Both terms are words in $\mathfrak{lie}_3$, but notice that the first term does not involve the letter x_3. This means that if the second term involves x_3 at all, i.e., if ρ_2 has tails on the second strand, then both terms have to be zero individually. Looking at the first term, we view ρ_2^{12} as a Lie word in x_1 and x_2, which does involve x_2 by our assumption above. We have $[\rho_2^{12}, a^{23}] = [x_2, \rho_2^{12}] = 0$, which implies ρ_2^{12} is a multiple of x_2, in other words, ρ is a single arrow on the second strand. This is ruled out by the assumption that $k \geq 2$.

If the second term does not involve x_3 at all, then ρ_2 has no tails on the second strand, hence it is of degree 1, but again $k \geq 2$. We have proven that the "tree part" of ρ is zero.

Now assume that ρ is a linear combination of wheels. Wheels have only tails, so the first, second and fourth terms of (37) are zero due to the tails commute relation. What remains is $[\rho^{13}, a^{23}] = 0$. We assert that this is true if and only if each linear component of ρ has all of its tails on one strand.

To prove this, recall each wheel of ρ^{13} represents a cyclic word in letters x_1 and x_3. The map $r : \rho^{13} \mapsto [\rho^{13}, a^{23}]$ is a map $\mathfrak{tr}_2 \to \mathfrak{tr}_3$, which sends each cyclic word in letters x_1 and x_3 to the sum of all ways of substituting $[x_2, x_3]$ for one of the x_3's in the word. Note that if we write out the commutators, then all terms that have x_2 between two x_3's cancel. Hence all remaining terms will be cyclic words in x_1 and x_3 with one occurrence of x_2 in between an x_1 and an x_3.

We construct an almost-inverse r' to r: for a cyclic word w in $\mathfrak{tr}_3$ with one occurrence of x_2, let r' be the map that deletes x_2 from w and maps it to the resulting word in $\mathfrak{tr}_2$ if x_2 is followed by x_3 in w, and maps it to 0 otherwise. On the rest of $\mathfrak{tr}_3$ the map r' may be defined to be 0.

The composition $r'r$ takes a cyclic word in x_1 and x_3 to itself multiplied by the number of times a letter x_3 follows a letter x_1 in it. The kernel of this map can consist only of cyclic words that do not contain the sub-word x_3x_1; namely, these are the words of the form x_3^k or x_1^k. Such words are indeed in the kernel of r, so these make up exactly the kernel of r. This is what we wanted to prove: all wheels in the "wheel part" have all their tails on one strand.

This concludes the proof of the claim, and the proof of the theorem.

6. w-Tangled Foams

NEW.

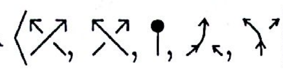
If you have come this far, you must have noticed the approximate Bolero spirit of this article. In every chapter a new instrument comes to play; the overall theme remains the same, but the composition is more and more intricate. In this chapter we add "foam vertices" to our w-tangles (and a few lesser things as well) and ask the same questions we asked before; primarily, "is there a homomorphic expansion?". As we shall see, in the current context this question is more or less equivalent (details to come) to the Alekseev-Torossian [AT] version of the Kashiwara-Vergne [KV] problem.

Is it just "more or less"?

6.1. The Circuit Algebra of w-Tangled Foams.

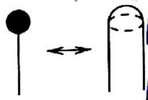
For reasons we will reluctantly acknowledge later in this section (see Comment 6.2), we will present the circuit algebra of w-tangled foams via its Reidemeister-style diagrammatic description (accompanied by a local topological interpretation) rather than as an entirely topological construct.

Definition 6.1. Let $w\mathit{TF}^r$ (where r stands for "restricted", to be explained in Section 6.5) be the algebraic structure

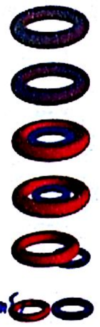
$$w\mathit{TF}^r = \mathrm{CA}\left\langle \right.$$  $$\left| \begin{array}{c}\text{w-relations as in}\\ \text{Section 6.1.2}\end{array} \right| \begin{array}{c}\text{w-operations as}\\ \text{in Section 6.1.3}\end{array} \left.\right\rangle .$$

Hence $w\mathit{TF}^r$ is the circuit algebra generated by the generators listed above and described below, modulo the relations described in Section 6.1.2, and augmented with several "auxiliary operations", which are a part of the algebraic structure of $w\mathit{TF}^r$ but are not a part of its structure as a circuit algebra, as described in Section 6.1.3.

6.1.1. *The generators of* $w\mathit{TF}^r$. There is topological meaning to each of the generators of $w\mathit{TF}^r$: they each stand for a certain local feature of (framed) knotted ribbon tubes in $\mathbb{R}^4$. As in Section 5.4, we require the tubes to be oriented as 2-dimensional surfaces, and also to have a distinguished core with a 1-dimensional orientation (direction).



The crossings are as explained in Section 2.2.2 and Section 5.4: the under-strand denotes the ring flying through, or the "thin" tube. Remember that there really are four kinds of crossing, but in the circuit algebra the two not shown are generated by the two that are shown. The bulleted end denotes a cap on the tube, as in the figure on the right.



The last two generators denote "foam vertices". As the notation suggests, a vertex can be thought of as "half of a crossing". To make this precise using the flying rings interpretation, the first singular vertex represents the movie shown on the left: the ring corresponding to the right strand approaches the ring represented by the left strand from below, flies inside it, and then the two rings fuse (as opposed to a crossing where the ring coming from the right would continue to fly out to above and to the left of the other one). The second vertex is the movie where a ring splits into a smaller and a larger ring, and the small one flies out to the left and below the big one.

The reader might object that there really are four types of vertices of each of the "fuse" and "split" kinds (since there are four types of crossings), as shown in Figure 23. However, looking at the fuse vertices, observe that the last two of these can be obtained from the first two by composing with virtual crossings, which are circuit algebra artifacts. We later (see Remark 6.4 in Section 6.1.3) show that the second fuse vertex can also be obtained from the first using $w\mathit{TF}^r$ operations. In fact, we will see that this lets us obtain all the "split" vertices from the first generating vertex as well so the second generating vertex is not necessary, we only included it as a generator for convenience.

The sign of a vertex can be defined the same way as the sign of a crossing (see Section 5.4). We will sometimes refer to the first generator vertex as "the positive vertex" and to the second one as "the negative vertex". The band notation for vertices is the same as for crossings: the fully colored band stands for the thin (inner) ring.

6.1.2. *The relations of* $w\mathit{TF}^r$. In addition to the usual R2, R3, and OC moves of Figure 6, we need more relations to describe the behavior of the additional features. These are as follows:

A cap means capping the tube represented by a strand or shrinking a flying ring to a point. Hence, a cap on the through strand can be "pulled out" from a crossing, but not a

I object to the distinction between "fuse" & "split"

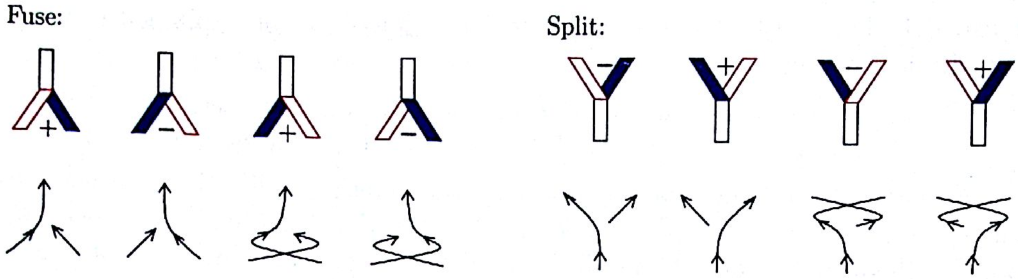
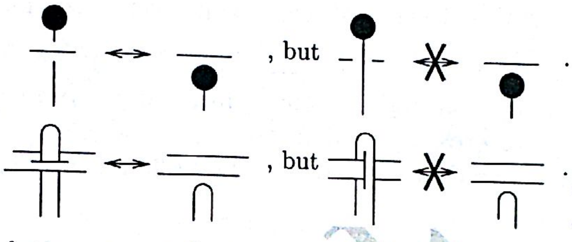


Figure 23. Vertex types in $w\!TF^r$.

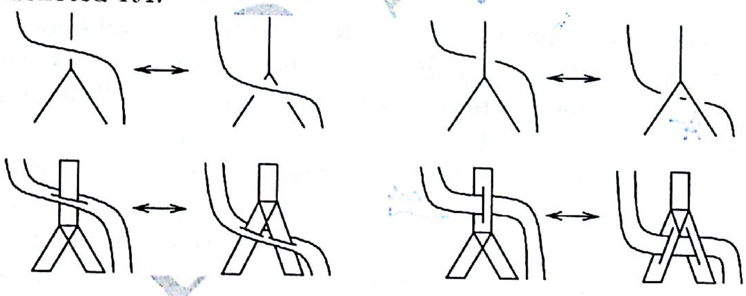
cap on the thick (or over) strand (in any orientation of the strands), as shown below. We decote [denote] this relation by CP, for Cap Pull-out.

CP:



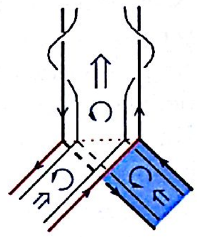
The Reidemeister 4 relations assert that a strand can be moved under or over a crossing, as shown in the picture below. The ambiguously drawn vertices in the picture denotes a vertex of any kind (as described in Section 6.1.1), and the strands can be oriented arbitrarily. The local topological (flying ring) interpretations can be read from the pictures below. These relations will be denoted $R4$.

R4:



Comment 6.2. We have presented the space $w\!TF^r$ as a circuit algebra generated by certain pictures and factored out by some relations. We have given local topological meaning to the pictures and explained how the topological intuition justifies the relations, but we only conjecture that the generators and relations above provide a reidemeister theory for knotted ribbon tubes in $\mathbb{R}^4$.

Comment 6.3. As a result of the previous comment, framing has not played a role here, except to explain the lack of a Reidemeister 1 relation. In the following section though, we will need a notion of framing to define the unzip (tube doubling) operation. Framing is a continuous up-to-homotopy choice of unit normal vector at every point of the tube. We do not allow any such choice, however. Recall that the knotted tubes we consider have a "filling", whith only "ribbon" self-intersections. When we double a tube by pushing it off itself slightly in the direction of the framing, we want this ribbon property to be preserved. This is eqivalent to saying that the ring obtained by pushing off any given girth of the tube in the framing direction is not linked with the tube. In the flying ring language, the framing translates to a "companion ring" to each ring, which can fly parallel inside, outside above and below it and change these positions, but is never linked with it:

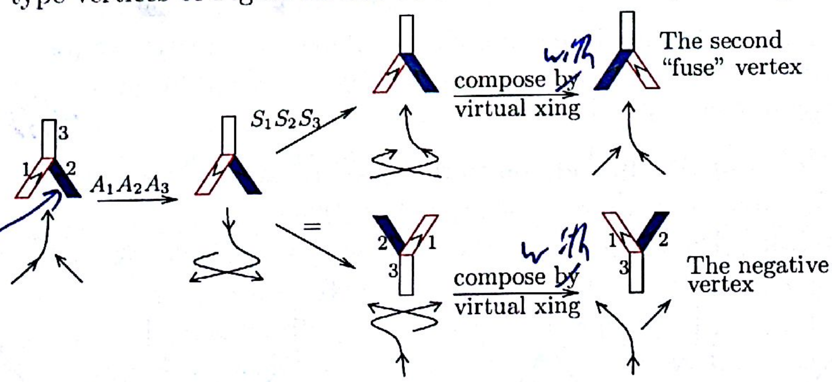


Without loss of generality, we restrict ourselves to framing choices as follows: fixing a t coordinate cuts out a ring (S^1) from each tube, choices of unit normal vectors along this ring are continuous maps $S^1 \to S^1$. We require that each of these maps be constant, in other words it is enough to specify the framing along the core of the tube. Hence the blackboard framing of a line diagram gives rise to a well-defined framing of the tube. We require framings to match at the vertices, with the normal vectors pointing either directly towards or away from the center of the singular ring; while the orientations of the three tubes may or may not match. An example of a vertex with the orientations and framings shown is on the right. Note that the framings on the two sides of each band are mirror images of each other, as they should be.

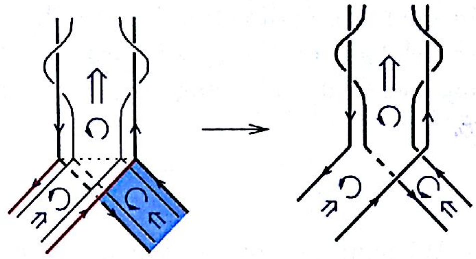
6.1.3. *The auxiliary operations of wTF^r.* The structure wTF^r is, by definition, a circuit algebra. In addition it is equipped with several extra operations.

The first one of these is the familiar orientation switch. We will, as mentioned in Section 5.4, distinguish between switching both the 2D and 1D orientations, or just the strand (1D) direction. ~~Orientation switch,~~ denoted S_e, means the total switch (of both orientations) of the strand e, while we call the operation of only reversing the strand direction ~~"antipode"~~ and denote it by A_e.

Remark 6.4. Note that as promised in Section 6.1.1, the second "fuse" type vertex of Figure 23 and the negative generator vertex can be obtained from the positive generator vertex by antipode and orientation switch operations, as shown below. This also implies that all the other "split" type vertices of Figure 23 can be obtained from the positive generator vertex.

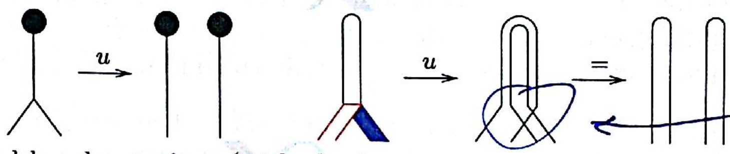


Perhaps the most interesting of the auxiliary operations is unzip, or tube doubling. This is done by pushing the tube off itself slightly in the framing direction. At each of the vertices at the two ends of the doubled tube there are two tubes to be attached to the doubled tube. At each end, the normal vectors pointed either directly towards or away from the center, so there is an "inside" and an "outside" ending ring. The two tubes to be attached also come as an "inside" and an "outside" one, which defines which one to attach to which. An example (in band notation) is shown on the right. Unzip can only be done if the 1D and 2D orientations match at both ends, as shown in the example.



We restrict unzip to strands whose two ending vertices are of different signs. This is a somewhat artificial condition which we impose to get equations eqvivalent to the [AT] equations, but will remove in Section 6.5.

A related operation, *disk unzip*, is unzip done on a capped strand, pushing the tube off in the direction of the framing, as before. An example in the line and band notations (with the framing suppressed) is shown on the right.



Finally, we allow the deletion of "long linear" strands, meaning strands that do not end in a vertex or cap on either side.

The goal, as before, is to construct a homomorphic expansion for wTF^r. However, first we need to understand its target space, the projectivization "proj wTF^r".

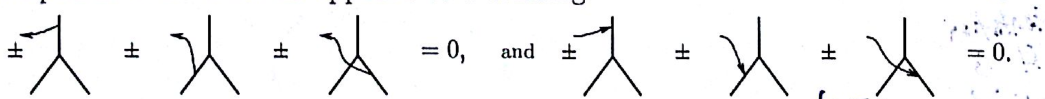
6.2. The projectivization. Mirroring the previous section, we describe the projectivization as a circuit algebra on certain generators modulo a number of relations.

$\mathcal{A}^w = \mathrm{CA}\langle$  $\mid$ relations as in Section 6.2.1 $\mid$ operations as in Section 6.2.2 $\rangle$.

In other words, $\mathcal{A}^w$ is the circuit algebra of arrow diagrams on trivalent (or foam) skeletons with caps. Note that all but the first of the generators are skeleton features, the single arrow is the only generator of degree 1.

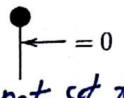
6.2.1. *The relations of* $\mathcal{A}^w$. In addition to the usual $\overrightarrow{4T}$ and TC relations (see Section 2.3), diagrams in $\mathcal{A}^w$ satisfy the following additional relations:

Vertex invariance, denoted by VI, are relations arising the same way as $\overrightarrow{4T}$ does, but with the participation of a vertex as opposed to a crossing:



The other end of the arrow, is in the same place throughout the relation, somewhere outside the picture shown. The signs are positive whenever the strand on which the arrow ends is directed towards the vertex, and negative when directed away. The ambiguously drawn vertex means either of the two vertices, but the same one throughout.

The CP relation (a cap can be pulled out from under a strand but not from over, Section 6.1.2) means that heads near a cap are zero, as shown on the right. Denote this relation also by CP. (Though note that a tail near a cap is not set to 0!!)



As in the previous sections, and in particular in Definition 3.13, we define a "w-Jacobi diagram" (or just "arrow diagram") on a foam skeleton by allowing trivalent chord vertices. Denote the circuit algebra of formal linear combinations of arrow diagrams by $\mathcal{A}^{wt}$. We have the following bracket-rise theorem:

Theorem 6.5. *The obvious inclusion of diagrams induces a circuit algebra isomorphism $\mathcal{A}^w \cong \mathcal{A}^{wt}$. Furthermore, the $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations of Figure 12 hold in $\mathcal{A}^{wt}$.*

Proof. Same as the proof of Theorem 3.15. □

As in Section 5.1, the primitive elements of $\mathcal{A}^w$ are connected diagrams, namely trees and wheels. Before moving on to the auxiliary operations of $\mathcal{A}^w$, let us take note of two useful observations:

Lemma 6.6. *$\mathcal{A}^w(\uparrow)$ is generated by wheels.*

Proof. Any arrow diagram with an arrow head at its top is zero by the Cap Pull-out (CP) relation. If D is an arrow diagram that has a head somewhere on the skeleton but not at the top, then one can use repeated $\overrightarrow{STU}$ relations to commute the head to the top at the cost of diagrams with one fewer skeleton head.

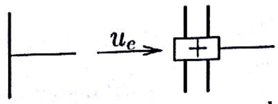
Iterating this procedure, we can get rid of all arrow heads, and hence write D as a linear combination of diagrams having no heads on the skeleton. All connected components of such diagrams are wheels. □

Lemma 6.7. $\mathcal{A}^w(Y) = \mathcal{A}^w(\uparrow_2)$, *where $\mathcal{A}^w(Y)$ stands for the space of arrow diagrams on a vertex with any orientation of the strands, and as before $\mathcal{A}^w(\uparrow_2)$ is the space of arrow diagrams on two strands.*

Proof. We can use the vertex invariance (VI) relation to push all arrow heads and tails from the "trunk" of the vertex to the other two strands. □

6.2.2. *The auxiliary operations of $\mathcal{A}^w$.* Recall from Section 5.4 that the orientation switch S_e (i.e. changing both the 1D and 2D orientations of a strand) always changes the sign of a crossing involving the strand e. Hence, letting F denote any $w\mathit{TF}^r$, the induced arrow diagrammatic operation is a map $S_e : \mathcal{A}^w(F) \to \mathcal{A}^w(S_e(F))$ which acts by multiplying each arrow diagram by (-1) raised to the number of arrow endings on e (counting both heads and tails).

The antipode operation A_e (i.e. switching only the strand direction), on the other hand, only changes the sign of a crossing when the strand being switched is the under- (or through) strand. (See section 5.4 for pictures and explanation.) Therefore, diagrammatic A_e acts by switching the direction of e and multiplying each arrow diagram by (-1) raised to the number of *arrow heads* on e. Note that in $\mathcal{A}(\uparrow_n)$ taking the antipode on every strand gives the adjoint map of Definition 5.17.



The arrow diagram operations induced by unzip and disc unzip (both to be denoted u_e, and interpreted appropriately according to whether the strand e is capped) act in the same manner: they are maps $u_e : \mathcal{A}^w(F) \to \mathcal{A}^w(u_e(F))$, where each arrow ending (head or tail) on e is mapped

On the right side of the equation above $e^{uD}e^{c(x+y)}e^{-uD}$ reminds us of Equation (42), however we cannot use (42) directly as we live in a different space now. In particular, x there meant an arrow from strand 1 to strand 3, while here it means a one-wheel on (capped) strand 1, and similarly for y. Fortunately, there is a map $\sigma : \mathcal{A}^w(\uparrow_3) \to \mathcal{A}^w(\uparrow_2)$, where σ "closes the third strand and turns it into a chord (or internal) strand, and caps the first two strands", as shown on the right. This map is well defined (in fact, it kills almost all relations, and turns one $\overrightarrow{STU}$ into an $\overrightarrow{IHX}$). Under this map, using our abusive notation, $\sigma(x) = x$ and $\sigma(y) = y$.

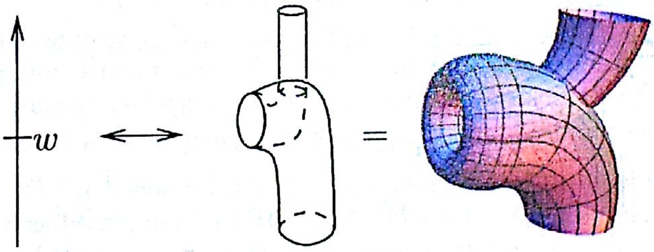
Now we can apply Equation (42) and get $e^b e^{c(\log e^x e^y)} = e^{c(x)+c(y)}$, which, using that tails commute, implies

$$b = c(x) + c(y) - c(\log e^x e^y) \tag{44}$$

Now suppose we have found a group-like homomorphic expansion, that is, solutions $V = e^b e^{uD}$ and $C = e^c$ to equations (38) (39) and (40). Then $F := e^D$ satisfies the Alekseev-Torossian equations (41): e^D acts on $\mathfrak{lie}_2$ by conjugation by e^{uD}, so the first part of (41) is satisfied by (42), and the second half is implied by (43) and (44).

On the other hand, suppose that we have found $F \in \mathrm{TAut}_2$ and $a \in \mathrm{tr}_1$ satisfying (41). Then take $D := \log F$, $b := \frac{-j(e^D)}{2}$, and $c \in \tilde{\delta}^{-1}(b)$. Then $V = e^b e^{uD}$ and $C = e^c$ satisfy the equations for homomorphic expansions (38), (39) and (40).

6.5. **The wen.** A topological feature of w-tangled foams which we excluded from the theory above is the wen. The wen was introduced in 2.5.4 as a Klein bottle cut apart; it amounts to changing the 2D orientation of a tube, as shown in the picture below:

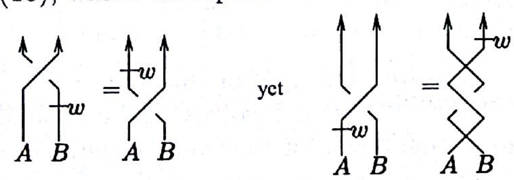


In this section we study the circuit algebra of w-Tangled Foams with the wen rightfully included as a generator, and denote this space by $w\mathcal{TF}$.

Perhaps the circuit algebra for WTF should respect the 2D orientations of the connectors?

6.5.1. *The wen relations.* Adding the wen as a generator means we also have to additional relations involving the wen, as follows:

The interaction of a wen and a crossing has already been mentioned in Section 2.5.4, and is described by Equation (16), which we repeat here for convenience:



Recall that in flying ring language, a wen is a ring flipping over. It does not matter whether ring B flips first and then flies through ring A or vice versa. However, the movies in which ring A first flips and then ring B flies through it, or B flies through A first and then A flips

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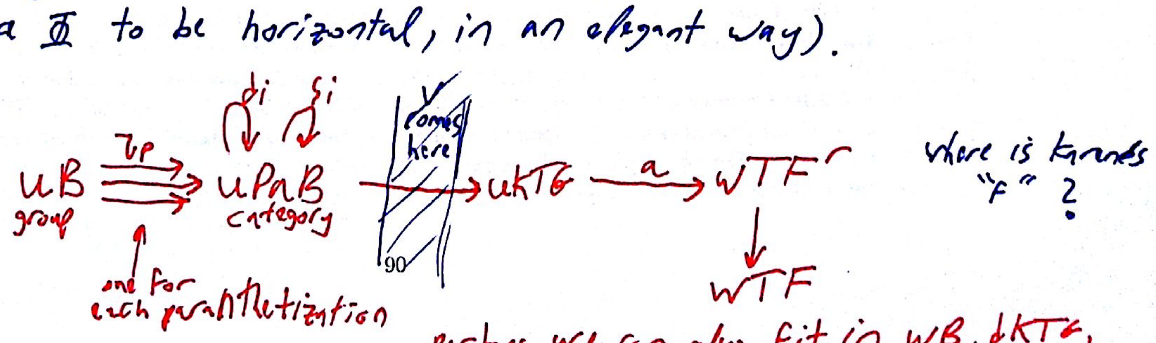
A general question/suggestion/idea: Is there a homomorphic expansion for the "Monster structure"

{uPaB → uwKTG → wTF} } That's the little monster

(This will force Φ to be horizontal, in an elegant way).

Big monster:



perhaps we can also fit in wB, dKTG, and maybe more....