister 4 equation (41), the unitarity equation (42), and the cap equation (43).

Suppose we have found such solutions and write $V = e^b e^{uD}$, where $b \in \mathrm{tr}_2^s$, $D \in \mathfrak{tder}_2 \oplus \mathfrak{a}_2$, and where u is the map $u\colon \mathfrak{tder}_2 \to \mathcal{A}^{sw}(\uparrow_2)$ which plants the head of a tree above all of its tails, as introduced in Section 5.2. V can be written in this form without loss of generality because wheels can always be brought to the bottom of a diagram (at the possible cost of more wheels). Furthermore, V is group-like and hence it can be written in exponential form. Similarly, write $C = e^c$ with $c \in \mathfrak{tr}_1^s$.

Note that $u(\mathfrak{a}_2)$ is central in $\mathcal{A}^{sw}(\uparrow_2)$ and that replacing a solution (V, C) by $(e^{u(a)}V, C)$ for any $a \in \mathfrak{a}_2$ does not interfere with any of the equations (41), (42) or (43). Hence we may assume that D does not contain any single arrows, that is, $D \in \mathfrak{tder}_2$. Also, a solution (V, C) in $\mathcal{A}^{sw}$ can be lifted to a solution in $\mathcal{A}^w$ by simply setting the degree one terms of b and c to be zero. It is easy to check that this $b \in \mathfrak{tr}_2$ and $c \in \mathfrak{tr}_1$ along with D still satisfy the equations. (In fact, in $\mathcal{A}^w$ (42) and (43) respectively imply that b is zero in degree 1, and that the degree 1 term of c is arbitrary, so we may as well assume it to be zero.) In light of this we declare that $b \in \mathfrak{tr}_2$ and $c \in \mathfrak{tr}_1$.

The hard Reidemeister 4 equation (41) reads $V^{12}R^{(12)3} = R^{23}R^{13}V^{12}$. Denote the arrow from strand 1 to strand 3 by x, and the arrow from strand 2 to strand 3 by y. Substituting the known value for R and rearranging, we get

$$e^b e^{uD} e^{x+y} e^{-uD} e^{-b} = e^y e^x.$$

Equivalently, $e^{uD}e^{x+y}e^{-uD} = e^{-b}e^y e^x e^b$. Now on the right side there are only tails on the first two strands, hence e^b commutes with $e^y e^x$, so $e^{-b}e^b$ cancels. Taking logarithm of both sides we obtain $e^{uD}(x+y)e^{-uD} = \log e^y e^x$. Now for notational alignment with [AT] we switch strands 1 and 2, which exchanges x and y so we obtain:

$$e^{uD^{21}}(x+y)e^{-uD^{21}} = \log e^x e^y. \tag{45}$$

The unitarity of V (Equation (42)) translates to $1 = e^b e^{uD}(e^b e^{uD})^*$, where $*$ denotes the adjoint map (Definition 5.18). Note that the adjoint switches the order of a product and acts trivially on wheels. Also, $e^{uD}(e^{uD})^* = J(e^D) = e^{j(e^D)}$, by Proposition 5.19. So we have $1 = e^b e^{j(e^D)} e^b$. Multiplying by e^{-b} on the right and by e^b on the left, we get $1 = e^{2b}e^{j(e^D)}$, and again by switching strand 1 and 2 we arrive at

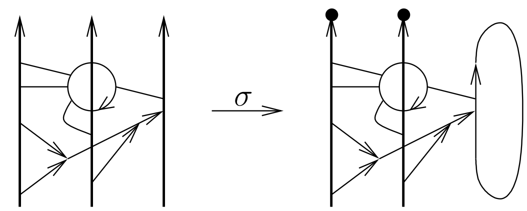
$$1 = e^{2b^{21}} e^{j(e^{D^{21}})}. \tag{46}$$

As for the cap equation, if $C^1 = e^{c(x)}$ and $C^2 = e^{c(y)}$, then $C^{12} = e^{c(x+y)}$. Note that wheels on different strands commute, hence $e^{c(x)}e^{c(y)} = e^{c(x)+c(y)}$, so the cap equation reads

$$e^b e^{uD} e^{c(x+y)} = e^{c(x)+c(y)}.$$

As this equation lives in the space of arrow diagrams on two *capped* strands, we can multiply the left side on the right by e^{-uD}: uD has its head at the top, so it is 0 by the Cap relation, hence $e^{uD} = 1$ near the cap. Hence,

$$e^b e^{uD} e^{c(x+y)} e^{-uD} = e^{c(x)+c(y)}.$$



On the right side of the equation above $e^{uD}e^{c(x+y)}e^{-uD}$ reminds us of Equation (45), however we cannot use (45) directly as we live in a different space now. In particular, x there meant an arrow from strand 1 to strand 3, while here it means a one-wheel on (capped) strand 1, and similarly for y. Fortunately, there is a map $\sigma\colon \mathcal{A}^{sw}(\uparrow_3) \to \mathcal{A}^{sw}(\overset{\bullet}{\uparrow}_2)$, where σ "closes the third strand and turns it into a chord (or internal) strand, and caps the first two strands", as shown on the right. This map is well defined (in fact, it kills almost all relations, and turns one $\overrightarrow{STU}$ into an $\overrightarrow{IHX}$). Under this map, using our abusive notation, $\sigma(x) = x$ and $\sigma(y) = y$.

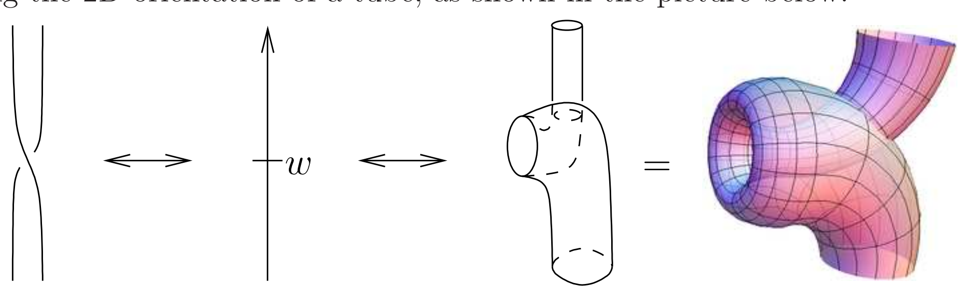
Now we can apply Equation (45) and get $e^b e^{c(\log e^y e^x)} = e^{c(x)+c(y)}$, which, using that tails commute, implies $b = c(x) + c(y) - c(\log e^y e^x)$. Switching strands 1 and 2, we obtain

$$b^{21} = c(x) + c(y) - c(\log e^x e^y) \tag{47}$$

In summary, we can use (V, C) to produce $F := e^{D^{21}}$ (sorry[36]) which satisfies the Alekseev-Torossian equations (44): $e^{D^{21}}$ acts on $\mathfrak{lie}_2$ by conjugation by $e^{uD^{21}}$, so the first part of (44) is implied by (45). The second half of (44) is true due to (46) and (47).

On the other hand, suppose that we have found $F \in \mathrm{TAut}_2$ and $a \in \mathrm{tr}_1$ satisfying (44). Then set $D^{21} := \log F$, $b^{21} := \frac{-j(e^{D^{21}})}{2}$, and $c \in \tilde{\delta}^{-1}(b^{21})$, in particular $c = -\frac{a}{2}$ works. Then $V = e^b e^{uD}$ and $C = e^c$ satisfy the equations for homomorphic expansions (41), (42) and (43). $\square$

6.5. **The wen.** A topological feature of w-tangled foams which we excluded from the theory so far is the wen w. The wen was introduced in 2.5.4 as a Klein bottle cut apart; it amounts to changing the 2D orientation of a tube, as shown in the picture below:

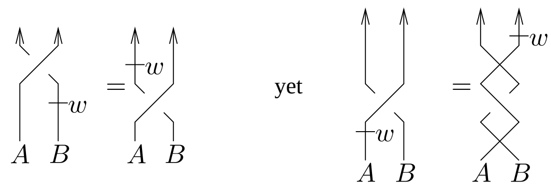


[36] We apologize for the annoying $2 \leftrightarrow 1$ transposition in this equation, which makes some later equations, especially (52), uglier than they could have been. There is no depth here, just mis-matching conventions between us and Alekseev-Torossian.

In this section we study the circuit algebra of w-Tangled Foams with the wen rightfully included as a generator, and denote this space by *wTF*.

6.5.1. *The relations and auxiliary operations of wTF.* Adding the wen as a generator means we have to impose additional relations involving the wen to keep our topological heuristics intact, as follows:

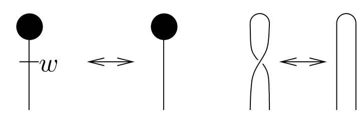
The interaction of a wen and a crossing has already been mentioned in Section 2.5.4, and is described by Equation (16), which we repeat here for convenience:

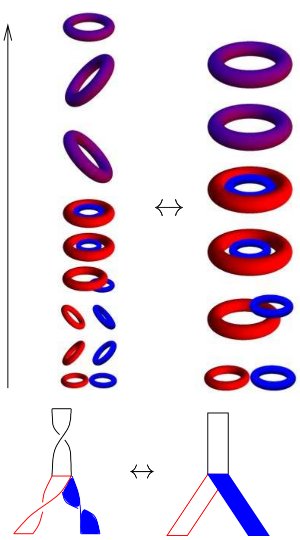


Recall that in flying ring language, a wen is a ring flipping over. It does not matter whether ring B flips first and then flies through ring A or vice versa. However, the movies in which ring A first flips and then ring B flies through it, or B flies through A first and then A flips differ in the fly-through direction, which is cancelled by virtual crossings, as in the figure above. We will refer to these relations as the Flip Relations, and abbreviate them by FR.

A double flip is homotopic to no flip, in other words two consecutive wens equal no wen. Let us denote this relation by W^2, for Wen squared. Note that this relation explains why there are no "left and right wens".

A cap can slide through a wen, hence a capped wen disappears, as shown on the right, to be denoted CW.





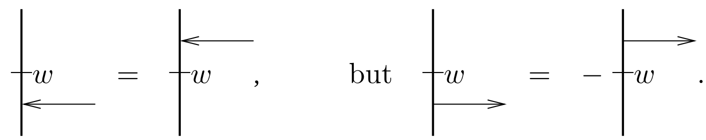
The last wen relation describes the interaction of wens and vertices. Recall that there are four types of vertices with the same strand orientation: among the bottom two bands (in the pictures on the left) there is a non-filled and a filled band (corresponding to over/under in the strand diagrams), meaning the "large" ring and the "small" one which flies into it before they merge. Furthermore, there is a top and a bottom band (among these bottom two, with apologies for the ambiguity in overusing the word bottom): this denotes the fly-in direction (flying in from below or from above). Conjugating a vertex by three wens switches the top and bottom bands, as shown in the figure on the left: if both rings flip, then merge, and then the merged ring flips again, this is homotopic to no flips, except the fly-in direction (from below or from above) has changed. We are going to denote this relation by TV, for "twisted vertex".

The auxiliary operations are the same as for wTF^o: orientation switches, adjoints, deletion of long linear strands, cap unzips and unzips[37]. Thus, informally we can say that $wTF = (wTF^o + \text{wens})/\text{FR}, W^2, \text{CW}, \text{TV}$.

[37] We need not specify how to unzip an edge e that carries a wen. To unzip such e, first use the TV relation to slide the wen off e.

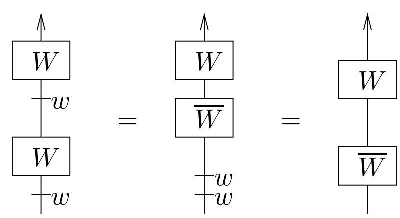
6.5.2. *The projectivization.* The projectivization of *wTF* (still denoted $\mathcal{A}^{sw}$) is the same as the projectivization for wTF^o but with the wen added as a generator (a degree 0 skeleton feature), and with extra relations describing the behaviour of the wen. Of course, the relations describing the interaction of wens with the other skeleton features (W^2, TV, and CW) still apply, as well as the old RI, $\overrightarrow{4T}$, and TC relations.

In addition, the Flip Relations FR imply that wens "commute" with arrow heads, but "anti-commute" with tails. We also call these FR relations:

FR: 

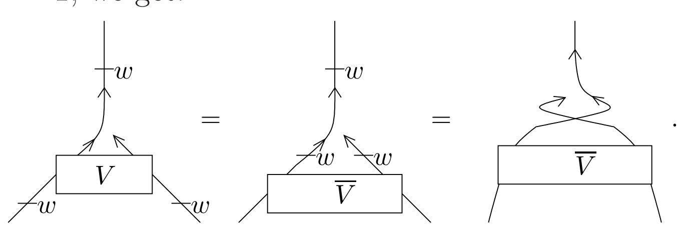
6.5.3. *The homomorphic expansion.* The goal of this section is to prove that there exists a homomorphic expansion Z for *wTF*. This involves solving a similar system of equations to Section 6.3, but with an added unknown for the value of the wen, as well as added equations arising from the wen relations. Let $W \in \mathcal{A}(\uparrow_1)$ denote the Z-value of the wen, and let us agree that the arrow diagram W always appears just above the wen on the skeleton. In fact, we are going to show that there exists a homomorphic expansion with $W = 1$.

As two consecutive wens on the skeleton cancel, we obtain the equation shown in the picture and explained below:



The Z-value of two consecutive wens on a strand is a skeleton wen followed by W followed by a skeleton wen and another W. Sliding the bottom W through the skeleton wen "multiplies each tail by (-1)". Let us denote this operation by "bar", i.e. for an arrow diagram D, $\overline{D} = D \cdot (-1)^{\#\text{ of tails in } D}$. Cancelling the two skeleton wens, we obtain $\overline{W}W = 1$. If $W = 1$ then this equation is certainly satisfied.

Now recall the Twisted Vertex relation of Section 6.5.1. Note that the negative the Z-value of the vertex on the right hand side of the relation can be written as $S_1S_2A_1A_2(V) = \overline{(V)}$. (Compare with Remark 6.3.) On the other hand, applying Z to the left hand side of the relation, assuming $W = 1$, we get:



Thus, the equation arising from the twisted vertex relation with $W = 1$ is automatically satisfied.

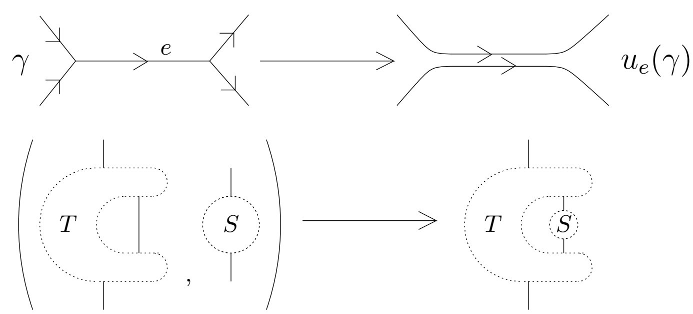
The CW (Capped Wen) relation says that a cap can slide through a wen. The value of the wen is 1, but the wen as a skeleton feature anti-commutes with tails (this is the Flip

Relation of Section 6.2.1). The value of the cap C is made up of only wheels (Lemma 6.6), hence the CW relation translates to the equation $\overline{C} = C$, which is equivalent to saying that C consists only of even wheels.

The fact that this is possible follows from Proposition 6.2 of [AT]: the value of the cap is $C = e^c$, where can be set to $c = -\frac{a}{2}$, as explained in the proof of Theorem 6.9. Here a is such that $\tilde{\delta}(a) = jF$ as in Equation 44. A power series f so that $a = \operatorname{tr} f$ (where tr is the trace which turns words into cyclic words) is called the Duflo function of F. In Proposition 6.2 Alekseev and Torossian show that the even part of f is $\frac{1}{2}\frac{ln(e^{x/2}-e^{-x/2})}{x}$, and that for any f with this even part there is a corresponding solution F of the generalized KV problem. In particular, f can be assumed to be even, namely the power series above, and hence it can be guaranteed that C consists of even wheels only. Thus we have proven the following:

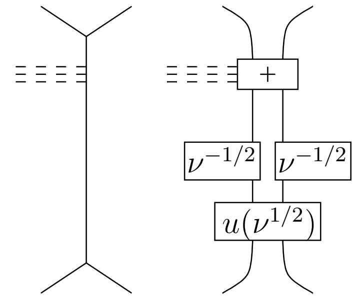
Theorem 6.10. *There exists a group-like homomorphic expansion $Z : wTF \to \mathcal{A}^{sw}$.* □

6.6. **The relationship with u-Knotted Trivalent Graphs.** The "$u$sual", or classical topological objects corresponding to wTF are loosely speaking Knotted Trivalent Graphs, or KTGs. A trivalent graph is a graph with three edges meeting at each vertex, equipped with a cyclic orientation of the three half-edges at each vertex. KTGs are framed embeddings of trivalent graphs into $\mathbb{R}^3$, regarded up to isotopies. The skeleton of a KTG is the trivalent graph (as a combinatorial object) behind it. For a detailed introduction to KTGs see for example [BND1]. Here we only recall the most important facts. The reader might recall that in Section 3 we only dealt with long w-knots, as the w-theory of round knots is essentially trivial (see Theorem 3.18). A similar issue arises with "w-knotted trivalent graphs". Hence, the space we are really interested in is "long KTGs", in other words trivalent $(1,1)$-tangles whose "top end" is connected to the "bottom end" by some path along the tangle.



Long KTGs form an algebraic structure with the operations orientation switch; edge unzip (as shown on the right); and tangle insertion (I.e., inserting a small copy of a $(1,1)$-tangle S into the middle of some strand of a $(1,1)$-tangle T, also shown on the right. It is a slightly weaker operation than the connected sum of [BND1]). The projectivization of the space of long KTGs is the space $\mathcal{A}^u$ of chord diagrams on long trivalent graph skeleta, modulo the $4T$ and vertex invariance (VI) relations. The induced operations on $\mathcal{A}^u$ are as expected: orientation switch multiplies a chord diagram by (-1) to the number of chord endings on the edge. Edge unzips u_e maps a chord diagram with k chord endings on the edge e to a sum of 2^k diagrams where each chord ending has a choice between the two daughter edges. Finally, tangle insertion induces the insertion of chord diagrams.

In [BND1] the authors prove that there is no *homomorphic* expansion for KTGs. This theorem, as well as the proof, applies to long KTGs with slight modifications. There is a well-known expansion constructed by extending the Kontsevich integral to KTGs and renormalizing at the vertices. There are several constructions that do this ([MO], [CL], [Da]), and not all of these are "compatible" with a corresponding Z^w. For now, let us choose one (any) such expansion and following the notation of [BND1] denote it by Z^{old}. It turns out that any of the above Z^{old} is almost homomorphic but not quite: they all intertwine the orientation switch, strand delete and tangle composition operations with their chord-diagrammatic counterparts, but commutativity with unzip fails by a controlled amount, as shown on the right. Here ν denotes the "invariant of the unknot", the value of which was conjectured in [BGRT1] and proven in [BLT].



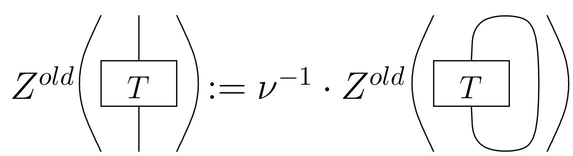
$Z^{old}(\gamma)$ $\qquad$ $Z^{old}(u(\gamma))$

In [BND1] the authors fix this anomaly by slightly changing the space of KTGs and adding some extra combinatorics ("dots" on the edges), and construct a homomorphic expansion for this new space by a slight adjustment of Z^{old}. Here we are going to use a similar but different adjustment of the space of long KTGs, namely breaking the symmetry of the vertices and restricting the domain of unzip.

In this model, denoted by *sKTG* for "signed long KTGs", each vertex has a distinguished edge coming out of it (denoted by a thick line in Figure 26), as well as a sign. Our pictorial convention will be that a vertex drawn in a "λ" shape with all strands oriented up and the top strand distinguished is always positive and a vertex drawn in a "Y" shape with strands oriented up and the bottom strand distinguished is always negative, as in Figure 26.

Orientation switch of either of the non-distinguished strands changes the sign of the vertex, switching the orientation of the distinguished strand does not. Unzip of an edge is only allowed if the edge is distinguished at both of its ends and the vertices at either end are of opposite signs.

The homomorphic expansion $Z^u\colon sKTG \to \mathcal{A}^u$ is computed from Z^{old} as follows. First of all we need to interpret Z^{old} as an invariant of $(1,1)$-tangles. This is done by connecting the top and bottom ends by a non-interacting long strand followed by a normalization, as shown on the right. By "multiplying by ν^{-1}" we mean that after computing Z^{old} we insert ν^{-1} on the long strand.



To compute Z^u from Z^{old} the following normalizations are added near the vertices, as in Figure 26. Note that in that figure the symbol c denotes a horizontal chord going from left to right, and $e^{\pm c/4}$ denotes the exponential of $\pm c/4$ in a sense similar to the exponentiation of arrows in Equation (15).

Checking that Z^u is a homomorphic expansion is a simple calculation using the almost homomorphicity of Z^{old}, which we leave to the reader. Now let us move on the the question of compatibility between Z^u and Z^w (from now on we are going to refer to the homomorphic expansion of *wTF* –called Z in the previous section– as Z^w to avoid confusion).

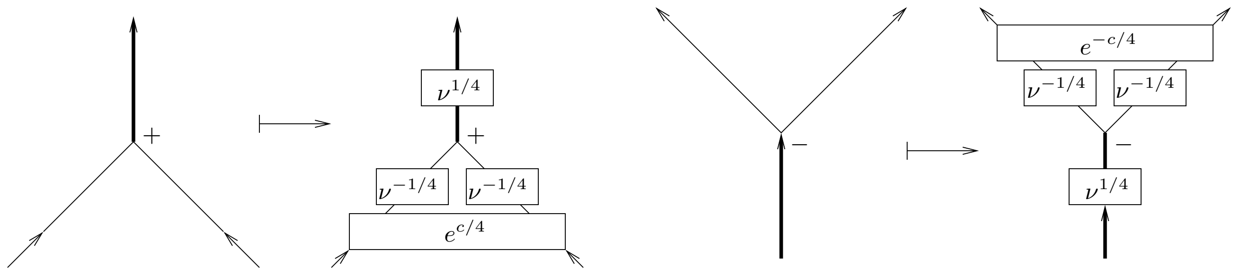


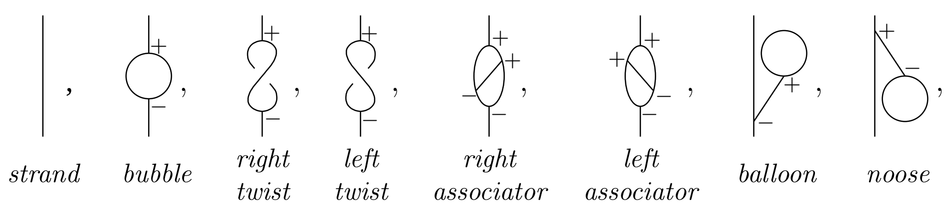
Figure 26. Normalizations for Z^u at the vertices.

$$\begin{tikzcd} sKTG \arrow[r, "a"] \arrow[d, "Z^u"'] & wTF \arrow[d, "Z^w"] \\ \mathcal{A}^u \arrow[r, "\alpha"] & \mathcal{A}^{sw} \end{tikzcd}$$

There is a map $a\colon$ *sKTG* $\to$ *wTF*, given by interpreting *sKTG* diagrams as *wTF* diagrams. In particular, positive vertices (of edge orientations shown above) are interpreted as the positive *wTF* vertex and negative vertices as the negative . The induced map $\alpha\colon \mathcal{A}^u \to \mathcal{A}^{sw}$ is defined as in Section 5.3, that is, α maps each chord to the sum of its two possible orientations. Now we can ask the question whether the square on the left commutes, or more precisely, whether we can choose Z^u and Z^w so that it does.

As a first step to answering this question, we prove that *sKTG* is finitely generated (and therefore Z^u is determined by its values on finitely many generators, and these values will later be compared with the values V and C that determine Z^w):

Proposition 6.11. *The algebraic structure sKTG is finitely generated by the following list of elements:*

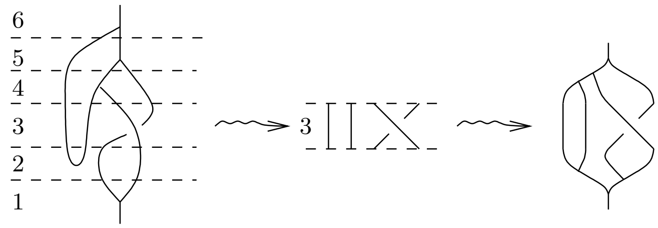


Proof. First of all note that throughout this proof (in fact even in the statement of the proposition) we are ignoring the issue of strand orientations. We can do this as orientation switches are allowed in *sKTG* without restriction. We are also going to omit vertex signs from the pictures given the pictorial convention stated before.

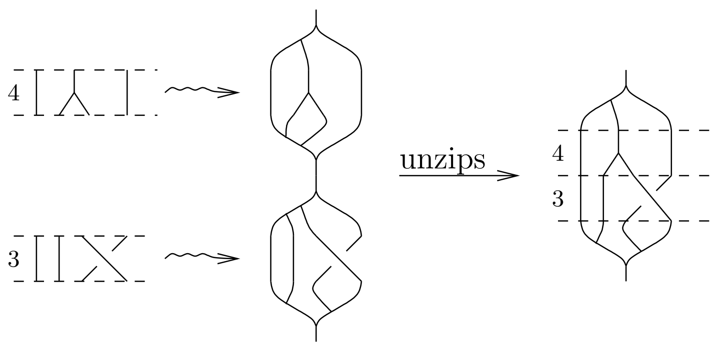
We need to prove that any *sKTG* (call it G) can be built from the generators above using *sKTG* operations. To show this, consider a Morse drawing of G, that is, a planar projection of G with a height function so that all singularities along the strands are Morse and so that every "feature" of the projection (local minima and maxima, crossings and vertices) occurs at a different height.

The idea in short is to decompose G into levels of this Morse drawing where at each level only one "feature" occurs. The levels themselves are not *sKTG*'s, but we show that the composition of the levels can be achieved by composing their "closed-up" *sKTG* versions followed by some unzips. Each feature gives rise to a generator by "closing up" extra ends at its top and bottom. We then show that we can construct each level using the generators and the tangle insert operation.

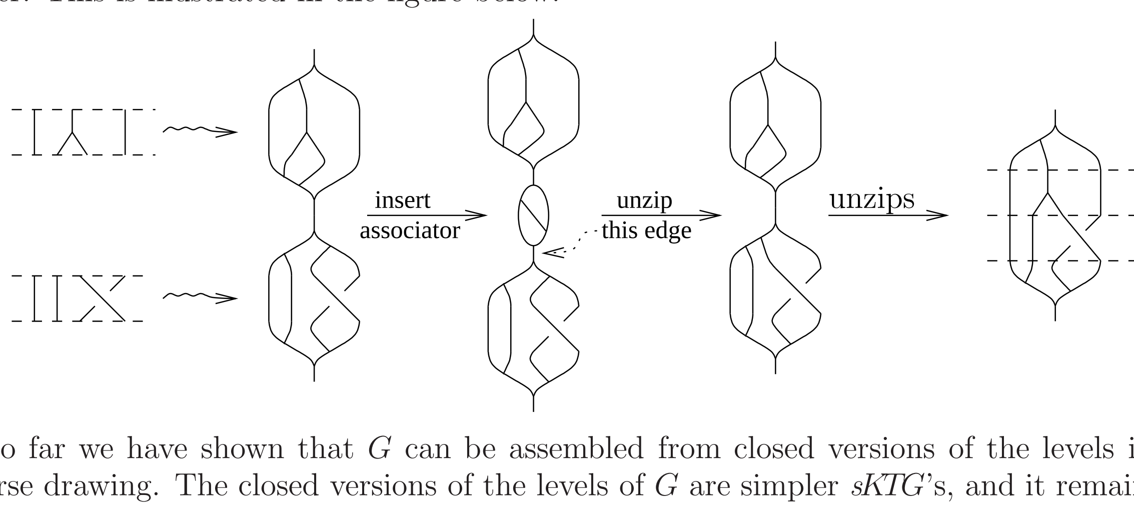
So let us decompose G into a composition of trivalent tangles, each of which has one "feature" and (possibly) some straight vertical strands. An example is shown in the figure below. Note that these tangles are not necessarily $(1,1)$-tangles, and hence need not be elements of *sKTG*. However, we can turn each of them into a $(1,1)$-tangle by "closing up" their tops and bottoms by arbitrary trees. In the example below we show this for one level of the Morse-drawn *sKTG* containing a crossing and two vertical strands.



Now we can compose the *sKTG*'s obtained from closing up each level, as tangle composition is a special case of tangle insertion. Each tree that we used to close up the tops and bottoms of levels determines a "parenthesization" of the strand endings. If these parenthesizations match on the top of each level with the bottom of the next, then we can recreate tangle composition of the levels by composing their closed versions followed by a number of unzips performed on the connecting trees. This is illustrated in the example below, for two consecutive levels of the *sKTG* of the previous example.

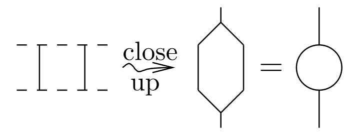


If the trees used to close up consecutive levels correspond to different parenthesizations, then we can use insertion of the left and right associators (the last two pictures of the list of generators in the statement of the theorem) to change one parenthesization to match the other. This is illustrated in the figure below.

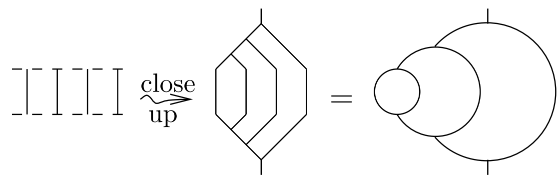


So far we have shown that G can be assembled from closed versions of the levels in its Morse drawing. The closed versions of the levels of G are simpler *sKTG*'s, and it remains to show that these can be obtained from the generators using *sKTG* operations.

Let us examine what each level might look like. First of all, in the absence of any "features" a level might be a single strand, in which case it is the first generator itself. Two parallel strands when closed up become the "bubble", as shown on the right.

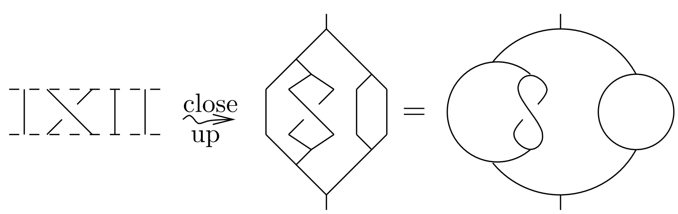


Now suppose that a level consists of n parallel strands, and that the trees used to close it up on the top and bottom are horizontal mirror images of each other, as shown below (if not, then this can be achieved by associator insertions and unzips). We want to show that this *sKTG* can be obtained from the generators using *sKTG* operations. Indeed, this can be achieved by repeatedly inserting bubbles into a bubble, as shown:



A level consisting of a single crossing becomes a left or right twist when closed up (depending on the sign of the crossing). Similarly, a single vertex becomes a bubble. A level can not contain a single minimum or maximum by itself, since we required that the top end of an *sKTG* be connected to its bottom end via a path. Hence, any minimum or maximum must be accompanied by at least one through strand. A maximum with one through strand becomes the balloon after closing up, and a minimum with one through strand becomes the noose.

It remains to see that the *sKTG*'s obtained when closing up simple features accompanied by more through strands can be built from the generators. This is achieved by inserting the corresponding generators into nested bubbles (bubbles inserted into bubbles), as in the example shown below. Recall that the trees (parenthesizations) used for the closing up process can be changed arbitrarily by inserting associators and unzipping, and hence we are free to use the most convenient tree in the example below. This completes the proof.



$\square$

We are now equipped to answer the main question of this section:

Theorem 6.12. *There exists a homomorphic expansion for the combined algebraic structure* $\left(\mathit{sKTG} \xrightarrow{a} \mathit{wTF}\right)$*. In other words, there exist homomorphic expansions Z^u and Z^w for which the square on the right commutes.*

$$\begin{array}{ccc} \mathit{sKTG} & \xrightarrow{a} & \mathit{wTF} \\ \Big\downarrow{\scriptstyle Z^u} & & \Big\downarrow{\scriptstyle Z^w} \\ \mathcal{A}^u & \xrightarrow{\alpha} & \mathcal{A}^{sw} \end{array} \tag{48}$$

Before moving on to the proof let us state and prove the following Lemma, to be used repeatedly in the proof of the theorem.

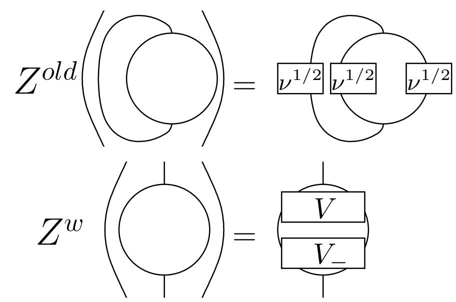
Lemma 6.13. *If a and b are group-like elements in $\mathcal{A}^{sw}(\uparrow_n)$, then $a = b$ if and only if $\pi(a) = \pi(b)$ and $aa^* = bb^*$. Here π is the projection induced by $\pi\colon \mathcal{P}^w(\uparrow_n) \to \mathfrak{tder}_n \oplus \mathfrak{a}_n$*

(see Section 5.2), and $*$ refers to the adjoint map of Definition 5.18. In the notation of this section $*$ is applying the adjoint A on all strands.

Proof. Write $a = e^w e^{uD}$ and $b = e^{w'} e^{uD'}$, where $w \in \mathfrak{tr}_n$, $D \in \mathfrak{tder}_n \oplus \mathfrak{a}_n$ and $u\colon \mathfrak{tder}_n \oplus \mathfrak{a}_n \to \mathcal{P}_n$ is the "upper" map of Section 5.2. Assume that $\pi(a) = \pi(b)$ and $aa^* = bb^*$. Since $\pi(a) = e^D$ and $\pi(b) = e^{D'}$, we conclude that $D = D'$. Now we compute $aa^* = e^w e^{uD} e^{-lD} e^w = e^w e^{j(D)} e^w$, where $j\colon \mathfrak{tder}_n \to \mathfrak{tr}_n$ is the map defined in Section 5.1 of [AT] and discussed in 5.19 of this paper. Now note that both w and $j(D)$ are elements of $\mathfrak{tr}_n$, hence they commute, so $aa^* = e^{2w+j(D)}$. Thus, $aa^* = bb^*$ means that $e^{2w+j(D)} = e^{2w'+j(D)}$, which implies that $w = w'$ and $a = b$. $\square$

Proof of Theorem 6.12. Since *sKTG* is finitely generated, we only need to check that the square (48) commutes for each of the generators.

Proof of commutativity of (48) *for the strand and the bubble.* For the single strand commutativity is obvious: both the Z^u and Z^w values are trivial.



We claim that the Z^u value of the bubble is also trivial. By connecting the top and bottom of the bubble we obtain a "theta-graph", and Z^{old} of a theta graph has $\nu^{1/2}$ on each strand, as shown on the right (for a computation see [BND1] for example). After applying the vertex normalizations of Figure 26, everything cancels, so the Z^u-value of the bubble is trivial. As for Z^w, the value of the bubble is V_-V, as shown, which equals to 1 by the Unitarity property of V, Equation (42). This proves the commutativity of the square for bubbles. $\square$

Proof of commutativity of (48) *for the twists.* First note that the Z^u-value of the right twist is $R^u = e^{c/2}$, where c denotes a single chord between the two twisted strands (see [BND1] for details). Hence the commutativity of Z^u and Z^w for the right twist is equivalent to the "Twist Equation" $\alpha(R^u) = V^{-1}RV^{21}$, where $R = e^{a_{12}}$ is the Z^w-value of the crossing, that is, the exponential of a single arrow pointing from strand 1 to strand 2. By definition of α, $\alpha(R^u) = e^{\frac{1}{2}(a_{12}+a_{21})}$, where a_{12} and a_{21} are single arrows pointing from strand 1 to 2 and 2 to 1, respectively. So the Twist Equation becomes

$$e^{\frac{1}{2}(a_{12}+a_{21})} = V^{-1}RV^{21}. \tag{49}$$

If V is to give rise to a homomorphic expansion Z^w that is compatible with Z^u, then V has to satisfy the Twist Equation in addition to the previous equations (41),(42) and (43). To prove that such a V exists, we use Lemma 6.13. Lemma 6.13 implies that it is enough to find a V which satisfies the Twist Equation "on tree level" (i.e., after applying π), and for which the adjoint condition of the Lemma holds.

Let us start with the adjoint condition. Multiplying the left hand side of the Twist Equation by its adjoint, we get

$$e^{\frac{1}{2}(a_{12}+a_{21})}\left(e^{\frac{1}{2}(a_{12}+a_{21})}\right)^* = e^{\frac{1}{2}(a_{12}+a_{21})}e^{-\frac{1}{2}(a_{12}+a_{21})} = 1.$$

As for the right hand side, we have to compute $V^{-1}RV^{21}(V^{21})^*R^*(V^{-1})^*$. Since V is unitary (Equation (42)), $VV^* = V \cdot A_1A_2(V) = 1$. Now $R = e^{a_{12}}$, so $R^* = e^{-a_{12}} = R^{-1}$, hence the expression on the right hand side also simplifies to 1, as needed.

As for the "tree level" of the Twist Equation, recall that in Section 6.3 we deduced the existence of a solution to all the previous equations from Alekseev and Torossian's solution $F \in \mathrm{TAut}_2$ to the Kashiwara–Vergne equations [AT]. We produced V from F by setting $F = e^{D^{21}}$ with $D \in \mathfrak{tder}_2^s$, $b := \frac{-j(F)}{2} \in \mathfrak{tr}_2$ and $V := e^b e^{uD}$, so F is "the tree part" of V, up to re-numbering strands. Substituting this into the Twist Equation we obtain:

$$e^{\frac{1}{2}(a_{12}+a_{21})} = e^{-uD}e^{-b}e^{a_{12}}e^{b^{21}}e^{uD^{21}}. \tag{50}$$

Applying π, we get

$$e^{\frac{1}{2}(a_{12}+a_{21})} = e^{-uD}e^{a_{12}}e^{uD^{21}} = (F^{21})^{-1}e^{a_{12}}F.$$

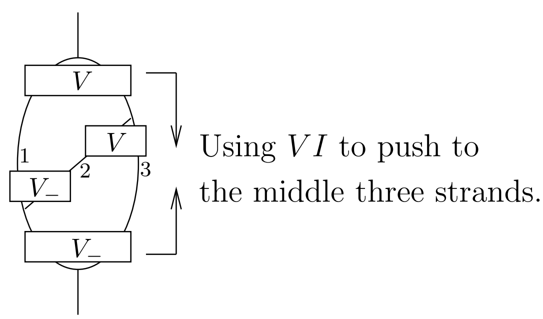
The existence of a solution F of the KV equations which also satisfies the above is equivalent to the existence of "symmetric solutions of the Kashiwara-Vergne problem" discussed and proven in Sections 8.2 and 8.3 of [AT] (note that in [AT] R denotes $e^{a_{21}}$). $\square$

Proof of commutativity of (48) *for the associators.* Let us recall that a Drinfel'd associator is a group-like element of $\mathcal{A}^u(\uparrow_3)$ satisfying the so-called pentagon and positive and negative hexagon equations, as well as a non-degeneracy and mirror skew-symmetry property. For a detailed explanation see Section 4 of [BND1]; associators were first defined in [Dr2]. The Z^u-value of the generator shown in the statement of Proposition 6.11 called "right associator" is a Drinfel'd associator. The proof of this statement is the same as the proof of Theorem 4.2 of [BND1], with minor modifications. (I.e., the graphs have positive and negative vertices as opposed to "dots and crosses" on the edges. Note that the vertex re-normalizations for the four vertices of an associator cancel each other out). Let us call this associator Φ.

What we need to show is that there exists a V satisfying all previous equations including the Twist Equation (49), so that

$$\alpha(\Phi) = V_-^{(12)3}V_-^{12}V^{23}V^{1(23)} \qquad \text{in} \qquad \mathcal{A}^{sw}(\uparrow_3), \tag{51}$$

where $\alpha\colon \mathcal{A}^u \to \mathcal{A}^{sw}$ is the map defined in Section 5.3, and keeping in mind that $V_- = V^{-1}$. The reasoning behind this equation is shown in the figure below.



We proceed in a similar manner as we did for the Twist Equation, treating the "tree and wheel parts" separately using Lemma 6.13. As Φ is by definition group-like, let us denote $\Phi =: e^{\phi}$.

First we verify that the "wheel level" adjoint condition holds. Starting with the right hand side of Equation (51), the unitarity $VV^* = 1$ of V implies that

$$V_-^{(12)3}V_-^{12}V^{23}V^{1(23)}(V^{1(23)})^*(V^{23})^*(V_-^{12})^*(V_-^{(12)3})^* = 1.$$

For the left hand side of (51) we need to study $e^{\alpha(\phi)}(e^{\alpha(\phi)})^*$ and show that it equals 1 as well. This is assured if we pick a Z^u for which Φ is a group-like horizontal chord associator (possible for example using [CL], as mentioned at the beginning of this section). Indeed restricted to the α-images of horizontal chords $*$ is multiplication by -1, and as it is an anti-Lie morphism, this fact extends to the Lie algebra generated by α-images of horizontal chords. Hence $e^{\alpha(\phi)}(e^{\alpha(\phi)})^* = e^{\alpha(\phi)}e^{\alpha(\phi)^*} = e^{\alpha(\phi)}e^{-\alpha(\phi)} = 1$.

On to the tree part. Applying π to Equation (51) we obtain

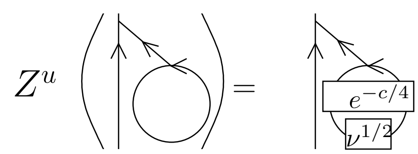
$$e^{\pi\alpha(\phi)} = (F^{3(12)})^{-1}(F^{21})^{-1}F^{32}F^{(23)1} = e^{-D^{(12)3}}e^{-D^{12}}e^{D^{23}}e^{D^{1(23)}}$$
$$\text{in } \mathrm{SAut}_3 := \exp(\mathfrak{sder}_3) \subset \mathrm{TAut}_3\,. \quad (52)$$

This is Equation (26) of [AT], up to re-numbering strands 1 and 2 as 2 and 1[38]. To prove it in our context, we need the following fact from [AT] (their Theorem 7.5, Propositions 9.2 and 9.3 combined):

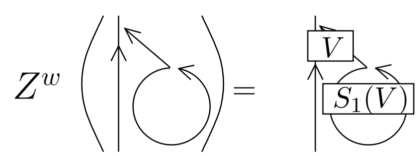
Fact 6.14. *If $\Phi' = e^{\phi'}$ is an associator in* SAut_3 *so that $j(\Phi') = 0$*[39] *then Equation (52) has a solution $F = e^{D^{21}}$ which is also a solution to the KV equations, and all such solutions are symmetric (i.e. verify the Twist Equation (50)).* □

To use this fact, we need to show that $\Phi' := \pi\alpha(\Phi)$ is an associator in SAut_3 and that $j(\Phi') = j(\pi\alpha(\Phi)) = 0$. The latter is the unitarity of Φ which is already proven. The former follows from the fact that Φ is an associator and the fact (Theorem 5.20) that the image of $\pi\alpha$ is contained in $\mathfrak{sder}$ (ignoring degree 1 terms, which are not present in an associator anyway). □

Proof of commutativity of (48) *for the balloon and the noose.* Connecting the top and bottom end of the noose picture creates a "dumbbell graph", and Z^{old} of the dumbbell is a ν placed on each of the circles with nothing on the line connecting them. Applying the vertex normalizations and the ν^{-1} normalization on the long strand, we obtain that Z^u of the noose has chords only on the circle, namely $e^{-c/4}\nu^{1/2}$, as shown on the right. We leave it to the reader to check this, keeping in mind the fact that in $\mathcal{A}^u$, any chord diagram with chord endings on a bridge in the graph (i.e., an edge whose deletion increases the number of connected components) is zero. Also keep in mind that the bottom vertex is not a positive vertex: the orientation of the left strand is switched, so we have to apply an orientation switch operation of that strand to the value of the normalization. As $S(\nu) = \nu$, this only affects the sign of the exponent. A similar computation can be done for the balloon, where the result is $e^{c/4}\nu^{1/2}$ on the circle.



Z^w on the other hand assigns a V value to each vertex, one of which has its first strand orientation switched as shown in the figure on the right. The top copy of V appearing there cancels: pushing arrow heads and tails onto the noose using VI results in two terms that have opposite signs but are otherwise equal (we can slide arrow



[38]Note that in [AT] "Φ' is an associator" means that Φ' satisfies the pentagon equation, mirror skew-symmetry, and positive and negative hexagon equations in the space SAut_3. These equations are stated in [AT] as equations (25), (29), (30), and (31), and the hexagon equations are stated with strands 1 and 2 re-named to 2 and 1 as compared to [Dr2] and [BND1]. This is consistent with $F = e^{D^{21}}$.

[39]The condition $j(\phi') = 0$ is equivalent to the condition $\Phi \in KRV_3^0$ in [AT]. The relevant definitions in [AT] can be found in Remark 4.2 and at the bottom of page 434 (before Section 5.2).

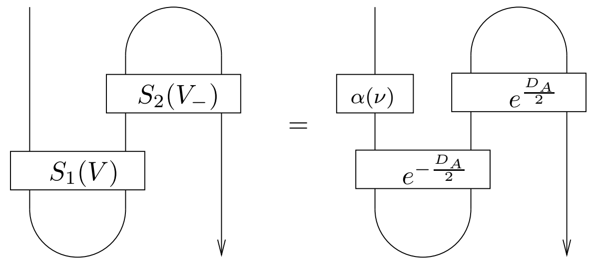


Figure 27. The product equation.

heads/tails across the $S_1(V)$ term as anything concentrated on one strand is a combination of wheels and D_A arrows, and we can slide across these using $\overrightarrow{4T}$/TC).

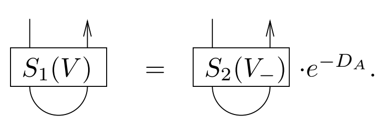
Hence, what we need to show is that the two equations below hold, arising from the noose and the balloon, respectively.

We will start by proving that the product of these two equations, shown in Figure 27, is satisfied. Note that any local (small) arrow diagram on a single strand is central in $\mathcal{A}^{sw}(\uparrow_n)$: a diagram on one strand can be written in terms of only wheels and isolated arrows, both of which commute with both arrow heads and tails by $\overrightarrow{4T}$ and TC. Hence we can slide and merge the $\alpha(\nu)$ terms as we wish.

To show that the product equation is satisfied, consider Figure 28. We start with the *wTF* on the top left and either apply Z^w followed by unzipping the edges marked by stars, or first unzip the same edges and then apply Z^w. We use that by the compatibility with associators, Z^w of an associator is $\alpha(\Phi)$. Since Z^w is homomorphic, the two results in the bottom right corner must agree. (Note that two of the four unzips we perform are "illegal", as the strand directions don't match. However, it is easy to get around this issue by inserting small bubbles at the top of the balloon and the bottom of the noose, and switching the appropriate edge orientations before and after the unzips. The Z^w-value of a bubble is 1, hence this will not effect the computation and so we ignore the issue for simplicity.)

We conclude that to prove that the product equation of Figure 27 is satisfied, it is enough to show that the left equality of Figure 29 holds. Note that in Figure 29 the inverse is taken in $\mathcal{A}^{sw}(\uparrow_1)$. As both sides of this equation are in the image of α, it is enough to prove the pre-image of the equation in $\mathcal{A}^u$, as shown on the right of Figure 29. That equation in turn follows from an argument identical to that of Figure 28 but carried out in *sKTG* and $\mathcal{A}^u$, using that Z^u is homomorphic with respect to tangle insertion. This finishes the proof that the product of the noose and balloon equations holds.

What remains is to show that the noose and balloon equations hold individually. In light of the results so far, it is sufficient to show that



(53)

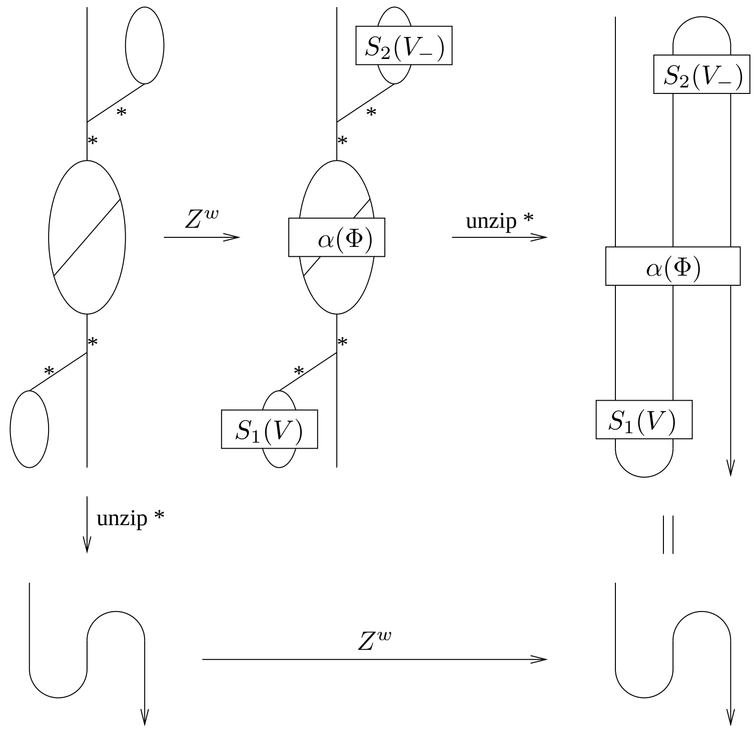


Figure 28. Unzipping a noose and a balloon to a squiggle.

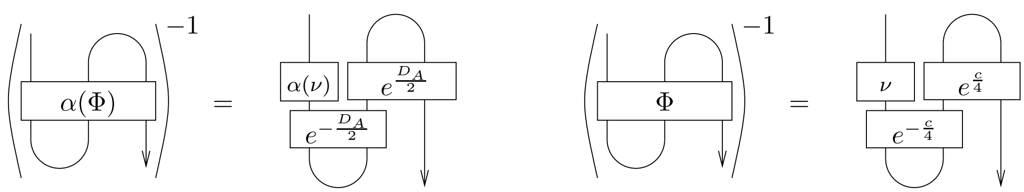


Figure 29. The reduced noose and balloon equation.

As stated in Theorem 3.16, $\mathcal{A}^{sw}(\uparrow_1)$ is the polynomial algebra freely generated by the arrow D_A and wheels of degrees 2 and higher. Since V is group-like, the "one-strand version" of $S_1(V)$ (resp. $S_2(V_-)$) shown in Equation (53) is an exponential e^{A_1} (resp. e^{A_2}) with $A_1, A_2 \in \mathcal{A}^{sw}(\uparrow_1)$. We want to show that $e^{A_1} = e^{A_2} \cdot e^{-D_A}$, equivalently that $A_1 = A_2 - D_A$.

In degree 1, this can be done by explicit verification. Let $A_1^{\geq 2}$ and $A_2^{\geq 2}$ denote the degree 2 and higher parts of A_1 and A_2, respectively. We claim that capping the strand at both its top and its bottom takes e^{A_1} to $e^{A_1^{\geq 2}}$, and similarly e^{A_2} to $e^{A_2^{\geq 2}}$. (In other words, capping kills arrows but leaves wheels un-changed.) This can be proven similarly to the proof of Lemma 6.6, but using

$$F' := \sum_{k_1,k_2=0}^{\infty} \frac{(-1)^{k_1+k_2}}{k_1!k_2!} D_A^{k_1+k_2} S_L^{k_1} S_R^{k_2}$$

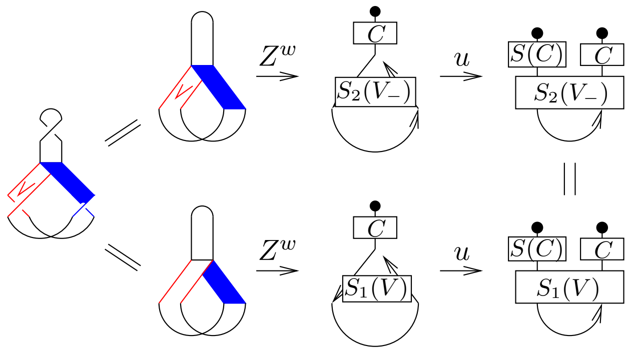
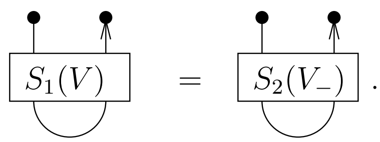


Figure 30. The proof of Equation (54). Note that the unzips are "illegal", as the strand directions don't match. This can be fixed by inserting a small bubble at the bottom of the noose and doing a number of orientation switches. As this doesn't change the result or the main argument, we suppress the issue for simplicity. Equation (54) is obtained from this result by multiplying by $S(C)^{-1}$ on the bottom and by C^{-1} on the top.

in place of F in the proof. What we want to show, then, is that



(54)

The proof of this is shown in Figure 30. □

Having verified the commutativity of (48) for all the generators of *sKTG* appearing in Proposition 6.11, we have concluded the proof of Theorem 6.12. □

Recall from Section 5.3 that there is no commutative square linking $Z^u\colon \mathit{uT} \to \mathcal{A}^u$ and $Z^w\colon \mathit{wT} \to \mathcal{A}^{sw}$, for the simple reason that the Kontsevich integral for tangles Z^u is not canonical, but depends on a choice of parenthesizations for the "bottom" and the "top" strands of a tangle T. Yet given such choices, a tangle T can be "closed" as within the proof of Proposition 6.11 into an *sKTG* which we will denote G. For G a commutativity statement does hold as we have just proven. The Z^u and Z^w invariants of T and of G differ only by a number of vertex-normalizations and vertex-values on skeleton-trees at the bottom or at the top of G, and using VI, these values can slide so they are placed on the original skeleton of T. This is summarized as the following proposition:

Proposition 6.15. *Let n and n' be natural numbers. Given choices c and and c' of parenthesizations of n and n' strands respectively, there exists invertible elements $C \in \mathcal{A}^{sw}(\uparrow_n)$ and $C' \in \mathcal{A}^{sw}(\uparrow_{n'})$ so that for any u-tangle T with n "bottom" ends and n' "top" ends we have*

$$\alpha Z^u_{c,c'}(T) = C^{-1} Z^w(aT) C',$$

where $Z^u_{c,c'}$ denotes the usual Kontsevich integral of T with bottom and top parenthesizations c and c'.

For u-braids the above proposition may be stated with $c = c'$ and then C and C' are the same.

7. Odds and Ends

7.1. **What means "closed form"?** As stated earlier, one of my hopes for this paper is that it will lead to closed-form formulae for tree-level associators. The notion "closed-form" in itself requires an explanation (see footnote 3). Is e^x a closed form expression for $\sum_{n=0}^{\infty}\frac{x^n}{n!}$, or is it just an artificial name given for a transcendental function we cannot otherwise reduce? Likewise, why not call some tree-level associator Φ^{tree} and now it is "in closed form"?

For us, "closed-form" should mean "useful for computations". More precisely, it means that the quantity in question is an element of some space $\mathcal{A}^{cf}$ of "useful closed-form thingies" whose elements have finite descriptions (hopefully, finite and short) and on which some operations are defined by algorithms which terminate in finite time (hopefully, finite and short). Furthermore, there should be a finite-time algorithm to decide whether two descriptions of elements of $\mathcal{A}^{cf}$ describe the same element[40]. It is even better if the said decision algorithm takes the form "bring each of the two elements in question to a canonical form by means of some finite (and hopefully short) procedure, and then compare the canonical forms verbatim"; if this is the case, many algorithms that involve managing a large number of elements become simpler and faster.

Thus for example, polynomials in a variable x are always of closed form, for they are simply described by finite sequences of integers (which in themselves are finite sequences of digits), the standard operations on polynomials ($+$, $\times$, and, say, $\frac{d}{dx}$) are algorithmically computable, and it is easy to write the "polynomial equality" computer program. Likewise for rational functions and even for rational functions of x and e^x.

On the other hand, general elements Φ of the space $\mathcal{A}^{\text{tree}}(\uparrow_3)$ of potential tree-level associators are not closed-form, for they are determined by infinitely many coefficients. Thus iterative constructions of associators, such as the one in [BN3] are computationally useful only within bounded-degree quotients of $\mathcal{A}^{\text{tree}}(\uparrow_3)$ and not as all-degree closed-form formulae. Likewise, "explicit" formulae for an associator Φ in terms of multiple ζ-values (e.g. [LM1]) are not useful for computations as it is not clear how to apply tangle-theoretic operations to Φ (such as $\Phi\mapsto\Phi^{1342}$ or $\Phi\mapsto(1\otimes\Delta\otimes 1)\Phi$) while staying within some space of "objects with finite description in terms of multiple ζ-values". And even if a reasonable space of such objects could be defined, it remains an open problem to decide whether a given rational linear combination of multiple ζ-values is equal to 0.

7.2. **Arrow Diagrams to Degree 2.** Just as an example, in this section we study the spaces $\mathcal{A}^-(\uparrow)$, $\mathcal{A}^{s-}(\uparrow)$, $\mathcal{A}^{r-}(\uparrow)$, $\mathcal{P}^-(\uparrow)$, $\mathcal{A}^-(\bigcirc)$, $\mathcal{A}^{s-}(\bigcirc)$, and $\mathcal{A}^{r-}(\bigcirc)$ in degrees $m\leq 2$ in detail, both in the "v" case and in the "w" case (the "u" case has been known since long).

7.2.1. *Arrow Diagrams in Degree 0.* There is only one degree 0 arrow diagram, the empty diagram D_0 (see Figure 31). There are no relations, and thus $\{D_0\}$ is the basis of all $\mathcal{G}_0\mathcal{A}^-(\uparrow)$ spaces and its obvious closure, the empty circle, is the basis of all $\mathcal{G}_0\mathcal{A}^-(\bigcirc)$ spaces. D_0 is the unit 1, yet $\Delta D_0 = D_0\otimes D_0 = 1\otimes 1 \neq D_0\otimes 1 + 1\otimes D_0$, so D_0 is not primitive and $\dim\mathcal{G}_0\mathcal{P}^-(\uparrow)=0$.

[40] In our context, if it is hard to decide within the target space of an invariant whether two elements are equal or not, the invariant is not too useful in deciding whether two knotted objects are equal or not.

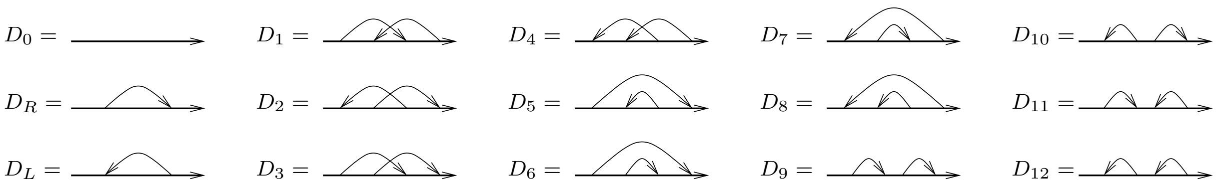


Figure 31. The 15 arrow diagrams of degree at most 2.

7.2.2. *Arrow Diagrams in Degree 1.* There is only two degree 1 arrow diagrams, the "right arrow" diagram D_R and the "left arrow" diagram D_L (see Figure 31). There are no $6T$ relations, and thus $\{D_R, D_L\}$ is the basis of $\mathcal{G}_1\mathcal{A}^-(\uparrow)$. Modulo RI, $D_L = D_R$ and hence $D_A := D_L = D_R$ is the single basis element of $\mathcal{G}_1\mathcal{A}^{s-}(\uparrow)$. Both D_R and D_L vanish modulo FI, so $\dim \mathcal{G}_1\mathcal{A}^{r-}(\uparrow) = \dim \mathcal{G}_1\mathcal{A}^{r-}(\bigcirc) = 0$. Both D_R and D_L are primitive, so $\dim \mathcal{G}_1\mathcal{P}^-(\uparrow) = 2$. Finally, the closures $\bar{D}_R$ and $\bar{D}_L$ of D_R and D_L are equal, so $\mathcal{G}_1\mathcal{A}^{s-}(\bigcirc) = \mathcal{G}_1\mathcal{A}^-(\bigcirc) = \langle \bar{D}_R \rangle = \langle \bar{D}_L \rangle = \langle \bar{D}_A \rangle$.

7.2.3. *Arrow Diagrams in Degree 2.* There are 12 degree 2 arrow diagrams, which we denote $D_1, \ldots, D_{12}$ (see Figure 31). There are six $6T$ relations, corresponding to the 6 ways of ordering the 3 vertical strands that appear in a $6T$ relation (see Figure 3) along a long line. The ordering (ijk) becomes the relation $D_3 + D_9 + D_3 = D_6 + D_3 + D_6$. Likewise, $(ikj) \mapsto D_6 + D_1 + D_{11} = D_3 + D_5 + D_1$, $(jik) \mapsto D_{10} + D_2 + D_6 = D_2 + D_5 + D_3$, $(jki) \mapsto D_4 + D_7 + D_1 = D_8 + D_1 + D_{11}$, $(kij) \mapsto D_2 + D_7 + D_4 = D_{10} + D_2 + D_8$, and $(kji) \mapsto D_8 + D_4 + D_8 = D_4 + D_{12} + D_4$. After some linear algebra, we find that $\{D_1, D_2, D_6, D_8, D_9, D_{11}, D_{12}\}$ form a basis of $\mathcal{G}_2\mathcal{A}^v(\uparrow)$, and that the remaining diagrams reduce to the basis as follows: $D_3 = 2D_6 - D_9$, $D_4 = 2D_8 - D_{12}$, $D_5 = D_9 + D_{11} - D_6$, $D_7 = D_{11} + D_{12} - D_8$, and $D_{10} = D_{11}$. In $\mathcal{G}_2\mathcal{A}^{sv}(\uparrow)$ we further have that $D_5 = D_6$, $D_7 = D_8$, and $D_9 = D_{10} = D_{11} = D_{12}$, and so $\mathcal{G}_2\mathcal{A}^{sv}(\uparrow)$ is 3-dimensional with basis D_1, D_2, and $D_3 = \ldots = D_{12}$. In $\mathcal{G}_2\mathcal{A}^{rv}(\uparrow)$ we further have that $D_{5-12} = 0$. Thus $\{D_1, D_2\}$ is a basis of $\mathcal{G}_2\mathcal{A}^{rv}(\uparrow)$.

There are 3 OC relations to write for $\mathcal{G}_2\mathcal{A}^w(\uparrow)$: $D_2 = D_{10}$, $D_3 = D_6$, and $D_4 = D_8$. Along with the $6T$ relations, we find that $\{D_1, D_3 = D_6 = D_9, D_2 = D_5 = D_7 = D_{10} = D_{11}, D_4 = D_8 = D_{12}\}$ is a basis of $\mathcal{G}_2\mathcal{A}^w(\uparrow)$. Similarly $\{D_1, D_2 = \ldots = D_{12}\}$ is a basis of the two-dimensional $\mathcal{G}_2\mathcal{A}^{sw}(\uparrow)$. When we mod out by FI, only one diagram remains non-zero in $\mathcal{G}_2\mathcal{A}^{rw}(\uparrow)$ and it is D_1.

We leave the determination of the primitives and the spaces with a circle skeleton as an exercise to the reader.

8. Glossary of notation

Greek letters, then Latin, then symbols:

α	maps $\mathcal{A}^u \to \mathcal{A}^v$ or $\mathcal{A}^u \to \mathcal{A}^w$	2.5.5
Δ	cloning, co-product	2.5.1.2, 4.2
δ	Satoh's tube map	3.1.1
δ_A	a formal D_A	3.8
θ	inversion, antipode	2.5.1.1
ι	an inclusion $wB_n \to wB_{n+1}$	2.2.3
ι	interpretation map	3.8, 3.8.2
ι	inclusion $\mathfrak{tr}_n \to \mathcal{P}^w(\uparrow_n)$	5.2
λ	a formal EZ	3.8
ν	the invariant of the unknot	6.6
ξ_i	the generators of F_n	2.2.3
π	the projection $\mathcal{P}^w(\uparrow_n) \to \mathfrak{a}_n \oplus \mathfrak{tder}_n$	5.2
Σ	a virtual surface	3.1.1
σ_i	a crossing between adjacent strands	2.1.1
σ_{ij}	strand i crosses over strand j	2.1.2
ς	the skeleton morphism	2.1.1
ϕ	log of an associator	6.6
Φ	an associator	6.6
(φ^i)	a basis of $\mathfrak{g}^*$	3.6
ψ_β	"operations"	4.1
ω_1	a formal 1-wheel	3.8
$\mathfrak{a}_n$	n-dimensional Abelian Lie algebra	5.2
$\mathcal{A}$	a candidate projectivization	4.3
$\mathcal{A}(G)$	associated graded of G	2.3.2
$\mathcal{A}^{sv}$	$\mathcal{D}^v$ mod 6T, RI	5.1
$\mathcal{A}^{sw}$	$\mathcal{D}^w$ mod $\overrightarrow{4T}$, TC, RI	5.1
$\mathcal{A}^{sw}$	proj wTF^o	6.2
$\mathcal{A}^{sw}$	proj wTF	6.5.2
$\mathcal{A}^{(s)w}$	$\mathcal{A}^w$ and/or $\mathcal{A}^{sw}$	6.2
$\mathcal{A}^u$	chord diagrams mod rels for KTGs	6.6
$\mathcal{A}^v$	$\mathcal{D}^v$ mod 6T	5.1
$\mathcal{A}^w$	$\mathcal{D}^w$ mod $\overrightarrow{4T}$, TC	5.1
$\mathcal{A}^w$	proj wTF^o without RI	6.2
$\mathcal{A}^-(\uparrow_n)$	$\mathcal{A}^-$ for pure n-tangles	5.2
$\mathcal{A}^-_n$	$\mathcal{D}^v_n$ mod relations	2.3.1
$\mathcal{A}^{-t}$	$\mathcal{A}^-$ allowing trivalent vertices	3.5
$\mathcal{A}^-(\uparrow)$	$\mathcal{D}^v(\uparrow)$ mod relations	3.2
$\mathcal{A}^-(\bigcirc)$	$\mathcal{A}^-(\uparrow)$ for round skeletons	3.3
$\mathcal{A}^u$	usual chord diagrams	3.9
$A(K)$	the Alexander polynomial	3.7
A_e	1D orientation reversal	6.1.3
$\overrightarrow{AS}$	arrow-AS relations	3.5
Ass	associative words	5.2
Ass^+	non-empty associative words	5.2

a	maps $u \to v$ or $u \to w$	2.5.5
a_{ij}	an arrow from i to j	2.3.1
$\mathcal{B}^w$	unitrivalent arrow diagrams	3.5
$\mathcal{B}^w_n$	n-coloured unitrivalent arrow diagrams	5.2
B	the matrix $T(\exp(-xS) - I)$	3.8
b^k_{ij}	structure constants of $\mathfrak{g}^*$	3.6
C	the invariant of a cap	6.3
CC	the Commutators Commute relation	3.5
CP	the Cap-Pull relation	6.1.2, 6.2
CW	Cap-Wen relations	6.5.1
c	a chord in $\mathcal{A}^u$	6.6
$\mathfrak{der}$	Lie-algebra derivations	5.2
$\mathcal{D}^v$, $\mathcal{D}^w$	arrow diagrams for v/w-tangles	5.1
$\mathcal{D}^v_n$	arrow diagrams for braids	2.3.1
$\mathcal{D}^{-t}$	$\mathcal{D}^-$ allowing trivalent vertices	3.5
$\mathcal{D}^v(\uparrow)$	arrow diagrams long knots	3.2
D_A	either D_L or D_R	3.5
D_L	left-going isolated arrow	3.5
D_R	right-going isolated arrow	3.5
div	the "divergence"	5.2
d_k	strand deletion	2.5.1.4
d_i	the direction of a crossing	3.7
E	the Euler operator	3.8
$\tilde{E}$	the normalized Euler operator	3.8
F	a map $\mathcal{A}^w \to \mathcal{A}^w$	6.2
F	the main [AT] unknown	6.4
FI	Framing Independence	3.3
FR	Flip Relations	6.5.1, 6.5.2
F_n	the free group	2.2.3
FA_n	the free associative algebra	2.5.1.5
fil	a filtered structure	4.3
$\mathfrak{g}$	a finite-dimensional Lie algebra	3.6
$\mathcal{G}_m$	degree m piece	2.3.1
$\mathcal{I}$	augmentation ideal	2.3.2, 4.2
$I\mathfrak{g}$	$\mathfrak{g}^* \rtimes \mathfrak{g}$	3.6
IAM	Infinitesimal Alexander Module	3.8, 3.8.2
IAM^0	IAM, before relations	3.8.2
$\overrightarrow{IHX}$	arrow-IHX relations	3.5
i_u	an inclusion $F_n \to wB_{n+1}$	2.2.3
J	a map $\mathrm{TAut}_n \to \exp(\mathfrak{tr}_n)$	5.2
j	a map $\mathrm{TAut}_n \to \mathfrak{tr}_n$	5.2
$\mathcal{K}^u$	usual knots	3.9
KTG	Knotted Trivalent Graphs	6.6

$\mathfrak{lie}_n$	free Lie algebra	5.2
l	a map $\mathfrak{tder}_n \to \mathcal{P}^w(\uparrow_n)$	5.2
M	the "mixed" move	3.1
$\mathcal{O}$	an "algebraic structure"	4.1
OC	the Overcrossings Commute relation	2.2
$\mathcal{P}^w_n$	primitives of $\mathcal{B}^w_n$	5.2
$\mathcal{P}^-(\uparrow)$	primitives of $\mathcal{A}^-(\uparrow)$	3.2
$\mathcal{P}^-(\uparrow_n)$	primitives of $\mathcal{A}^-(\uparrow_n)$	5.2
PvB_n	the group of pure v-braids	2.1.1
PwB_n	the group of pure w-braids	2.2
proj	projectivization	4.2
$\mathcal{R}$	the relations in IAM	3.8.2
R	$Z(\times)$	2.4
R	the ring $\mathbb{Z}[X, X^{-1}]$	3.8.2
R_1	the augmentation ideal of R	3.8.2
R	the invariant of a crossing	6.3
RI	Rotation number Independence	3.2
R123	Reidemeister moves	3.1
R4	a Reidemeister move for foams/graphs	6.1.2
$\mathrm{R1}^s$	the "spun" R1 move	3.1
$\mathfrak{sder}$	special derivations	5.3
$\mathcal{S}$	the circuit algebra of skeletons	4.4
SAut_n	the group $\exp(\mathfrak{sder}_n)$	6.6
$S(K)$	a matrix of signs	3.7
S_k	complete orientation reversal	5.5
S_e	complete orientation reversal	6.1.3
S_n	the symmetric group	2.1.1
$\overrightarrow{STU}$	arrow-STU relations	3.5
s_i	a virtual crossing between adjacent strands	2.1.1
s_i	the sign of a crossing	3.7
$sKTG$	signed long KTGs	6.6
sl	self-linking	3.1
TV	Twisted Vertex relations	6.5.1
$\mathfrak{tder}$	tangential derivations	5.2
$\mathfrak{tr}_n$	cyclic words	5.2
$\mathfrak{tr}^s_n$	cyclic words mod degree 1	5.2
$\mathcal{T}^w_{\mathfrak{g}}$	a map $\mathcal{A}^w \to \mathcal{U}(I\mathfrak{g})$	3.6
TAut_n	the group $\exp(\mathfrak{tder}_n)$	5.2
TC	Tails Commute	2.3.1
$T(K)$	the "trapping" matrix	3.7
$\mathcal{U}$	universal enveloping algebra	3.6
UC	Undercrossings Commute	2.2
u	a map $\mathfrak{tder}_n \to \mathcal{P}^w(\uparrow_n)$	5.2
u_e	strand unzips	6.1.3
u_k	strand unzips	2.5.1.6
uB_n	the (usual) braid group	2.1.1
uT	u-tangles	5.3
V	a finite-type invariant	2.3.1
V, V^+	the invariant of a (positive) vertex	6.3
V^-	the invariant of a negative vertex	6.3
II	Vertex Invariance	6.2
VR123	virtual Reidemeister moves	3.1
vB_n	the virtual braid group	2.1.1
vT	v-tangles	5.1
$vT(\uparrow_n)$	pure n-component v-tangles	5.2
W	$Z(w)$	6.5.3
W_m	weight system	2.3.1
W^2	Wen squared	6.5.1
w	the map $x^k \mapsto w_k$	3.7
w	the wen	6.5
w_i	flip ring $\#i$	2.2.1
w_k	the k-wheel	3.5
wB_n	the group of w-braids	2.2
wT	w-tangles	5.1
$wT(\uparrow_n)$	pure n-component w-tangles	5.2
wTF	w-tangled foams with wens	6.5
wTF^o	orientable w-tangled foams	6.1
X	an indeterminate	3.7
X_n, $\tilde{X}_n$	moduli of horizontal rings	2.2.1
x_i	the generators of FA_n	2.5.1.5
(x_j)	a basis of $\mathfrak{g}$	3.6
Y_n, $\tilde{Y}_n$	moduli of rings	2.5.4
Z	expansions	throughout
$Z_{\mathcal{A}}$	an $\mathcal{A}$-expansion	4.3
Z^u	the Kontsevich integral	3.9
4T	$4T$ relations	6.6
$\overrightarrow{4T}$	$\overrightarrow{4T}$ relations	2.3.1
$6T$	$6T$ relations	2.3.1
$\times$, $\times$	semi-virtual crossings	2.3.1
$/\!\!/$	right action	2.2.3
$\uparrow$	a "long" strand	throughout
$\uparrow$	the quandle operation	4.2
$\uparrow_2$	doubled $\uparrow$	4.2
$*$	the adjoint on $\mathcal{A}^w(\uparrow_n)$	5.2

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Department of Mathematics, University of Toronto, Toronto Ontario M5S 2E4, Canada

E-mail address: `drorbn@math.toronto.edu, zsuzsi@math.toronto.edu`

URL: `http://www.math.toronto.edu/~drorbn, http://www.math.toronto.edu/zsuzsi`