Comments on pages 62–66.

# FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS: FROM ALEXANDER TO KASHIWARA AND VERGNE

DROR BAR-NATAN AND ZSUZSANNA DANCSO

Abstract. w-Knots, and more generally, w-knotted objects (w-braids, w-tangles, etc.) make a class of knotted objects which is wider but weaker than their "usual" counterparts. To get (say) w-knots from u-knots, one has to allow non-planar "virtual" knot diagrams, hence enlarging the the base set of knots. But then one imposes a new relation, the "overcrossings commute" relation, further beyond the ordinary collection of Reidemeister moves, making w-knotted objects a bit weaker once again.

The group of w-braids was studied (under the name "welded braids") by Fenn, Rimanyi and Rourke [FRR] and was shown to be isomorphic to the McCool group [Mc] of "basis-conjugating" automorphisms of a free group $F_n$ — the smallest subgroup of $\mathrm{Aut}(F_n)$ that contains both braids and permutations. Brendle and Hatcher [BH], in work that traces back to Goldsmith [Gol], have shown this group to be a group of movies of flying rings in $\mathbb{R}^3$. Satoh [Sa] studied several classes of w-knotted objects (under the name "weakly-virtual") and has shown them to be closely related to certain classes of knotted surfaces in $\mathbb{R}^4$. So w-knotted objects are algebraically and topologically interesting.

In this article we study finite type invariants of several classes of w-knotted objects. Following Berceanu and Papadima [BP], we construct a homomorphic universal finite type invariant of w-braids, and hence show that the McCool group of automorphisms is "1-formal". We also construct a homomorphic universal finite type invariant of w-tangles. We find that the universal finite type invariant of w-knots is more or less the Alexander polynomial (details inside).

Much as the spaces $\mathcal{A}$ of chord diagrams for ordinary knotted objects are related to metrized Lie algebras, we find that the spaces $\mathcal{A}^w$ of "arrow diagrams" for w-knotted objects are related to not-necessarily-metrized Lie algebras. Many questions concerning w-knotted objects turn out to be equivalent to questions about Lie algebras. Most notably we find that a homomorphic universal finite type invariant of w-knotted trivalent graphs is essentially the same as a solution of the Kashiwara-Vergne [KV] conjecture and much of the Alekseev-Torossian [AT] work on Drinfel'd associators and Kashiwara-Vergne can be re-interpreted as a study of w-knotted trivalent graphs.

The true value of w-knots, though, is likely to emerge later, for we expect them to serve as a warmup example for what we expect will be even more interesting — the study of virtual knots, or v-knots. We expect v-knotted objects to provide the global context whose projectivization (or "associated graded structure") will be the Etingof-Kazhdan theory of deformation quantization of Lie bialgebras [EK].



*Date*: Aug. 8, 2012.

1991 *Mathematics Subject Classification.* 57M25.

*Key words and phrases.* virtual knots, w-braids, w-knots, w-tangles, knotted graphs, finite type invariants, Alexander polynomial, Kashiwara-Vergne, associators, free Lie algebras.

This work was partially supported by NSERC grant RGPIN 262178. Electronic version, videos (wClips) and related files at [BND], `http://www.math.toronto.edu/~drorbn/papers/WKO/`.

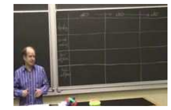
Contents


1. Introduction 3
1.1. Dreams 3
1.2. Stories 3
1.3. The Bigger Picture 4
1.4. Plans 5
1.5. wClips 5
1.6. Acknowledgement 5
2. w-Braids 5
2.1. Preliminary: Virtual Braids, or v-Braids. 6
2.2. On to w-Braids 8
2.3. Finite Type Invariants of v-Braids and w-Braids 12
2.4. Expansions for w-Braids 15
2.5. Some Further Comments 16
3. w-Knots 21
3.1. v-Knots and w-Knots 21
3.2. Finite Type Invariants of v-Knots and w-Knots 23
3.3. Some Dimensions 25
3.4. Expansions for w-Knots 26
3.5. Jacobi Diagrams, Trees and Wheels 27
3.6. The Relation with Lie Algebras 31
3.7. The Alexander Polynomial 33
3.8. Proof of Theorem 3.27 35
3.9. The Relationship with u-Knots 42
4. Algebraic Structures, Projectivizations, Expansions, Circuit Algebras 42
4.1. Algebraic Structures 43
4.2. Projectivization 44
4.3. Expansions and Homomorphic Expansions 46
4.4. Circuit Algebras 47
5. w-Tangles 50
5.1. v-Tangles and w-Tangles 50
5.2. $\mathcal{A}^w(\uparrow_n)$ and the Alekseev-Torossian Spaces 52
5.3. The Relationship with u-Tangles 57
5.4. The local topology of w-tangles 58
5.5. Good properties and uniqueness 60
6. w-Tangled Foams 62
6.1. The Circuit Algebra of w-Tangled Foams 62
7. Odds and Ends 66
7.1. What means “closed form”? 66
7.2. The Injectivity of $i_u : F_n \to wB_{n+1}$ 67
7.3. Finite Type Invariants of v-Braids and w-Braids, in some Detail 67
7.4. Finite type invariants of w-braids 69
7.5. Arrow Diagrams to Degree 2 69
8. Glossary of notation 70
References 70

## 1. Introduction

1.1. **Dreams.** I have a dream[1], at least partially founded on reality, that many of the difficult algebraic equations in mathematics, especially those that are written in graded spaces, more especially those that are related in one way or another to quantum groups [Dr1], and even more especially those related to the work of Etingof and Kazhdan [EK], can be understood, and indeed, would appear more natural, in terms of finite type invariants of various topological objects.

**wClip 120111-1**



**starts**

**wClips are explained in Section 1.5.**

I believe this is the case for Drinfel'd's theory of associators [Dr2], which can be interpreted as a theory of well-behaved universal finite type invariants of parenthesized tangles[2] [LM2, BN3], and even more elegantly, as a theory of universal finite type invariants of knotted trivalent graphs [Da].

I believe this is the case for Drinfel'd's "Grothendieck-Teichmuller group" [Dr3] which is better understood as a group of automorphisms of a certain algebraic structure, also related to universal finite type invariants of parenthesized tangles [BN6].

And I'm optimistic, indeed I believe, that sooner or later the work of Etingof and Kazhdan [EK] on quantization of Lie bialgebras will be re-interpreted as a construction of a well-behaved universal finite type invariant of virtual knots [Ka2] or of some other class of virtually knotted objects. Some steps in that direction were taken by Haviv [Hav].

I have another dream, to construct a useful "Algebraic Knot Theory". As at least a partial writeup exists [BN8], I'll only state that an important ingredient necessary to fulfill that dream would be a "closed form"[3] formula for an associator, at least in some reduced sense. Formulas for associators or reduced associators were in themselves the goal of several studies undertaken for various other reasons [LM1, Lie, Kur, Lee].

1.2. **Stories.** Thus I was absolutely delighted when in January 2008 Anton Alekseev described to me his joint work [AT] with Charles Torossian — he told me they found a relationship between the Kashiwara-Vergne conjecture [KV], a cousin of the Duflo isomorphism (which I already knew to be knot-theoretic [BLT]), and associators taking values in a space called $\mathfrak{sder}$, which I could identify as "tree-level Jacobi diagrams", also a knot-theoretic space related to the Milnor invariants [BN2, HM]. What's more, Anton told me that in certain quotient spaces the Kashiwara-Vergne conjecture can be solved explicitly; this should lead to some explicit associators!

So I spent the following several months trying to understand [AT], and this paper is a summary of my efforts. The main thing I learned is that the Alekseev-Torossian paper, and with it the Kashiwara-Vergne conjecture, fit very nicely with my first dream recalled above, about interpreting algebra in terms of knot theory. Indeed much of [AT] can be reformulated as a construction and a discussion of a well-behaved universal finite type invariant $Z$ of a certain class of knotted objects (which I will call here w-knotted), a certain natural quotient of the space of virtual knots (more precisely, virtual trivalent tangles). And my hopes remain high that later I (or somebody else) will be able to exploit this relationship in directions



[1] Understanding an author's history and psychology ought never be necessary to understand his/her papers, yet it may be helpful. Nothing material in the rest of this paper relies on Section 1.1.

[2] "$q$-tangles" in [LM2], "non-associative tangles" in [BN3].

[3] The phrase "closed form" in itself requires an explanation. See Section 7.1.

compatible with my second dream recalled above, on the construction of an "algebraic knot theory".

The story, in fact, is prettier than I was hoping for, for it has the following additional qualities:

- w-Knotted objects are quite interesting in themselves: as stated in the abstract, they are related to combinatorial group theory via "basis-conjugating" automorphisms of a free group $F_n$, to groups of movies of flying rings in $\mathbb{R}^3$, and more generally, to certain classes of knotted surfaces in $\mathbb{R}^4$. The references include [BH, FRR, Gol, Mc, Sa].
- The "chord diagrams" for w-knotted objects (really, these are "arrow diagrams") describe formulas for invariant tensors in spaces pertaining to not-necessarily-metrized Lie algebras in much of the same way as ordinary chord diagrams for ordinary knotted objects describe formulas for invariant tensors in spaces pertaining to metrized Lie algebras. This observation is bound to have further implications.
- Arrow diagrams also describe the Feynman diagrams of topological BF theory [CCM, CCFM] and of a certain class of Chern-Simons theories [Na]. Thus it is likely that our story is directly related to quantum field theory[4].
- When composed with the map from knots to w-knots, $Z$ becomes the Alexander polynomial. For links, it becomes an invariant stronger than the multi-variable Alexander polynomial which contains the multi-variable Alexander polynomial as an easily identifiable reduction. On other w-knotted objects $Z$ has easily identifiable reductions that can be considered as "Alexander polynomials" with good behaviour relative to various knot-theoretic operations — cablings, compositions of tangles, etc. There is also a certain specific reduction of $Z$ that can be considered as the "ultimate Alexander polynomial" — in the appropriate sense, it is the minimal extension of the Alexander polynomial to other knotted objects which is well behaved under a whole slew of knot theoretic operations, including the ones named above.

1.3. **The Bigger Picture.** Parallel to the w-story run the possibly more significant u-story and v-story. The u-story is about u-knots, or more generally, u-knotted objects (braids, links, tangles, etc.), where "u" stands for <u>u</u>sual; hence the u-story is about ordinary knot theory. The v-story is about v-knots, or more generally, v-knotted objects, where "v" stands for <u>v</u>irtual, in the sense of Kauffman [Ka2].

The three stories, u, v, and w, are different from each other. Yet they can be told along similar lines — first the knots (topology), then their finite type invariants and their "chord diagrams" (combinatorics), then those map into certain universal enveloping algebras and similar spaces associated with various classes of Lie algebras (low algebra), and finally, in order to construct a "good" universal finite type invariant, in each case one has to confront a certain deeper algebraic subject (high algebra). These stories are summarized in a table form in Figure 1.

u-Knots map into v-knots, and v-knots map into w-knots[5]. The other parts of our stories, the "combinatorics" and "low algebra" and "high algebra" rows of Figure 1, are likewise related, and this relationship is a crucial part of our overall theme. Thus we cannot and will

[4]Some non-perturbative relations between BF theory and w-knots was discussed by Baez, Wise and Crans [BWC].

[5]Though the composition "$u \to v \to w$" is not 0. In fact, the map $u \to w$ is injective.

| | u-Knots → | v-Knots → | w-Knots |
|---|---|---|---|
| Topology | Ordinary (usual) knotted objects in 3D — braids, knots, links, tangles, knotted graphs, etc. | Virtual knotted objects — "algebraic" knotted objects, or "not specifically embedded" knotted objects; knots drawn on a surface, modulo stabilization. | Ribbon knotted objects in 4D; "flying rings". Like v, but also with "overcrossings commute". |
| Combinatorics | Chord diagrams and Jacobi diagrams, modulo $4T$, $STU$, $IHX$, etc. | Arrow diagrams and v-Jacobi diagrams, modulo $6T$ and various "directed" $STU$s and $IHX$s, etc. | Like v, but also with "tails commute". Only "two in one out" internal vertices. |
| Low Algebra | Finite dimensional metrized Lie algebras, representations, and associated spaces. | Finite dimensional Lie bi-algebras, representations, and associated spaces. | Finite dimensional co-commutative Lie bi-algebras (i.e., $\mathfrak{g} \ltimes \mathfrak{g}^*$), representations, and associated spaces. |
| High Algebra | The Drinfel'd theory of associators. | Likely, quantum groups and the Etingof-Kazhdan theory of quantization of Lie bi-algebras. | The Kashiwara-Vergne-Alekseev-Torossian theory of convolutions on Lie groups and Lie algebras. |

**Figure 1.** The u-v-w Stories

not tell the w-story in isolation, and while it is central to this article, we will necessarily also include some episodes from the u and v series. **wClip 120111-1 ends**

1.4. **Plans.** Our order of proceedings is: w-braids (pp. 5), w-knots (pp. 21), generalities (pp. 42), w-tangles (pp. 50), w-tangled foams (pp. 62), and then some odds and ends (pp. 66). For more detailed information consult the "Section Summary" paragraph at the beginning of each of the sections. A glossary of notation is on page 70.

1.5. **wClips.** Alongside this paper there is a series of video clips explaining parts of it. The series as a whole can be found at [BND]; references to specific clips and specific times within clips appear at the margin of this paper. We thank Peter Lee for contributing wClip:120201 and Karene Chu for contributing wClip:120314. **wClip 120118-1**

**at 0:03:10**

1.6. **Acknowledgement.** We wish to thank Anton Alekseev, Jana Archibald, Scott Carter, Karene Chu, Iva Halacheva, Joel Kamnitzer, Lou Kauffman, Peter Lee, Louis Leung, Dylan Thurston, and Lucy Zhang for comments and suggestions.

## 2. w-Braids

**Section Summary.** This section is largely a compilation of existing literature, though we also introduce the language of arrow diagrams that we use throughout the rest of the paper. We define v-braids and then w-braids and survey their relationship with basis-conjugating automorphisms of free groups and with "the group of flying rings in $\mathbb{R}^3$" (really, a group of knotted tubes in $\mathbb{R}^4$). We then play the usual game of introducing finite type invariants, weight systems, chord diagrams (arrow diagrams, for this case), and 4T-like relations. Finally we define and construct a universal finite type invariant for w-braids. It turns out that the only algebraic tool we need to use is the formal exponential function $\exp(a) := \sum a^n/n!$.

**wClip 120111-2**

**starts**

2.1. **Preliminary: Virtual Braids, or v-Braids.** Our main object of study for this section, w-braids, are best viewed as "virtual braids" [Ba, KL, BB], or v-braids, modulo one additional relation. Hence we start with v-braids.

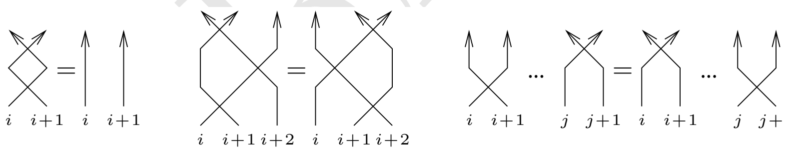
It is simplest to define v-braids in terms of generators and relations, either algebraically or pictorially. This can be done in at least two ways — the easier-at-first but philosophically-less-satisfactory "planar" way, and the harder to digest but morally more correct "abstract" way.[6]

2.1.1. *The "Planar" Way.* For a natural number $n$ set $vB_n$ to be the group generated by symbols $\sigma_i$ $(1 \leq i \leq n-1)$, called "crossings" and graphically represented by an overcrossing ⤫ "between strand $i$ and strand $i+1$" (with inverse ⤬)[7], and $s_i$, called "virtual crossings" and graphically represented by a non-crossing, ✕, also "between strand $i$ and strand $i+1$", subject to the following relations:

- The subgroup of $vB_n$ generated by the virtual crossings $s_i$ is the symmetric group $S_n$, and the $s_i$'s correspond to the transpositions $(i, i+1)$. That is, we have

$$s_i^2 = 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad \text{and if } |i-j| > 1 \text{ then} \qquad s_i s_j = s_j s_i. \tag{1}$$

  In pictures, this is



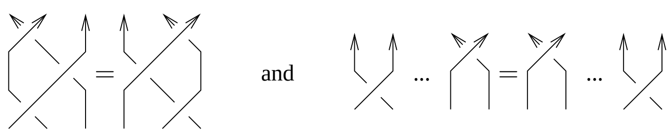

(2)

  Note that we read our braids from bottom to top.

- The subgroup of $vB_n$ generated by the crossings $\sigma_i$'s is the usual braid group $uB_n$, and $\sigma_i$ corresponds to the braiding of strand $i$ over strand $i+1$. That is, we have

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad \text{and if } |i-j| > 1 \text{ then} \qquad \sigma_i \sigma_j = \sigma_j \sigma_i. \tag{3}$$

  In pictures, dropping the indices, this is




(4)

[6]Compare with a similar choice that exists in the definition of manifolds, as either appropriate subsets of some ambient Euclidean spaces (module some equivalences) or as abstract gluings of coordinate patches (modulo some other equivalences). Here in the "planar" approach of Section 2.1.1 we consider v-braids as "planar" objects, and in the "abstract approach" of Section 2.1.2 they are just "gluings" of abstract "crossings", not drawn anywhere in particular.

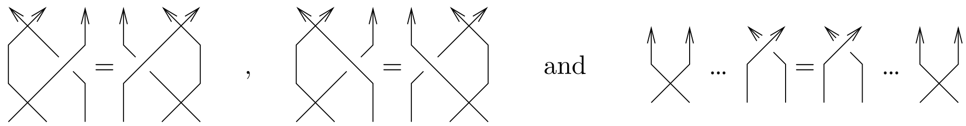
[7]We sometimes refer to ⤫ as a "positive crossing" and to ⤬ as a "negative crossing".

The first of these relations is the "Reidemeister 3 move"[8] of knot theory. The second is sometimes called "locality in space" [BN3].

- Some "mixed relations",

$$s_i\sigma_{i+1}^{\pm 1}s_i = s_{i+1}\sigma_i^{\pm 1}s_{i+1}, \qquad \text{and if } |i-j|>1 \text{ then} \qquad s_i\sigma_j = \sigma_j s_i. \tag{5}$$

In pictures, this is




(6)

*Remark* 2.1. The "skeleton" of a v-braid $B$ is the set of strands appearing in it, retaining the association between their beginning and ends but ignoring all the crossing information. More precisely, it is the permutation induced by tracing along $B$, and even more precisely it is the image of $B$ via the "skeleton morphism" $\varsigma : vB_n \to S_n$ defined by $\varsigma(\sigma_i) = \varsigma(s_i) = s_i$ (or pictorially, by $\varsigma(\overcrossing) = \varsigma(\times) = \times$). Thus the symmetric group $S_n$ is both a subgroup and a quotient group of $vB_n$.

Like there are pure braids to accompany braids, there are pure virtual braids as well:

**Definition 2.2.** A pure v-braid is a v-braid whose skeleton is the identity permutation; the group $PvB_n$ of all pure v-braids is simply the kernel of the skeleton morphism $\varsigma : vB_n \to S_n$.

We note the sequence of group homomorphisms

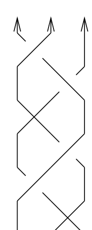
$$1 \longrightarrow PvB_n \hookrightarrow vB_n \xrightarrow{\varsigma} S_n \longrightarrow 1. \tag{7}$$

This sequence is exact and split, with the splitting given by the inclusion $S_n \hookrightarrow vB_n$ mentioned above (1). Therefore we have that

$$vB_n = PvB_n \rtimes S_n. \tag{8}$$

2.1.2. *The "Abstract" Way.* The relations (2) and (6) that govern the behaviour of virtual crossings precisely say that virtual crossings really are "virtual" — if a piece of strand is routed within a braid so that there are only virtual crossings around it, it can be rerouted in any other "virtual only" way, provided the ends remain fixed (this is Kauffman's "detour move" [Ka2, KL]). Since a v-braid $B$ is independent of the routing of virtual pieces of strand, we may as well never supply this routing information.

Thus for example, a perfectly fair verbal description of the (pure!) v-braid on the right is "strand 1 goes over strand 3 by a positive crossing then likewise positively over strand 2 then negatively over 3 then 2 goes positively over 1". We don't need to specify how strand 1 got to be near strand 3 so it can go over it — it got there by means of virtual crossings, and it doesn't matter how. Hence we arrive at the following "abstract" presentation of $PvB_n$ and $vB_n$:




**Proposition 2.3.** *(E.g.* [Ba]*)*

(1) *The group $PvB_n$ of pure v-braids is isomorphic to the group generated by symbols $\sigma_{ij}$ for $1 \le i \ne j \le n$ (meaning "strand $i$ crosses over strand $j$ at a positive crossing"[9]),*

wClip
120118-2

starts

[8]The Reidemeister 2 move is the relations $\sigma_i\sigma_i^{-1} = 1$ which is part of the definition of "a group". There is no Reidemeister 1 move in the theory of braids.

[9]The inverse, $\sigma_{ij}^{-1}$, is "strand $i$ crosses over strand $j$ at a negative crossing"

*subject to the third Reidemeister move and to locality in space (compare with* (3) *and* (4)*):*

$$\sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \qquad \textit{whenever} \qquad |\{i,j,k\}| = 3,$$
$$\sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \qquad \textit{whenever} \qquad |\{i,j,k,l\}| = 4.$$

**wClip 120111-2 ends**

(2) *If $\tau \in S_n$, then with the action $\sigma_{ij}^{\tau} := \sigma_{\tau i, \tau j}$ we recover the semi-direct product decomposition $vB_n = PvB_n \rtimes S_n$.* □

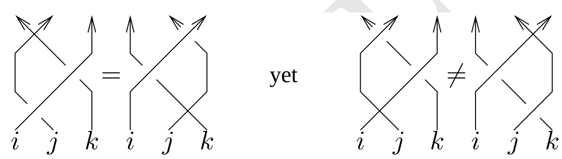
2.2. **On to w-Braids.** To define w-braids, we break the symmetry between over crossings and under crossings by imposing one of the "forbidden moves" virtual knot theory, but not the other:

$$\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}, \qquad \text{yet} \qquad s_i\sigma_{i+1}\sigma_i \neq \sigma_{i+1}\sigma_i s_{i+1}. \tag{9}$$

Alternatively,

$$\sigma_{ij}\sigma_{ik} = \sigma_{ik}\sigma_{ij}, \qquad \text{yet} \qquad \sigma_{ik}\sigma_{jk} \neq \sigma_{jk}\sigma_{ik}.$$

In pictures, this is



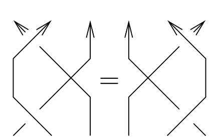

(10)

The relation we have just imposed may be called the "unforbidden relation", or, perhaps more appropriately, the "overcrossings commute" relation (OC). Ignoring the non-crossings[10] ⤫, the OC relation says that it is the same if strand $i$ first crosses over strand $j$ and then over strand $k$, or if it first crosses over strand $k$ and then over strand $j$. The "undercrossings commute" relation UC, the one we do not impose in (9), would say the same except with "under" replacing "over".

**Definition 2.4.** The group of w-braids is $wB_n := vB_n/OC$. Note that $\varsigma$ descends to $wB_n$ and hence we can define the group of pure w-braids to be $PwB_n := \ker \varsigma : wB_n \to S_n$. We still have a split exact sequence as at (7) and a semi-direct product decomposition $wB_n = PwB_n \rtimes S_n$.

*Exercise* 2.5. Show that the OC relation is equivalent to the relation

$$\sigma_i^{-1}s_{i+1}\sigma_i = \sigma_{i+1}s_i\sigma_{i+1}^{-1} \qquad \text{or}$$



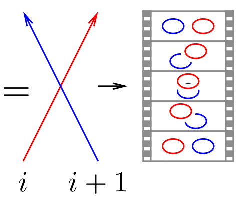

While mostly in this paper the pictorial / algebraic definition of w-braids (and other w-knotted objects) will suffice, we ought describe at least briefly 2-3 further interpretations of $wB_n$:

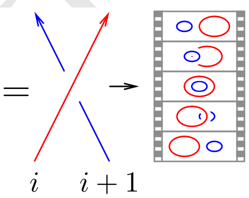
[10]Why this is appropriate was explained in the previous section.

2.2.1. *The group of flying rings.* Let $X_n$ be the space of all placements of $n$ numbered disjoint geometric circles in $\mathbb{R}^3$, such that all circles are parallel to the $xy$ plane. Such placements will be called horizontal[11]. A horizontal placement is determined by the centers in $\mathbb{R}^3$ of the $n$ circles and by $n$ radii, so $\dim X_n = 3n + n = 4n$. The permutation group $S_n$ acts on $X_n$ by permuting the circles, and one may think of the quotient $\tilde{X}_n := X_n/S_n$ as the space of all horizontal placements of $n$ unmarked circles in $\mathbb{R}^3$. The fundamental group $\pi_1(\tilde{X}_n)$ is a group of paths traced by $n$ disjoint horizontal circles (modulo homotopy), so it is fair to think of it as "the group of flying rings".

**Theorem 2.6.** *The group of pure w-braids $PwB_n$ is isomorphic to the group of flying rings $\pi_1(X_n)$. The group $wB_n$ is isomorphic to the group of unmarked flying rings $\pi_1(\tilde{X}_n)$.*

For the proof of this theorem, see [Gol, Sa] and especially [BH]. Here we will contend ourselves with pictures describing the images of the generators of $wB_n$ in $\pi_1(\tilde{X}_n)$ and a few comments:

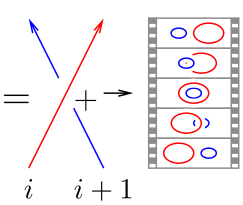
$s_i =$ 

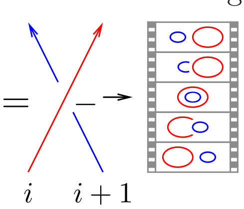

$\sigma_i =$ 


Thus we map the permutation $s_i$ to the movie clip in which ring number $i$ trades its place with ring number $i+1$ by having the two flying around each other. This acrobatic feat is performed in $\mathbb{R}^3$ and it does not matter if ring number $i$ goes "above" or "below" or "left" or "right" of ring number $i+1$ when they trade places, as all of these possibilities are homotopic. More interestingly, we map the braiding $\sigma_i$ to the movie clip in which ring $i+1$ shrinks a bit and flies through ring $i$. It is a worthwhile exercise for the reader to verify that the relations in the definition of $wB_n$ become homotopies of movie clips. Of these relations it is most interesting to see why the "overcrossings commute" relation $\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}$ holds, yet the "undercrossings commute" relation $\sigma_i^{-1}\sigma_{i+1}^{-1}s_i = s_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}$ doesn't. **wClip 120118-2 ends**

*Exercise* 2.7. To be perfectly precise, we have to specify the fly-through direction. In our notation, $\sigma_i$ means that the ring corresponding to the under-strand approaches the bigger ring representing the over-strand from below, flies through it and exists above. For $\sigma_i^{-1}$ we are "playing the movie backwards", i.e., the ring of the under-strand comes from above and exits below the ring of the over-strand.

Let "the signed $w$ braid group", $swB_n$, be the group of horizontal flying rings where both fly-through directions are allowed. This introduces a "sign" for each crossing $\sigma_i$:

$\sigma_{i+} =$ 


$\sigma_{i-} =$ 



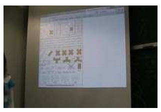

[11] For the group of non-horizontal flying rings see Section 2.5.4

In other words, $swB_n$ is generated by $s_i$, $\sigma_{i+}$ and $\sigma_{i-}$, for $i = 1, ..., n$. Check that in $swB_n$ $\sigma_{i-} = s_i\sigma_{i+}^{-1}s_i$, and this, along with the other obvious relations implies $swB_n \cong wB_n$.

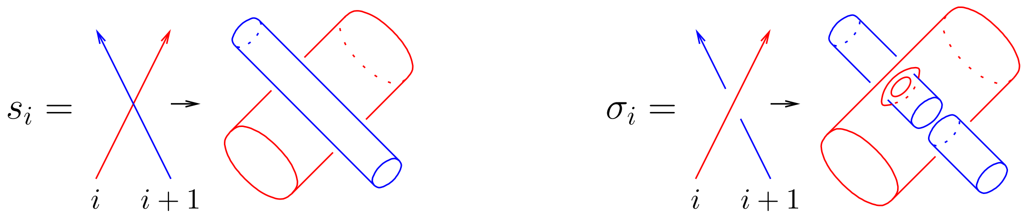
For a rigorous discussion of orientations and signs, see Section 5.4.

2.2.2. *Certain ribbon tubes in* $\mathbb{R}^4$. With time as the added dimension, a flying ring in $\mathbb{R}^3$ traces a tube (an annulus) in $\mathbb{R}^4$, as shown in the picture below:

**wClip 120118-3**



**at 0:00:00**




Note that we adopt here the drawing conventions of Carter and Saito [CS] — we draw surfaces as if they were projected from $\mathbb{R}^4$ to $\mathbb{R}^3$, and we cut them open whenever they are "hidden" by something with a higher fourth coordinate.

Note also that the tubes we get in $\mathbb{R}^4$ always bound natural 3D "solids" — their "insides", in the pictures above. These solids are disjoint in the case of $s_i$ and have a very specific kind of intersection in the case of $\sigma_i$ — these are transverse intersections with no triple points, and their inverse images are a meridional disk on the "thin" solid tube and an interior disk on the "thick" one. By analogy with the case of ribbon knots and ribbon singularities in $\mathbb{R}^3$ (e.g. [Ka1, Chapter V]) and following Satoh [Sa], we call this kind if intersections of solids in $\mathbb{R}^4$ "ribbon singularities" and thus our tubes in $\mathbb{R}^4$ are always "ribbon tubes".

2.2.3. *Basis conjugating automorphisms of* $F_n$. Let $F_n$ be the free (non-Abelian) group with generators $\xi_1, \ldots, \xi_n$. Artin's theorem (Theorems 15 and 16 of [Ar]) says that that the (usual) braid group $uB_n$ (equivalently, the subgroup of $wB_n$ generated by the $\sigma_i$'s) has a faithful right action on $F_n$. In other words, $uB_n$ is isomorphic to a subgroup $H$ of $\operatorname{Aut}^{\mathrm{op}}(F_n)$ (the group of automorphisms of $F_n$ with opposite multiplication; $\psi_1\psi_2 := \psi_2 \circ \psi_1$). Precisely, using $(\xi, B) \mapsto \xi /\!\!/ B$ to denote the right action of $\operatorname{Aut}^{\mathrm{op}}(F_n)$ on $F_n$, the subgroup $H$ consists of those automorphisms $B : F_n \to F_n$ of $F_n$ that satisfy the following two conditions:

(1) $B$ maps any generator $\xi_i$ to a conjugate of a generator (possibly different). That is, there is a permutation $\beta \in S_n$ and elements $a_i \in F_n$ so that for every $i$,

$$\xi_i /\!\!/ B = a_i^{-1}\xi_{\beta i}a_i. \tag{11}$$

(2) $B$ fixes the ordered product of the generators of $F_n$,

$$\xi_1\xi_2\cdots\xi_n /\!\!/ B = \xi_1\xi_2\cdots\xi_n.$$

McCool's theorem [Mc] says that the same holds true[12] if one replaces the braid group $uB_n$ with the bigger group $wB_n$ and drops the second condition above. So $wB_n$ is precisely the group of "basis-conjugating" automorphisms of the free group $F_n$, the group of those automorphisms which map any "basis element" in $\{\xi_1, \ldots, \xi_n\}$ to a conjugate of a (possibly different) basis element.

The relevant action is explicitly defined on the generators of $wB_n$ and $F_n$ as follows (with the omitted generators of $F_n$ always fixed):

$$(\xi_i, \xi_{i+1}) /\!\!/ s_i = (\xi_{i+1}, \xi_i) \qquad (\xi_i, \xi_{i+1}) /\!\!/ \sigma_i = (\xi_{i+1}, \xi_{i+1}\xi_i\xi_{i+1}^{-1}) \qquad \xi_j /\!\!/ \sigma_{ij} = \xi_i\xi_j\xi_i^{-1} \tag{12}$$

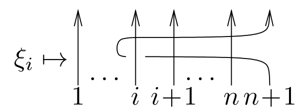
[12] Though see Warning 2.8.

It is a worthwhile exercise to verify that $/\!\!/$ respects the relations in the definition of $wB_n$ and that the permutation $\beta$ in (11) is the skeleton $\varsigma(B)$.

There is a more conceptual description of $/\!\!/$, in terms of the structure of $wB_{n+1}$. Consider the inclusions

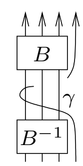
$$wB_n \overset{\iota}{\hookrightarrow} wB_{n+1} \overset{i_u}{\hookleftarrow} F_n. \tag{13}$$

Here $\iota$ is the map of $wB_n$ into $wB_{n+1}$ by adding an inert $(n+1)-$st strand (it is injective as it has a well defined one sided inverse — the deletion of the $(n+1)$-st strand). The inclusion $i_u$ of the free group $F_n$ into $wB_{n+1}$ is defined by $i_u(\xi_i) := \sigma_{i,n+1}$.




The image $i_u(F_n) \subset wB_{n+1}$ is the set of all w-braids whose first $n$ strands are straight and vertical, and whose $(n+1)$-st strand wanders among the first $n$ strands mostly virtually (i.e., mostly using virtual crossings), occasionally slipping under one of those $n$ strands, but never going over anything. In the "flying rings" picture of Section 2.2.1, the image $i_u(F_n) \subset wB_{n+1}$ can be interpreted as the fundamental group of the complement in $\mathbb{R}^3$ of $n$ stationary rings (which is indeed $F_n$) — in $i_u(F_n)$ the only ring in motion is the last, and it only goes under, or "through", other rings, so it can be replaced by a point object whose path is an element of the fundamental group. The injectivity of $i_u$ follows from this geometric picture. Putting the carriage ahead of the horses, we also sketch an algebraic proof of the injectivity of $i_u$ which uses the existence of $/\!\!/$ in Section 7.2.

One may explicitly verify that $i_u(F_n)$ is normalized by $\iota(wB_n)$ in $wB_{n+1}$ (that is, the set $i_u(F_n)$ is preserved by conjugation by elements of $\iota(wB_n)$). Thus the following definition (also shown as a picture on the right) makes sense, for $B \in wB_n \subset wB_{n+1}$ and for $\gamma \in F_n \subset wB_{n+1}$:



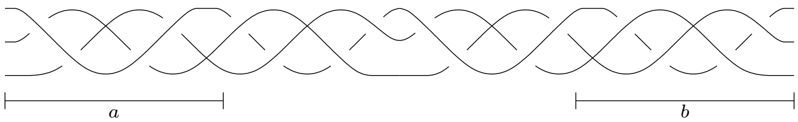

$$\gamma /\!\!/ B := i_u^{-1}(B^{-1}\gamma B) \tag{14}$$

It is a worthwhile exercise to recover the explicit formulas in (12) from the above definition.

*Warning* 2.8. People familiar with the Artin story for ordinary braids should be warned that even though $wB_n$ acts on $F_n$ and the action is induced from the inclusions in (13) in much of the same way as the Artin action is induced by inclusions $uB_n \overset{\iota}{\hookrightarrow} uB_{n+1} \overset{i}{\hookleftarrow} F_n$, there are also some differences, and some further warnings apply:

- In the ordinary Artin story, $i(F_n)$ is the set of braids in $uB_{n+1}$ whose first $n$ strands are unbraided (that is, whose image in $uB_n$ via "dropping the last strand" is the identity). This is not true for w-braids. For w-braids, in $i_u(F_n)$ the last strand always goes "under" all other strands (or just virtually crosses them), but never over.
- Thus unlike the isomorphism $PuB_{n+1} \cong PuB_n \ltimes F_n$, it is not true that $PwB_{n+1}$ is isomorphic to $PwB_n \ltimes F_n$.
- The Overcrossings Commute relation imposed in $wB$ breaks the symmetry between overcrossings and undercrossings. Thus let $i_o : F_n \to wB_n$ be the "opposite" of $i_u$, mapping into braids in which the last strand is always "over" or virtual. Then $i_o$ is not injective (its image is in fact Abelian) and its image is not normalized by $\iota(wB_n)$. So there is no "second" action of $wB_n$ on $F_n$ defined using $i_o$.
- For v-braids, both $i_u$ and $i_o$ are injective and there are two actions of $vB_n$ on $F_n$ — one defined by first projecting into w-braids, and the other defined by first projecting into v-braids modulo "Undercrossings Commute". Yet v-braids contain more information than

these two actions can see. The "Kishino" v-braid below, for example, is visibly trivial if either overcrossings or undercrossings are made to commute, yet by computing its Kauffman bracket we know it is non-trivial as a v-braid [BND, "The Kishino Braid"]:




(The commutator $ab^{-1}a^{-1}b$ of v-braids $a, b$ annihilated by OC/UC, respectively, with a minor cancellation.)

**Problem 2.9.** Is $PwB_n$ a semi-direct product of free groups? Note that both $PuB_n$ and $PvB_n$ are such semi-direct products: For $PuB_n$, this is the well known "combing of braids"; it follows from $PuB_n \cong PuB_{n-1} \ltimes F_{n-1}$ and induction. For $PvB_n$, it is a result stated in [Ba] (though my own understanding of [Ba] is incomplete).

*Remark* 2.10. Note that Gutiérrez and Krstić [GK] find "normal forms" for the elements of $PwB_n$, yet they do not decide whether $PwB_n$ is "automatic" in the sense of [Ep].

2.3. **Finite Type Invariants of v-Braids and w-Braids.** Just as we had two definitions for v-braids (and thus w-braids) in Section 2.1, we will give two (obviously equivalent) developments of the theory of finite type invariants of v-braids and w-braids — a pictorial/topological version in Section 2.3.1, and a more abstract algebraic version in Section 2.3.2.

2.3.1. *Finite Type Invariants, the Pictorial Approach.* In the standard theory of finite type invariants of knots (also known as Vassiliev or Goussarov-Vassiliev invariants) [Gou1, Vas, BN1, BN7] one progresses from the definition of finite type via iterated differences to chord diagrams and weight systems, to $4T$ (and other) relations, to the definition of universal finite type invariants, and beyond. The exact same progression (with different objects playing similar roles, and sometimes, when yet insufficiently studied, with the last step or two missing) is also seen in the theories of finite type invariants of braids [BN5], 3-manifolds [Oh, LMO, Le], virtual knots [GPV, Po] and of several other classes of objects. We thus assume that the reader has familiarity with these basic ideas, and we only indicate briefly how they are implemented in the case of v-braids and w-braids. Some further details are in Section 7.3.

Much like the formula $\doublepoint \to \overcrossing - \undercrossing$ of the Vassiliev-Goussarov fame, given a v-braid invariant $V : vB_n \to A$ valued in some Abelian group $A$, we extend it to "singular" v-braids, braids that contain "semi-virtual crossings" like $\semivirtualover$ and $\semivirtualunder$ using the formulas $V(\semivirtualover) := V(\overcrossing) - V(\virtualcrossing)$ and $V(\semivirtualunder) := V(\virtualcrossing) - V(\undercrossing)$ (see [GPV, Po]). We say that "$V$ is of type $m$" if its extension vanishes on singular v-braids having more than $m$ semi-virtual crossings. Up to invariants of lower type, an invariant of type $m$ is determined by its "weight system", which is a functional $W = W_m(V)$ defined on "$m$-singular v-braids modulo $\overcrossing = \undercrossing = \virtualcrossing$". Let us denote the vector space of all formal linear combinations of such equivalence classes by $\mathcal{G}_m\mathcal{D}^v_n$. Much as $m$-singular knots modulo $\overcrossing = \undercrossing$ can be identified with chord diagrams, the basis elements of $\mathcal{G}_m\mathcal{D}^v_n$ can be identified with pairs $(D, \beta)$, where $D$ is a horizontal arrow diagram and $\beta$ is a "skeleton permutation". See Figure 2.

We assemble the spaces $\mathcal{G}_m\mathcal{D}^v_n$ together to form a single graded space, $\mathcal{D}^v_n := \oplus_{m=0}^{\infty}\mathcal{G}_m\mathcal{D}^v_n$. Note that throughout this paper, whenever we write an infinite direct sum, we automatically complete it. Thus in $\mathcal{D}^v_n$ we allow infinite sums with one term in each homogeneous piece $\mathcal{G}_m\mathcal{D}^v_n$.

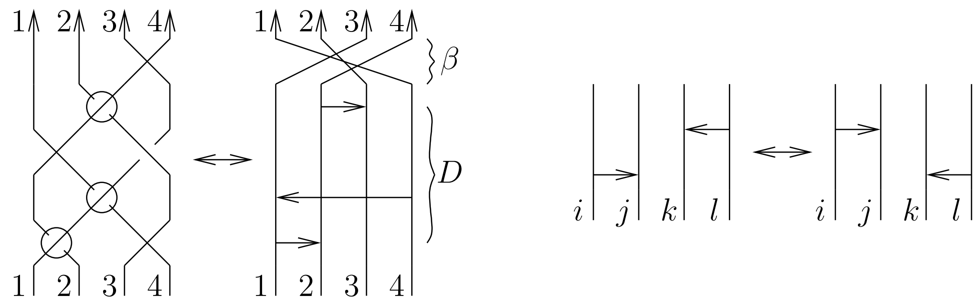
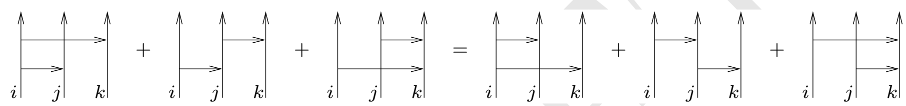
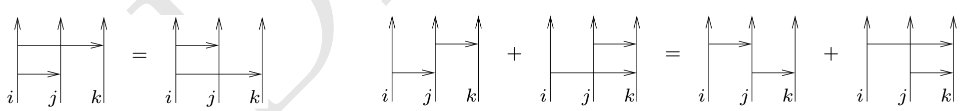



**Figure 2.** On the left, a 3-singular v-braid and its corresponding 3-arrow diagram. A self-explanatory algebraic notation for this arrow diagram is $(a_{12}a_{41}a_{23}, 3421)$. picture and in algebraic notation. Note that we regard arrow diagrams as graph-theoretic objects, and hence the two arrow diagrams on the right, whose underlying graphs are the same, are regarded as equal. In algebraic notation this means that we always impose the relation $a_{ij}a_{kl} = a_{kl}a_{ij}$ when the indices $i$, $j$, $k$, and $l$ are all distinct.




$$a_{ij}a_{ik} + a_{ij}a_{jk} + a_{ik}a_{jk} = a_{ik}a_{ij} + a_{jk}a_{ij} + a_{jk}a_{ik}$$
$$\text{or} \qquad [a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}] = 0$$

**Figure 3.** The $6T$ relation. Standard knot theoretic conventions apply — only the relevant parts of each diagram is shown; in reality each diagram may have further vertical strands and horizontal arrows, provided the extras are the same in all 6 diagrams. Also, the vertical strands are in no particular order — other valid $6T$ relations are obtained when those strands are permuted in other ways.




$$a_{ij}a_{ik} = a_{ik}a_{ij} \qquad\qquad a_{ij}a_{jk} + a_{ik}a_{jk} = a_{jk}a_{ij} + a_{jk}a_{ik}$$
$$\text{or} \quad [a_{ij}, a_{ik}] = 0 \qquad\qquad \text{or} \quad [a_{ij} + a_{ik}, a_{jk}] = 0$$

**Figure 4.** The TC and the $\overrightarrow{4T}$ relations.

In the standard finite-type theory for knots, weight systems always satisfy the $4T$ relation, and are therefore functionals on $\mathcal{A} := \mathcal{D}/4T$. Likewise, in the case of v-braids, weight systems satisfy the "$6T$ relation" of [GPV, Po], shown in Figure 3, and are therefore functionals on $\mathcal{A}^v_n := \mathcal{D}^v_n/6T$. In the case of w-braids, the "overcrossings commute" relation (9) implies the "Tails Commute" (TC) relation on the level of arrow diagrams, and in the presence of the TC relation, two of the terms in the $6T$ relation drop out, and what remains is the "$\overrightarrow{4T}$" relation. These relations are shown in Figure 4. Thus weight systems of finite type invariants of w-braids are linear functionals on $\mathcal{A}^w_n := \mathcal{D}^v_n/TC, \overrightarrow{4T}$.

The next question that arises is whether we have already found *all* the relations that weight systems always satisfy. More precisely, given a degree $m$ linear functional on $\mathcal{A}_n^v = \mathcal{D}_n^v/6T$ (or on $\mathcal{A}_n^w = \mathcal{D}_n^v/TC, \overrightarrow{4T}$), is it always the weight system of some type $m$ invariant $V$ of v-braids (or w-braids)? As in every other theory of finite type invariants, the answer to this question is affirmative if and only if there exists a "universal finite type invariant" (or simply, an "expansion") of v-braids (w-braids):

**Definition 2.11.** An expansion for v-braids (w-braids) is an invariant $Z : vB_n \to \mathcal{A}_n^v$ (or $Z : wB_n \to \mathcal{A}_n^w$) satisfying the following "universality condition":

- If $B$ is an $m$-singular v-braid (w-braid) and $D \in \mathcal{G}_m\mathcal{D}_n^v$ is its underlying arrow diagram as in Figure 2, then
$$Z(B) = D + (\text{terms of degree } > m).$$

Indeed if $Z$ is an expansion and $W \in \mathcal{G}_m\mathcal{A}^\star$,[13] the universality condition implies that $W \circ Z$ is a finite type invariant whose weight system is $W$. To go the other way, if $(D_i)$ is a basis of $\mathcal{A}$ consisting of homogeneous elements, if $(W_i)$ is the dual basis of $\mathcal{A}^\star$ and $(V_i)$ are finite type invariants whose weight systems are the $W_i$'s, then $Z(B) := \sum_i D_i V_i(B)$ defines an expansion.

In general, constructing a universal finite type invariant is a hard task. For knots, one uses either the Kontsevich integral or perturbative Chern-Simons theory (also known as "configuration space integrals" [BT] or "tinker-toy towers" [Th]) or the rather fancy algebraic theory of "Drinfel'd associators" (a summary of all those approaches is at [BS]). For homology spheres, this is the "LMO invariant" [LMO, Le] (also the "Århus integral" [BGRT]). For v-braids, we still don't know if an expansion exists. As we shall see below, the construction of an expansion for w-braids is quite easy.

2.3.2. *Finite Type Invariants, the Algebraic Approach.* For any group $G$, one can form the group algebra $\mathbb{F}G$ for some field $\mathbb{F}$ by allowing formal linear combinations of group elements and extending multiplication linearly. The *augmentation ideal* is the ideal generated by differences, or eqvivalently, the set of linear combinations of group elements whose coefficients sum to zero:
$$\mathcal{I} := \Big\{ \sum_{i=1}^{k} a_i g_i : a_i \in \mathbb{F}, g_i \in G, \sum_{i=1}^{k} a_i = 0 \Big\}.$$
Powers of the augmentation ideal provide a filtration of the group algebra. Let $\mathcal{A}(G) := \bigoplus_{m\geq 0} \mathcal{I}^m/\mathcal{I}^{m+1}$ be the associated graded space corresponding to this filtration.

**Definition 2.12.** An expansion for the group $G$ is a map $Z : G \to \mathcal{A}(G)$, such that the linear extension $Z : \mathbb{F}G \to \mathcal{A}(G)$ is filtration preserving and the induced map
$$\text{gr } Z : (\text{gr } \mathbb{F}G = \mathcal{A}(G)) \to (\text{gr } \mathcal{A}(G) = \mathcal{A}(G))$$
is the identity. An eqvivalent way to phrase this is that the degree $m$ piece of $Z$ restricted to $\mathcal{I}^m$ is the projection onto $\mathcal{I}^m/\mathcal{I}^{m+1}$.

*Exercise* 2.13. Verify that for the groups $PvB_n$ and $PwB_n$ the m-th power of the augmentation ideal coincides with resolutions of $m$-singular $v$- or $w$-braids (by a resolution we mean the formal linear combination where each semivirtual crossing is replaced by the appropriate

[13] $\mathcal{A}$ here denotes either $\mathcal{A}_n^v$ or $\mathcal{A}_n^w$, or in fact, any of many similar spaces that we will discuss later on.

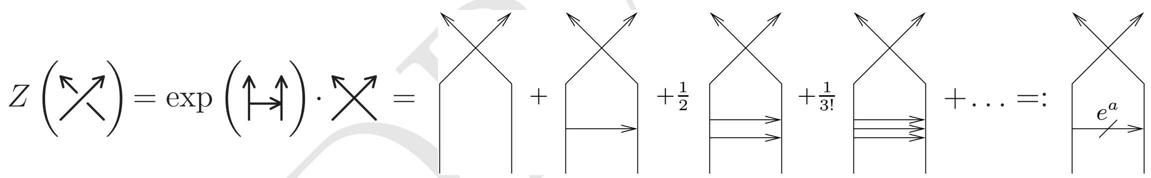
difference of a virtual and a regular crossing). Then check that the notion of expansion defined above is the same as that of Definition 2.11. FIX

Finally, note the functorial nature of the construction above. What we have described is a functor, called "projectivization" proj : $Groups \to GradedAlgebras$, which assigns to each group $G$ the graded algebra $\mathcal{A}(G)$. To each homomorphism $\phi : G \to H$, proj assigns the induced map gr $\phi : (\mathcal{A}(G) = \text{gr}\, \mathbb{F}G) \to (\mathcal{A}(H) = \text{gr}\, \mathbb{F}H)$.

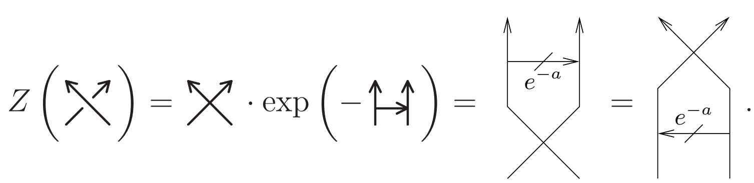
2.4. **Expansions for w-Braids.** The space $\mathcal{A}_n^w$ of arrow diagrams on $n$ strands is an associative algebra in an obvious manner: If the permutations underlying two arrow diagrams are the identity permutations, we simply juxtapose the diagrams. Otherwise we "slide" arrows through permutations in the obvious manner — if $\tau$ is a permutation, we declare that $\tau a_{(\tau i)(\tau j)} = a_{ij}\tau$. Instead of seeking an expansion $wB_n \to \mathcal{A}_n^w$, we set the bar a little higher and seek a "homomorphic expansion":

**Definition 2.14.** A homomorphic expansion $Z : wB_n \to \mathcal{A}_n^w$ is an expansion that carries products in $wB_n$ to products in $\mathcal{A}_n^w$.

To find a homomorphic expansion, we just need to define it on the generators of $wB_n$ and verify that it satisfies the relations defining $wB_n$ and the universality condition. Following [BP, Section 5.3] and [AT, Section 8.1] we set $Z(\times) = \times$ (that is, a transposition in $wB_n$ gets mapped to the same transposition in $\mathcal{A}_n^w$, adding no arrows) and $Z(\times) = \exp(\mapsto)\times$. This last formula is important so deserves to be magnified, explained and replaced by some new notation:


(15)

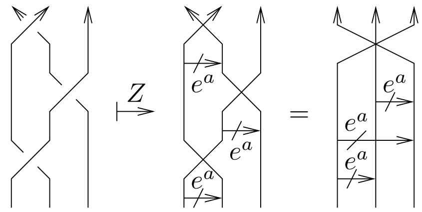
Thus the new notation $\xrightarrow{e^a}$ stands for an "exponential reservoir" of parallel arrows, much like $e^a = 1 + a + aa/2 + aaa/3! + \ldots$ is a "reservoir" of $a$'s. With the obvious interpretation for $\xrightarrow{e^{-a}}$ (the $-$ sign indicates that the terms should have alternating signs, as in $e^{-a} = 1 - a + a^2/2 - a^3/3! + \ldots$), the second Reidemeister move $\times\times = 1$ forces that we set



**Theorem 2.15.** *The above formulas define an invariant $Z : wB_n \to \mathcal{A}_n^w$ (that is, $Z$ satisfies all the defining relations of $wB_n$). The resulting $Z$ is a homomorphic expansion (that is, it satisfies the universality property of Definition 2.14).*

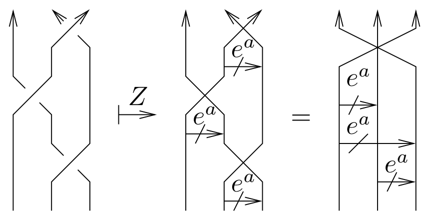
*Proof.* (Following [BP, AT]) For the invariance of $Z$, the only interesting relations to check are the Reidemeister 3 relation of (4) and the Overcrossings Commute relation of (10). For

Reidemeister 3, we have



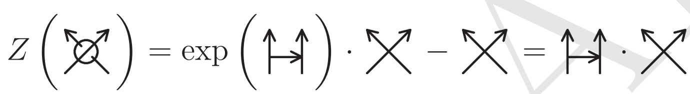
$$= e^{a_{12}}e^{a_{13}}e^{a_{23}}\tau \overset{1}{=} e^{a_{12}+a_{13}}e^{a_{23}}\tau \overset{2}{=} e^{a_{12}+a_{13}+a_{23}}\tau,$$

where $\tau$ is the permutation 321 and equality 1 holds because $[a_{12}, a_{13}] = 0$ by a Tails Commute (TC) relation and equality 2 holds because $[a_{12} + a_{13}, a_{23}] = 0$ by a $\overrightarrow{4T}$ relation. Likewise, again using TC and $\overrightarrow{4T}$,



$$= e^{a_{23}}e^{a_{13}}e^{a_{12}}\tau = e^{a_{23}}e^{a_{13}+a_{12}}\tau = e^{a_{23}+a_{13}+a_{12}}\tau,$$

and so Reidemeister 3 holds. An even simpler proof using just the Tails Commute relation shows that the Overcrossings Commute relation also holds. Finally, since $Z$ is homomorphic, it is enough to check the universality property at degree 1, where it is very easy:



+ (terms of degree $> 1$),

and a similar computation manages the ⤫ case. ☐

*Remark* 2.16. Note that the main ingredient of the above proof was to show that $R := Z(\sigma_{12}) = e^{a_{12}}$ satisfies the famed Yang-Baxter equation,

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12},$$

where $R^{ij}$ means "place $R$ on strands $i$ and $j$".

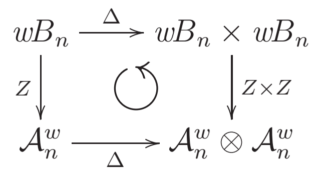
## 2.5. **Some Further Comments.**

2.5.1. *Compatibility with Braid Operations.* As with any new gadget, we would like to know how compatible the expansion $Z$ of the previous section is with the gadgets we already have; namely, with various operations that are available on w-braids and with the action of w-braids on the free group $F_n$ (Section 2.2.3).

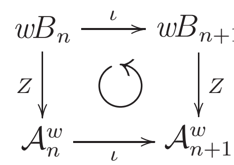
2.5.1.1. $Z$ is Compatible with Braid Inversion. Let $\theta$ denote both the "braid inversion" operation $\theta : wB_n \to wB_n$ defined by $B \mapsto B^{-1}$ and the "antipode" anti-automorphism $\theta : \mathcal{A}^w_n \to \mathcal{A}^w_n$ defined by mapping permutations to their inverses and arrows to their negatives (that is, $a_{ij} \mapsto -a_{ij}$). Then the diagram on the right commutes.

$$\begin{array}{ccc} wB_n & \xrightarrow{\theta} & wB_n \\ {\scriptstyle Z}\downarrow & \circlearrowleft & \downarrow{\scriptstyle Z} \\ \mathcal{A}^w_n & \xrightarrow[\theta]{} & \mathcal{A}^w_n \end{array}$$

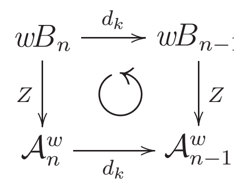
2.5.1.2. Braid Cloning and the Group-Like Property. Let $\Delta$ denote both the "braid cloning" operation $\Delta : wB_n \to wB_n wB_n$ defined by $B \mapsto (B,B)$ and the "co-product" algebra morphism $\Delta : \mathcal{A}^w_n \to \mathcal{A}^w_n \otimes \mathcal{A}^w_n$ defined by cloning permutations (that is, $\tau \mapsto \tau \otimes \tau$) and by treating arrows as primitives (that is, $a_{ij} \mapsto a_{ij} \otimes 1 + 1 \otimes a_{ij}$). Then the diagram on the right commutes. In formulas, this is $\Delta(Z(B)) = Z(B) \otimes Z(B)$, which is the statement "$Z(B)$ is group-like".



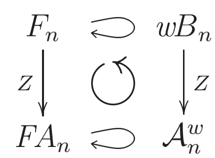

2.5.1.3. Strand Insertions. Let $\iota : wB_n \to wB_{n+1}$ be an operation of "inert strand insertion". Given $B \in wB_n$, the resulting $\iota B \in wB_{n+1}$ will be $B$ with one strand $S$ added at some location chosen in advance — to the left of all existing strands, or to the right, or starting from between the 3rd and the 4th strand of $B$ and ending between the 6th and the 7th strand of $B$; when adding $S$, add it "inert", so that all crossings on it are virtual (this is well defined). There is a corresponding inert strand addition operation $\iota : \mathcal{A}^w_n \to \mathcal{A}^w_{n+1}$, obtained by adding a strand at the same location as for the original $\iota$ and adding no arrows. It is easy to check that $Z$ is compatible with $\iota$; namely, that the diagram on the right is commutative.




2.5.1.4. Strand Deletions. Given $k$ between 1 and $n$, let $d_k : wB_n \to wB_{n-1}$ the operation of "removing the $k$th strand". This operation induces a homonymous operation $d_k : \mathcal{A}^w_n \to \mathcal{A}^w_{n-1}$: if $D \in \mathcal{A}^w_n$ is an arrow diagram, $d_k D$ is $D$ with its $k$th strand removed if no arrows in $D$ start or end on the $k$th strand, and it is 0 otherwise. It is easy to check that $Z$ is compatible with $d_k$; namely, that the diagram on the right is commutative.[14]




2.5.1.5. Compatibility with the action on $F_n$. Let $FA_n$ denote the (degree-completed) free associative (but not commutative) algebra on generators $x_1, \ldots, x_n$. Then there is an "expansion" $Z : F_n \to FA_n$ defined by $\xi_i \mapsto e^{x_i}$ (see [Lin] and the related "Magnus Expansion" of [MKS]). Also, there is a right action of $\mathcal{A}^w_n$ on $FA_n$ defined on generators by $x_i \tau = x_{\tau i}$ for $\tau \in S_n$ and by $x_j a_{ij} = [x_i, x_j]$ and $x_k a_{ij} = 0$ for $k \neq j$ and extended by the Leibniz rule to the rest of $FA_n$ and then multiplicatively to the rest of $\mathcal{A}^w_n$.

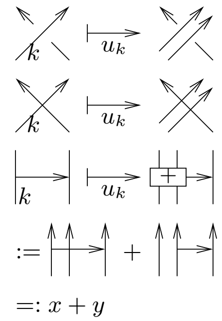



*Exercise* 2.17. Using the language of Section 4.2, verify that $FA_n = \operatorname{proj} F_n$ and that when the actions involved are regarded as instances of the algebraic structure "one monoid acting on another", we have that $\left(FA_n \circlearrowleft \mathcal{A}^w_n\right) = \operatorname{proj}\left(F_n \circlearrowleft wB_n\right)$. Finally, use the definition of the action in (14) and the commutative diagrams of paragraphs 2.5.1.1, 2.5.1.2 and 2.5.1.3 to show that the diagram of paragraph 2.5.1.5 is also commutative.

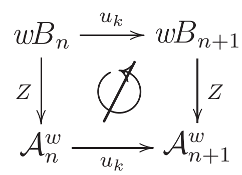


[14]Using the language of Section 4.2, "$d_k : wB_n \to wB_{n-1}$" is an algebraic structure made of two spaces ($wB_n$ and $wB_{n-1}$), two binary operations (braid composition in $wB_n$ and in $wB_{n-1}$), and one unary operation, $d_k$. After projectivization we get the algebraic structure $d_k : \mathcal{A}^w_n \to \mathcal{A}^w_{n-1}$ with $d_k$ as described above, and an alternative way of stating our assertion is to say that $Z$ is a morphism of algebraic structures. A similar remark applies (sometimes in the negative form) to the other operations discussed in this section.

2.5.1.6. Unzipping a Strand. Given $k$ between 1 and $n$, let $u_k : wB_n \to wB_{n+1}$ the operation of "unzipping the $k$th strand", briefly defined on the right[15]. The induced operation $u_k : \mathcal{A}_n^w \to \mathcal{A}_{n+1}^w$ is also shown on the right — if an arrow starts (or ends) on the strand being doubled, it is replaced by a sum of two arrows that start (or end) on either of the two "daughter strands" (and this is performed for each arrow independently; so if there are $t$ arrows touching the $k$th strands in a diagram $D$, then $u_k D$ will be a sum of $2^t$ diagrams).




In some sense, this whole paper as well as the work of Kashiwara and Vergne [KV] and Alekseev and Torossian [AT] is about coming to grips with the fact that $Z$ is **not** compatible with $u_k$ (that the diagram on the right is **not** commutative). Indeed, let $x := a_{13}$ and $y := a_{23}$ be as on the right, and let $s$ be the permutation 21 and $\tau$ the permutation 231. Then $d_1 Z(\times) = d_1(e^{a_{12}} s) = e^{x+y}\tau$ while $Z(d_1 \times) = e^y e^x \tau$. So the failure of $d_1$ and $Z$ to commute is the ill-behaviour of the exponential function when its arguments are not commuting, which is measured by the BCH formula, central to both [KV] and [AT].




2.5.2. *Power and Injectivity.* The following theorem is due to Berceanu and Papadima [BP, Theorem 5.4]; a variant of this theorem are also true for ordinary braids [BN2, Ko, HM], and can be proven by similar means:

**Theorem 2.18.** *$Z : wB_n \to \mathcal{A}_n^w$ is injective. In other words, finite type invariants separate w-braids.*

*Proof.* Follows immediately from the faithfulness of the action $F_n \circlearrowleft wB_n$, from the compatibility of $Z$ with this action, and from the injectivity of $Z : F_n \to FA_n$ (the latter is well known, see e.g. [MKS, Lin]). Indeed if $B_1$ and $B_2$ are w-braids and $Z(B_1) = Z(B_2)$, then $Z(\xi)Z(B_1) = Z(\xi)Z(B_2)$ for any $\xi \in F_n$, therefore $\forall \xi\, Z(\xi \,/\!\!/\, B_1) = Z(\xi \,/\!\!/\, B_2)$, therefore $\forall \xi\, \xi \,/\!\!/\, B_1 = \xi \,/\!\!/\, B_2$, therefore $B_1 = B_2$. $\square$

*Remark* 2.19. Apart from the obvious, that $\mathcal{A}_n^w$ can be computed degree by degree in exponential time, we do not know a simple formula for the dimension of the degree $m$ piece of $\mathcal{A}_n^w$ or a natural basis of that space. This compares unfavourably with the situation for ordinary braids (see e.g. [BN5]). Also compare with Problem 2.9 and with Remark 2.10.

2.5.3. *Uniqueness.* There is certainly not a unique expansion for w-braids — if $Z_1$ is an expansion and and $P$ is any degree-increasing linear map $\mathcal{A}^w \to \mathcal{A}^w$ (a "pollution" map), then $Z_2 := (I + P) \circ Z_1$ is also an expansion, where $I : \mathcal{A}^w \to \mathcal{A}^w$ is the identity. But that's all, and if we require a bit more, even that freedom disappears.

**Proposition 2.20.** *If $Z_{1,2} : wB_n \to \mathcal{A}_n^w$ are expansions then there exists a degree-increasing linear map $P : \mathcal{A}^w \to \mathcal{A}^w$ so that $Z_2 := (I + P) \circ Z_1$.*

*Proof.* (Sketch). Let $\widehat{wB_n}$ be the unipotent completion of $wB_n$. That is, let $\mathbb{Q}wB_n$ be the algebra of formal linear combinations of w-braids, let $\mathcal{I}$ be the ideal in $\mathbb{Q}wB_n$ be the ideal

[15]Unzipping a knotted zipper turns a single band into two parallel ones. This operation is also known as "strand doubling", but for compatibility with operations that will be introduced later, we prefer "unzipping".

generated by $\doublepoint = \overcrossing - \virtualcrossing$ and by $\doublepoint = \virtualcrossing - \undercrossing$, and set

$$\widehat{wB_n} := \varprojlim_{m\to\infty} \mathbb{Q}wB_n \,/\mathcal{I}^m \,.$$

$\widehat{wB_n}$ is filtered with $\mathcal{F}_m\widehat{wB_n} := \varprojlim_{m'>m} \mathcal{I}^m \,/\mathcal{I}^{m'}$ . An "expansion" can be re-interpreted as an "isomorphism of $\widehat{wB_n}$ and $\mathcal{A}^w_n$ as filtered vector spaces". Always, any two isomorphisms differ by an automorphism of the target space, and that's the essence of $I + P$. $\square$

**Proposition 2.21.** *If $Z_{1,2} : wB_n \to \mathcal{A}^w_n$ are homomorphic expansions that commute with braid cloning (paragraph 2.5.1.2) and with strand insertion (paragraph 2.5.1.3), then $Z_1 = Z_2$.*

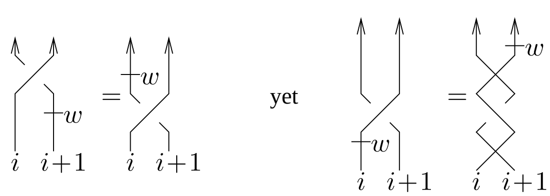
*Proof.* (Sketch). A homomorphic expansion that commutes with strand insertions is determined by its values on the generators $\overcrossing$, $\undercrossing$ and $\virtualcrossing$ of $wB_2$. Commutativity with braid cloning implies that these values must be (up to permuting the strands) group like, that is, the exponentials of primitives. But the only primitives are $a_{12}$ and $a_{21}$, and one may easily verify that there is only one way to arrange these so that $Z$ will respect $\virtualcrossing^2 = \virtualcrossing \cdot \virtualcrossing = 1$ and $\doublepoint \mapsto \rightarrowdiagram +$ (higher degree terms). $\square$

2.5.4. *The group of non-horizontal flying rings.* Let $Y_n$ denote the space of all placements of $n$ numbered disjoint oriented unlinked geometric circles in $\mathbb{R}^3$. Such a placement is determined by the centers in $\mathbb{R}^3$ of the circles, the radii, and a unit normal vector for each circle pointing in the positive direction, so $\dim Y_n = 3n + n + 3n = 7n$. $S_n \ltimes \mathbb{Z}_2^n$ acts on $Y_n$ by permuting the circles and mapping each circle to its image in either an orientation-preserving or an orientation-reversing way. Let $\tilde{Y}_n$ denote the quotient $Y_n/S_n \ltimes \mathbb{Z}_2^n$. The fundamental group $\pi_1(\tilde{Y}_n)$ can be thought of as the "group of flippable flying rings". Without loss of generality, we can assume that the basepoint is chosen to be a horizontal placement. We want to study the relationship of this group to $wB_n$.

**Theorem 2.22.** *$\pi_1(\tilde{Y}_n)$ is a $\mathbb{Z}_2^n$-extension of $wB_n$, generated by $s_i$, $\sigma_i$ $(1 \le i \le n-1)$, and $w_i$ ("flips"), for $1 \le i \le n$; with the relations as above, and in addition:*

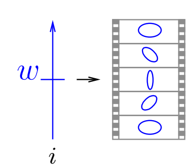
$$w_i^2 = 1; \qquad w_i w_j = w_j w_i; \qquad w_j s_i = s_i w_j \quad \textit{when } i \neq j, j+1;$$
$$w_i s_i = s_i w_{i+1}; \qquad w_{i+1} s_i = s_i w_i;$$
$$w_j \sigma_i = \sigma_i w_j \quad \textit{if } j \neq i, i+1; \quad w_{i+1}\sigma_i = \sigma_i w_i; \quad \textit{yet} \quad w_i\sigma_i = s_i\sigma_i^{-1}s_i w_{i+1}.$$

The two most interesting flip relations in pictures:




(16)

Instead of a proof, we provide some heuristics. Since each circle starts out in a horizontal position and returns to a horizontal position, there is an integer number of "flips" they do in between, these are the generators $w_i$, as shown on the right.

$w_i =$ 

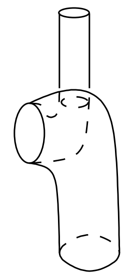

The first relation says that a double flip is homotopic to doing nothing. Technically, there are two different directions of flips, and they are the same via this (non-obvious but true) relation. The rest of the first line is obvious: flips of different rings commute, and if two rings fly around each other while another one flips, the order of these events can be switched by homotopy. The second line says that if two rings trade places with no interaction while one flips, the order of these events can be switched as well. However, we have to re-number the flip to conform to the strand labeling convention.

The only subtle point is how flips interact with crossings. First of all, if one ring flies through another while a third one flips, the order clearly does not matter. If a ring flies through another and also flips, the order can be switched. However, if ring $A$ flips and then ring $B$ flies through it, this is homotopic to ring $B$ flying through ring $A$ from the other direction and then ring $A$ flipping. In other words, commuting $\sigma_i$ with $w_i$ changes the "sign of the crossing" in the sense of Exercise 2.7. This gives the last, and the only truely non-commutative flip relation.

**wClip 120118-3**

**at 0:19:30**

To explain why the flip is denoted by $w$, let us consider the alternative description by ribbon tubes. A flipping ring traces a so called wen[16] in $\mathbb{R}^4$. A wen is a Klein bottle cut along a meridian circle, as shown. The wen is embedded in $\mathbb{R}^4$.



Finally, note that $\pi_1 Y_n$ is exactly the pure $w$-braid group $PwB_n$: since each ring has to return to its original position and orientation, each does an even number of flips. The flips (or wens) can all be moved to the bottoms of the braid diagram strands (to the bottoms of the tubes, to the beginning of words), at a possible cost, as specified by Equation **??**. Once together, they pairwise cancel each other. As a result, this group can be thought of as not containing wens at all.

2.5.5. *The Relationship with u-Braids.* For the sake of ignoring strand permutations, we restrict our attention to pure braids.

By Section 2.3.2, for any expansion $Z^u : PuB_n \to \mathcal{A}^u_n$ (where $PuB_n$ is the "usual" braid group and $\mathcal{A}^u_n$ is the algebra of horizontal chord diagrams on $n$ strands), there is a square of maps as shown on the right. Here, $Z^w$ is the expansion constructed in Section 2.4, the left vertical map $a$ is the composition of the inclusion and projection maps $PuB_n \to PvB_n \to PwB_n$.

$$\begin{array}{ccc} PuB & \xrightarrow{Z^u} & \mathcal{A}^u \\ \downarrow a & & \downarrow \alpha \\ PwB & \xrightarrow{Z^w} & \mathcal{A}^w \end{array}$$

The map $\alpha$ is the induced map by the functoriality of projectivisation, as noted after Exercise 2.13. The reader can verify that $\alpha$ maps each chord to the sum of its two possible directed versions.

Note that this square is *not* commutative for any choice of $Z^u$ even in degree 2: the image of a crossing under $Z^w$ is outside the image of $\alpha$.

More specifically, for any choice $c$ of a "parenthetization" of $n$ points, the KZ-construction / Kontsevich integral (see for example [BN3]) returns an expansion $Z^u_c$ of $u$-braids. As we shall see in section **??**, for any choice of $c$, the two compositions $\alpha \circ Z^u_c$ and $Z^w \circ a$ are "conjugate in a bigger space": there is a map $i$ from $\mathcal{A}^w$ to a larger space of "non-horizontal arrow diagrams", and in this space the images of the above composites are conjugate. However, we are not certain that $i$ is an injection, and whether the conjugation leaves the $i$-image of $\mathcal{A}^w$ invariant, and so we do not know if the two compositions differ merely by an outer automorphism of $\mathcal{A}^w$.

$$\begin{array}{ccc} PuB_n & \xrightarrow{Z^u_c} & \mathcal{A}^u_n \\ \downarrow a & & \downarrow \alpha \\ PwB_n & \xrightarrow{Z^w} & \mathcal{A}^w_n \end{array}$$



[16]The term wen was coined by Kanenobu and Shima in [KS]

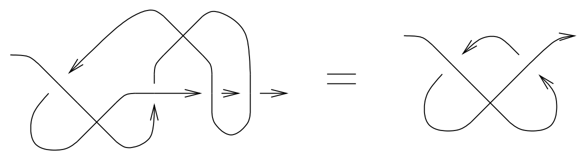


**Figure 5.** A long v-knot diagram with 2 virtual crossings, 2 positive crossings and 2 negative crossings. A positive-negative pair can easily be canceled using R2, and then a virtual crossing can be canceled using VR1, and it seems that the rest cannot be simplified any further.

## 3. w-Knots

**Section Summary.** We define v-knots and w-knots (long v-knots and long w-knots, to be precise). We determine the space of “chord diagrams” for w-knots to be the space $\mathcal{A}^w(\uparrow)$ of arrow diagrams modulo $\overrightarrow{4T}$ and TC relations. We show that $\mathcal{A}^w(\uparrow)$ can be re-interpreted as a space of trivalent graphs modulo STU- and IHX-like relations, and this allows us to completely determine $\mathcal{A}^w(\uparrow)$. With no difficulty at all we construct a universal finite type invariant for w-knots. With a bit of further difficulty we show that it is essentially equal to the Alexander polynomial.

**Knots are the wrong objects for study in knot theory,** v-knots are the wrong objects for study in the theory of v-knotted objects and w-knots are the wrong objects for study in the theory of w-knotted objects. Studying uvw-knots on their own is the parallel of studying cakes and pastries as they come out of the bakery — we sure want to make them our own, but the theory of desserts is more about the ingredients and how they are put together than about the end products. In algebraic knot theory this reflects through the fact that knots are not finitely generated in any sense (hence they must be made of some more basic ingredients), and through the fact that there are very few operations defined on knots (connected sums and satellite operations being the main exceptions), and thus most interesting properties of knots are transcendental, or non-algebraic, when viewed from within the algebra of knots and operations on knots [BN8].

The right objects for study in knot theory, or v-knot theory or w-knot theory, are thus the ingredients that make up knots and that permit a richer algebraic structure. These are braids, studied in the previous section, and even more so tangles and tangled graphs, studied in the following sections. Yet tradition has its place and the sweets are tempting, and I feel compelled to introduce some of the tools we will use in the deeper and healthier study of w-tangles and w-tangled foams in the limited but tasty arena of the baked goods of knot theory, the knots themselves.

### 3.1. v-Knots and w-Knots.

v-Knots may be understood either as knots drawn on surfaces modulo the addition or removal of empty handles [Ka2, Kup] or as “Gauss diagrams” (Remark 3.4), or simply “unimbedded but wired together” crossings modulo the Reidemeister moves ([Ka2, Rou] and Section 4.4). But right now we forgo the topological and the abstract and give only the “planar” (and somewhat less philosophically satisfying) definition of v-knots.

**Definition 3.1.** A “long v-knot diagram” is an arc smoothly drawn in the plane from $-\infty$ to $+\infty$, with finitely many self-intersections, divided into “virtual crossings” ⤫ and over- and under-crossings, ⤪ and ⤩, and regarded up to planar isotopy. A picture is worth more than

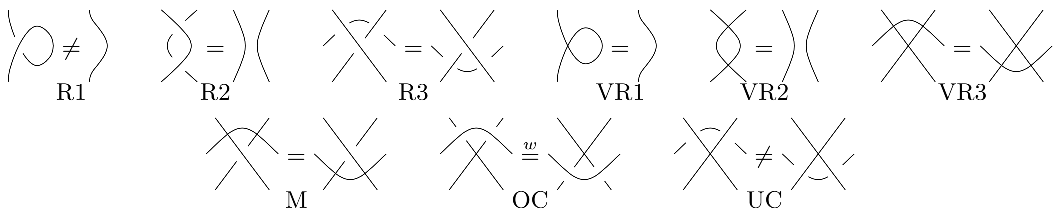



**Figure 6.** The relations defining v-knots and w-knots, along with two relations that are *not* imposed.

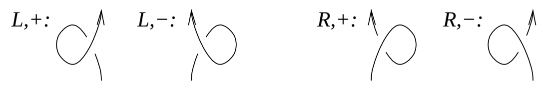



**Figure 7.** The positive and negative under-then-over kinks (left), and the positive and negative over-then-under kinks (right). In each pair the negative kink is the #-inverse of the positive kink.

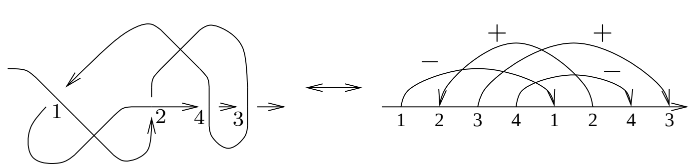
a more formal definition, and one appears in Figure 5. A "long v-knot" is an equivalence class of long v-knot diagrams, modulo the equivalence generated by the Reidemeister 2 and 3 moves (R2 and R3), the virtual Reidemeister 1 through 3 moves (VR1 through VR3), and by the mixed relations (M); all these are shown in Figure 6. Finally, "long w-knots" are obtained from long v-knots by also dividing by the Overcrossings Commute (OC) relation, also shown in Figure 6. Note that we never mod out by the Reidemeister 1 (R1) move or by the Undercrossings Commute relation (UC).



**Definition and Warning 3.2.** A "circular v-knot" is like a long v-knot, except parametrized by a circle rather than by a long line. Unlike the case of ordinary knots, circular v-knots are **not** equivalent to long v-knots. The same applies to w-knots.

**Definition and Warning 3.3.** Long v-knots form a monoid using the concatenation operation #. Unlike the case of ordinary knots, the resulting monoid is **not** Abelian. The same applies to w-knots.

*Remark* 3.4. A "Gauss diagram" is a straight "skeleton line" along with signed directed chords (signed "arrows") marked along it (more at [Ka2, GPV]). Gauss diagrams are in an obvious bijection with long v-knot diagrams; the skeleton line of a Gauss diagram corresponds to the parameter space of the v-knot, and the arrows correspond to the crossings, with each arrow heading from the upper strand to the lower strand, marked by the sign of the relevant crossing:




One may also describe the relations in Figure 6 as well as circular v-knots and other types of v-knots (as we will encounter later) in terms of Gauss diagrams with varying skeletons.

*Remark* 3.5. Since we do not mod out by R1, it is perhaps more appropriate to call our class of v-knots “framed long v-knots”, but since we care more about framed v-knots than about unframed ones, we reserve the unqualified name for the framed case, and when we do wish to mod out by R1 we will explicitly write “unframed long v-knots”. This said, note that the monoid of long v-knots is just a central extension by $\mathbb{Z}^2$ of the monoid of unframed long v-knots, and so studying the framed case is not very different from studying the unframed case. Indeed the four “kinks” of Figure 7 generate a central $\mathbb{Z}^2$ within long v-knots, and it is not hard to show that the sequence

$$1 \longrightarrow \mathbb{Z}^2 \longrightarrow \{\text{long v-knots}\} \longrightarrow \{\text{unframed long v-knots}\} \longrightarrow 1 \tag{17}$$

is split and exact. The same applies to w-knots.

*Exercise* 3.6. Show that a splitting of the sequence (17) is given by the “self-linking” invariants $sl = (sl_L, sl_R) : \{\text{long v-knots}\} \to \mathbb{Z}^2$ defined by

$$sl_L(K) := \sum_{\substack{\text{left crossings}\\ x \text{ in } D}} \operatorname{sign} x \qquad \text{and} \qquad sl_R(K) := \sum_{\substack{\text{right crossings}\\ x \text{ in } D}} \operatorname{sign} x,$$

where $D$ is a v-knot diagram, a “left crossing” (“right crossing”) is a crossing in which when traversing $D$, the lower strand is visited before (after) the upper strand, and the sign of a crossing $x$ is defined so as to agree with the signs in Figure 7.

*Remark* 3.7. w-Knots are strictly weaker than v-knots — a notorious example is the Kishino knot (e.g. [Dye]) which is non-trivial as a v-knot yet both it and its mirror are trivial as w-knots. Yet ordinary knots inject even into w-knots, as the Wirtinger presentation makes sense for w-knots and therefore w-knots have a “fundamental quandle” which generalizes the fundamental quandle of ordinary knots [Ka2], and as the fundamental quandle of ordinary knots separates ordinary knots [Joy].

Following Satoh [Sa] and using the same constructions as in Section 2.2.2, we can map w-knots to (“long”) ribbon tubes in $\mathbb{R}^4$ (and the relations in Figure 6 still hold). It is natural to expect that this map is an isomorphism; in other words, that the theory of w-knots provides a “Reidemeister framework” for long ribbon tubes in $\mathbb{R}^4$ — that every long ribbon tube is in the image of this map and that two “w-knot diagrams” represent the same long ribbon tube iff they differ by a sequence of moves as in Figure 6. This remains unproven, though very similar theorem about ribbon 2-spheres in $\mathbb{R}^4$ was proven by Winter [Win]. It is likely that Winter’s techniques are sufficient to give a Reidemeister framework for w-knots and for all other classes of w-knotted objects studied elsewhere in this paper.

3.2. **Finite Type Invariants of v-Knots and w-Knots.** Much as for v-braids and w-braids (Section 2.3) and much as for ordinary knots (e.g. [BN1]) we define finite type invariants for v-knots and for w-knots using an alternation scheme with $\semivirtualover \to \overcrossing - \virtualcrossing$ and $\semivirtualunder \to \undercrossing - \virtualcrossing$. That is, we extend any Abelian-group-valued invariant of v- or w-knots to v- or w-knots also containing “semi-virtual crossings” like $\semivirtualover$ and $\semivirtualunder$ using the above assignments, and we declare an invariant “of type $m$” if it vanishes on v- or w-knots with more than $m$ semi-virtuals. As for v- and w-braids and as for ordinary knots, such invariants have an “$m$th derivative”, their “weight system”, which is a linear functional on the space $\mathcal{A}^v(\uparrow)$ (for v-knots) or $\mathcal{A}^w(\uparrow)$ (for w-knots). We turn to the definition of these spaces:

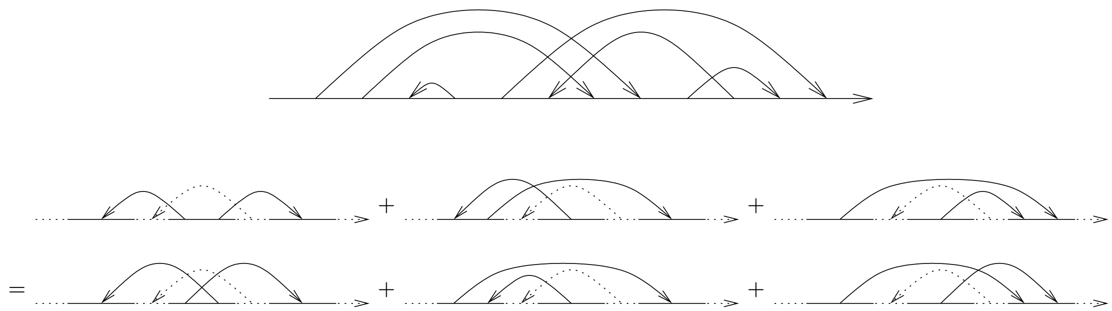


**Figure 8.** An arrow diagram of degree 6 and a 6T relation.

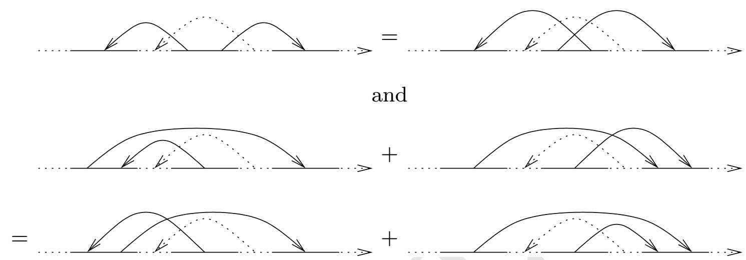



**Figure 9.** The TC and the $\overrightarrow{4T}$ relations for knots.



**Definition 3.8.** An "arrow diagram" is a chord diagram along a long line (called "the skeleton"), in which the chords are oriented (hence "arrows"). An example is in Figure 8. Let $\mathcal{D}^v(\uparrow)$ be the space of formal linear combinations of "arrow diagrams". Let $\mathcal{A}^v(\uparrow)$ be $\mathcal{D}^v(\uparrow)$ modulo all "6T relations", where a 6T relation is any (signed) combination of arrow diagrams obtained from the diagrams in Figure 3 by placing the 3 vertical strands there along a long line in any order, and possibly adding some further arrows in between. An example is in Figure 8. Let $\mathcal{A}^w(\uparrow)$ be the further quotient of $\mathcal{A}^v(\uparrow)$ by the "Tails Commute" (TC) relation, first displayed in Figure 4 and reproduced for the case of a long-line skeleton in Figure 9. Alternatively, $\mathcal{A}^w(\uparrow)$ is the space of formal linear combinations of arrow diagrams modulo TC and $\overrightarrow{4T}$ relations, displayed in Figures 4 and 9. Finally, grade $\mathcal{D}^v(\uparrow)$, $\mathcal{A}^v(\uparrow)$, and $\mathcal{A}^w(\uparrow)$ by declaring that the degree of an arrow diagram is the number of arrows in it.

As an example, the spaces $\mathcal{A}^v(\uparrow)$ and $\mathcal{A}^w(\uparrow)$ restricted to degrees up to 2 are studied in detail in Section 7.5.

In the same manner as in the theory of finite type invariants of ordinary knots (see especially [BN1, Section 3], the spaces $\mathcal{A}^{v,w}(\uparrow)$ carry much algebraic structure. The obvious juxtaposition product makes them into graded algebras. The product of two finite type invariants is a finite type invariant (whose type is the sum of the types of the factors); this induces a product for weight systems, and therefore a co-product $\Delta$ on arrow diagrams. In brief (and much the same as in the usual finite type story), the co-product $\Delta D$ of an arrow diagram $D$ is the sum of all ways of dividing the arrows in $D$ between a "left co-factor" and a "right co-factor". In summary,

**Proposition 3.9.** *$\mathcal{A}^v(\uparrow)$ and $\mathcal{A}^w(\uparrow)$ are co-commutative graded bi-algebras.*

By the Milnor-Moore theorem [MM] we find that $\mathcal{A}^v(\uparrow)$ and $\mathcal{A}^w(\uparrow)$ are the universal enveloping algebras of their Lie algebras of primitive elements. Denote these (graded) Lie algebras by $\mathcal{P}^v(\uparrow)$ and $\mathcal{P}^w(\uparrow)$.

When I grow up I'd like to understand $\mathcal{A}^v(\uparrow)$. At the moment I know only very little about it beyond the generalities of Proposition 3.9: in the next section some dimensions of low degree parts of $\mathcal{A}^v(\uparrow)$ are displayed, and given a finite dimensional Lie bialgebra and a finite dimensional representation thereof, we know how to construct linear functionals on $\mathcal{A}^v(\uparrow)$ (one in each degree) [Hav, Leu]. But we don't even know which degree $m$ linear functionals on $\mathcal{A}^v(\uparrow)$ are the weight systems of degree $m$ invariants of v-knots (that is, we have not solved the "Fundamental Problem" [BS] for v-knots).

As we shall see below, the situation is much brighter for $\mathcal{A}^w(\uparrow)$.

3.3. **Some Dimensions.** The table below lists what we could find about $\mathcal{A}^v$ and $\mathcal{A}^w$ by crude brute force computations in low degrees. We list degrees 0 through 7. The spaces we study are $\mathcal{A}^-(\uparrow)$, $\mathcal{A}^{r-}(\uparrow)$ which is $\mathcal{A}^-(\uparrow)$ moded out by "short" arrows [17], $\mathcal{P}^-(\uparrow)$ which is the space of primitives in $\mathcal{A}^-(\uparrow)$, and $\mathcal{A}^-(\bigcirc)$ and $\mathcal{A}^{r-}(\bigcirc)$, which are the same as $\mathcal{A}^-(\uparrow)$ and $\mathcal{A}^{r-}(\uparrow)$ except with closed knots (knots with a circle skeleton) replacing long knots. Each of these spaces we study in three variants: the "v" and the "w" variants, as well as the usual knots "u" variant which is here just for comparison. We also include a row "$\dim \mathcal{G}_m\mathcal{L}ie^-(\uparrow)$" for the dimensions of "Lie-algebraic weight systems". Those are not explained here; for details, see [BN1, Hav, Leu].

| $m$ | | 0 (See Section 7.5) | 1 (See Section 7.5) | 2 (See Section 7.5) | 3 | 4 | 5 | 6 | 7 | Comments |
|---|---|---|---|---|---|---|---|---|---|---|
| $\dim \mathcal{G}_m\mathcal{A}^-(\uparrow)$ | $u \mid v$ | 1 \| 1 | 1 \| 2 | 2 \| 7 | 3 \| 27 | 6 \| 139 | 10 \|? | 19 \|? | 33 \|? | 1 \| 2 |
| | $w$ | 1 | 2 | 4 | 7 | 12 | 19 | 30 | 45 | 3, 4 |
| $\dim \mathcal{G}_m\mathcal{L}ie^-(\uparrow)$ | $u \mid v$ | 1 \| 1 | 1 \| 2 | 2 \| 7 | 3 \| 27 | 6 \| $\geq$128 | 10 \|? | 19 \|? | 33 \|? | 1 \| 5 |
| | $w$ | 1 | 2 | 4 | 7 | 12 | 19 | 30 | 45 | 6 |
| $\dim \mathcal{G}_m\mathcal{A}^{r-}(\uparrow)$ | $u \mid v$ | 1 \| 1 | 0 \| 0 | 1 \| 2 | 1 \| 7 | 3 \| 42 | 4 \|? | 9 \|? | 14 \|? | 1 \| 7 |
| | $w$ | 1 | 0 | 1 | 1 | 2 | 2 | 4 | 4 | 3, 8 |
| $\dim \mathcal{G}_m\mathcal{P}^-(\uparrow)$ | $u \mid v$ | 0 \| 0 | 1 \| 2 | 1 \| 4 | 1 \| 15 | 2 \| 82 | 3 \|? | 5 \|? | 8 \|? | 1 \| 9 |
| | $w$ | 0 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 3 |
| $\dim \mathcal{G}_m\mathcal{A}^-(\bigcirc)$ | $u \mid v$ | 1 \| 1 | 1 \| 1 | 2 \| 2 | 3 \| 5 | 6 \| 19 | 10 \| 77 | 19 \|? | 33 \|? | 1 \| 10 |
| | $w$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 3 |
| $\dim \mathcal{G}_m\mathcal{A}^{r-}(\bigcirc)$ | $u \mid v$ | 1 \| 1 | 0 \| 0 | 1 \| 0 | 1 \| 1 | 3 \| 4 | 4 \| 17 | 9 \|? | 14 \|? | 1 \| 10 |
| | $w$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |



*Comments* 3.10.
(1) Much more is known computationally on the u-knots case. See especially [BN1, BN4, Kn, AS].
(2) These dimensions were computed by Louis Leung and myself using a program available at [BND, "Dimensions"]. Degree 5 is probably also within reach but we have not attempted to optimize our program.
(3) As we shall see in Section 3.5, the spaces associated with w-knots are understood to all degrees.
(4) To degree 4, these numbers were also verified by [BND, "Dimensions"].

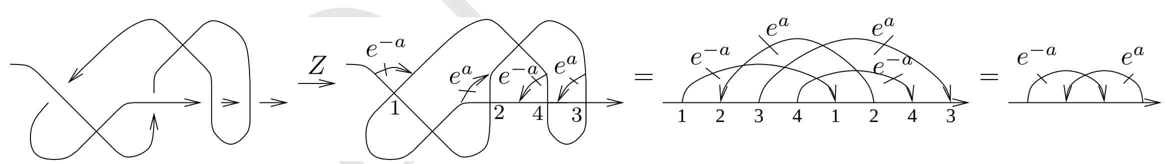
[17] That is, $\mathcal{A}^{r-}(\uparrow)$ is $\mathcal{A}^-(\uparrow)$ modulo "Framing Independence" (FI) relations [BN1]. It is the space related to finite type invariants of unframed knots, on which the first Reidemeister move is also imposed) in the same way as $\mathcal{A}^-(\uparrow)$ is related to framed knots.

(5) These dimensions were computed by Louis Leung and myself using a program available at [BND, "Arrow Diagrams and $gl(N)$"]. Note the match with the row above, and note that the degree 4 computation is still on going.
(6) See Section 3.6.
(7) These numbers were computed by [BND, "Dimensions"]. Contrary to the $\mathcal{A}^u$ case, $\mathcal{A}^{rv}$ is *not* the quotient of $\mathcal{A}^v$ by the ideal generated by degree 1 elements, and therefore the dimensions of the graded pieces of these two spaces cannot be deduced from each other using the Milnor-Moore theorem.
(8) The next few numbers in this sequence are 7,8,12,14,21.
(9) These dimensions were deduced from the dimensions of $\mathcal{G}_m\mathcal{A}^v(\uparrow)$ using the Milnor-Moore theorem.
(10) Computed by [BND, "Dimensions"]. Contrary to the $\mathcal{A}^u$ case, $\mathcal{A}^v(\bigcirc)$ and $\mathcal{A}^{rv}(\bigcirc)$ are *not* isomorphic to $\mathcal{A}^v(\uparrow)$ and $\mathcal{A}^{rv}(\uparrow)$ and separate computations are required.

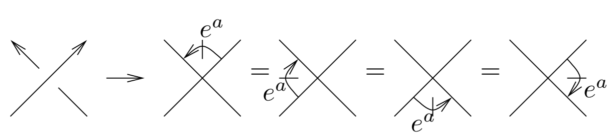
3.4. **Expansions for w-Knots.** The notion of "an expansion" (or "a universal finite type invariant") for w-knots (or v-knots) is defined in complete analogy with the parallel notion for ordinary knots (e.g. [BN1]), except replacing double points (⤫) with semi-virtual crossings (⤫ and ⤫) and replacing chord diagrams by arrow diagrams. Alternatively, it is the same as an expansion for w-braids (Definition 2.11), with the obvious replacement of w-braids by w-knots. Just as in the cases of ordinary knots and/or w-braids, the existence of an expansion $Z : \{\text{w-knots}\} \to \mathcal{A}^w(\uparrow)$ is equivalent to the statement "every weight system integrates", i.e., "every degree $m$ linear functional on $\mathcal{A}^w(\uparrow)$ is the $m$th derivative of a type $m$ invariant of long w-knots".

**Theorem 3.11.** *There exists an expansion* $Z : \{\textit{w-knots}\} \to \mathcal{A}^w(\uparrow)$.

*Proof.* It is best to define $Z$ by an example, and it is best to display the example only as a picture:

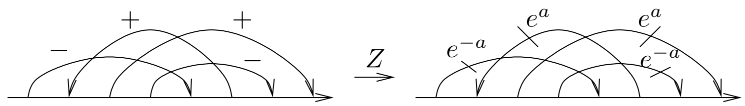



It is clear how to define $Z(K)$ in the general case — for every crossing in $K$ place an exponential reservoir of arrows (compare with (15)) next to that crossing, with the arrows heading from the upper strand to the lower strand, taking positive reservoirs ($e^a$, with $a$ symbolizing the arrow) for positive crossings and negative reservoirs ($e^{-a}$) for negative crossings, and then tug the skeleton until it looks like a straight line. Note that the Tails Commute relation in $\mathcal{A}^w$ is used to show that all reasonable ways of placing an arrow reservoir at a crossing (with its heading and sign fixed) are equivalent:




The same proof that shows the invariance of $Z$ in the braids case (Theorem 2.15) works here as well, and the same argument as in the braids case shows the universality of $Z$. □

*Remark* 3.12. Using the language of Gauss diagrams (Remark 3.4) the definition of $Z$ is even simpler. Simply map every positive arrow in a Gauss diagram to a positive ($e^a$) reservoir, and every negative one to a negative ($e^{-a}$) reservoir:




An expansion (a universal finite type invariant) is as interesting as its target space, for it is just a tool that takes linear functionals on the target space to finite type invariants on its domain space. The purpose of the next section is to find out how interesting is our present target space, $\mathcal{A}^w(\uparrow)$.

3.5. **Jacobi Diagrams, Trees and Wheels.** In studying $\mathcal{A}^w(\uparrow)$ we again follow the model set by ordinary knots. Compare the following definitions and theorem with [BN1, Section 3].

**Definition 3.13.** A "w-Jacobi diagram on a long line skeleton"[18] is a connected graph made of the following ingredients:

- A "long" oriented "skeleton" line. We usually draw the skeleton line a bit thicker for emphasis.
- Other directed edges, sometimes called "arrows".
- Trivalent "skeleton vertices" in which an arrow starts or ends on the skeleton line.
- Trivalent "internal vertices" in which two arrows end and one arrow begins. The internal vertices are "oriented" — of the two arrows that end in an internal vertices, one is marked as "left" and the other is marked as "right". In reality when a diagram is drawn in the plane, we almost never mark "left" and "right", but instead assume the "left" and "right" inherited from the plane, as seen from the outgoing arrow from the given vertex.

Note that we allow multiple arrows connecting the same two vertices (though at most two are possible, given connectedness and trivalence) and we allow "bubbles" — arrows that begin and end in the same vertex. Note that for the purpose of determining equality of diagrams the skeleton line is distinguished. The "degree" of a w-Jacobi diagram is half the number of trivalent vertices in it, including both internal and skeleton vertices. An example of a w-Jacobi diagram is in Figure 10.

**Definition 3.14.** Lat $\mathcal{D}^{wt}(\uparrow)$ be the graded vector space of formal linear combinations of w-Jacobi diagrams on a long line skeleton, and let $\mathcal{A}^{wt}(\uparrow)$ be $\mathcal{D}^{wt}(\uparrow)$ modulo the "$\overrightarrow{STU}_{1,2}$" and TC relations of Figure 11. Note that that each diagram appearing in each $\overrightarrow{STU}$ relation has a "central edge" $e$ which can serve as an "identifying name" for that $\overrightarrow{STU}$. Thus given a diagram $D$ with a marked edge $e$ which is either on the skeleton or which contacts the skeleton, there is an unambiguous $\overrightarrow{STU}$ relation "around" or "along" the edge $e$.

I like to call the following theorem "the bracket-rise theorem", for it justifies the introduction of internal vertices, and as should be clear from the $\overrightarrow{STU}$ relations and as will become even clearer in Section 3.6, internal vertices can be viewed as "brackets". Two other bracket-rise theorems are Theorem 6 of [BN1] and Ohtsuki's theorem, Theorem 4.9 of [Po].

[18]What a mouthful! We usually short this to "w-Jacobi diagram", or sometimes "arrow diagram" or just "diagram".

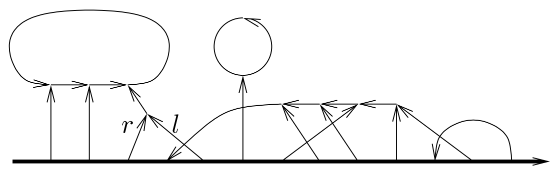



**Figure 10.** A w-Jacobi diagram on a long line skeleton of degree 11. It has a skeleton line at the bottom, 13 vertices along the skeleton (of which 2 are incoming and 11 are outgoing), 9 internal vertices (with only one explicitly marked with "left" ($l$) and "right" ($r$)) and one bubble. The four quadrivalent vertices that seem to appear in the diagram are just projection artifacts and graph-theoretically, they don't exist.

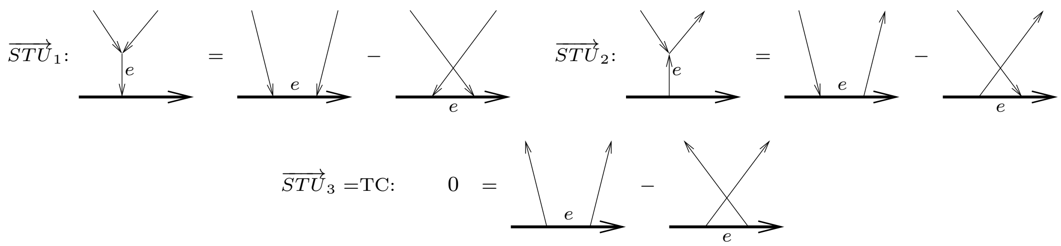



**Figure 11.** The $\overrightarrow{STU}_{1,2}$ and TC relations with their "central edges" marked $e$.

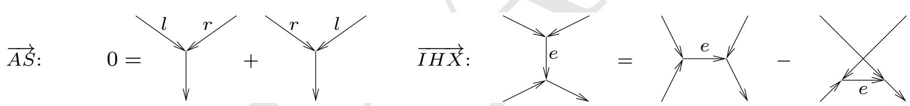
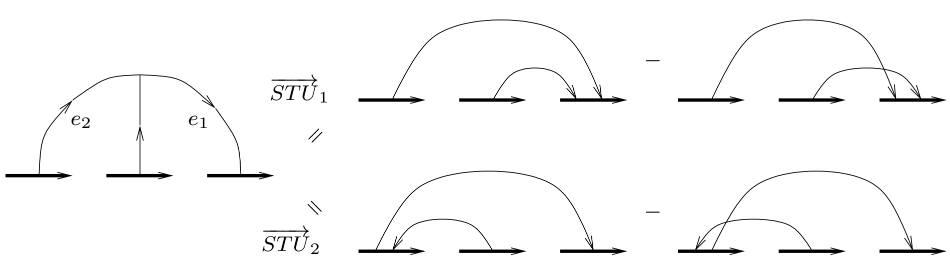



**Figure 12.** The $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations.

**Theorem 3.15** (bracket-rise)**.** *The obvious inclusion $\iota : \mathcal{D}^v(\uparrow) \to \mathcal{D}^{wt}(\uparrow)$ of arrow diagrams (Definition 3.8) into w-Jacobi diagrams descends to the quotient $\mathcal{A}^w(\uparrow)$ and induces an isomorphism $\bar\iota : \mathcal{A}^w(\uparrow) \xrightarrow{\sim} \mathcal{A}^{wt}(\uparrow)$. Furthermore, the $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations of Figure 12 hold in $\mathcal{A}^{wt}(\uparrow)$.*

*Proof.* The proof, joint with D. Thurston, is modeled after the proof of Theorem 6 of [BN1]. To show that $\iota$ descends to $\mathcal{A}^w(\uparrow)$ we just need to show that in $\mathcal{A}^{wt}(\uparrow)$, $\overrightarrow{4T}$ follows from $\overrightarrow{STU}_{1,2}$. Indeed, applying $\overrightarrow{STU}_1$ along the edge $e_1$ and $\overrightarrow{STU}_2$ along $e_2$ in the picture below, we get the two sides of $\overrightarrow{4T}$:



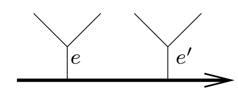

(18)

The fact that $\bar{\iota}$ is surjective is obvious; indeed, for diagrams in $\mathcal{A}^{wt}(\uparrow)$ that have no internal vertices there is nothing to show, for they are really in $\mathcal{A}^{w}(\uparrow)$. Further, by repeated use of $\overrightarrow{STU}_{1,2}$ relations, all internal vertices in any diagram in $\mathcal{A}^{wt}(\uparrow)$ can be removed (remember that the diagrams in $\mathcal{A}^{wt}(\uparrow)$ are always connected, and in particular, if they have an internal vertex they must have an internal vertex connected by an edge to the skeleton, and the latter vertex can be removed first).

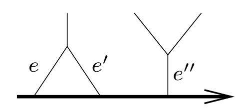
To complete the proof that $\bar{\iota}$ is an isomorphism it is enough to show that the "elimination of internal vertices" procedure of the last paragraph is well defined — that its output is independent of the order in which $\overrightarrow{STU}_{1,2}$ relations are applied in order to eliminate internal vertices. Indeed, this done, the elimination map would by definition satisfy the $\overrightarrow{STU}_{1,2}$ relations and thus descend to a well defined inverse for $\bar{\iota}$.

On diagrams with just one internal vertex, Equation (18) shows that all ways of eliminating that vertex are equivalent modulo $\overrightarrow{4T}$ relations, and hence the elimination map is well defined on such diagrams.

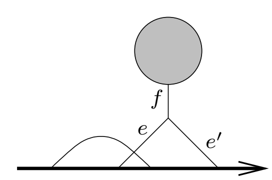
Now assume that we have shown that the elimination map is well defined on all diagrams with at most 7 internal vertices, and let $D$ be a diagram with 8 internal vertices[19]. Let $e$ and $e'$ be edges in $D$ that connect the skeleton of $D$ to an internal vertex. We need to show that any elimination process that begins with eliminating $e$ yields the same answer, modulo $\overrightarrow{4T}$, as any elimination process that begins with eliminating $e'$. There are several cases to consider.




**Case I.** $e$ and $e'$ connect the skeleton to *different* internal vertices of $D$. In this case, after eliminating $e$ we get a signed sum of two diagrams with exactly 7 internal vertices, and since the elimination process is well defined on such diagrams, we may as well continue by eliminating $e'$ in each of those, getting a signed sum of 4 diagrams with 6 internal vertices each. On the other hand, if we start by eliminating $e'$ we can continue by eliminating $e$, and we get the *same* signed sum of 4 diagrams with 6 internal vertices.




**Case II.** $e$ and $e'$ are connected to the same internal vertex $v$ of $D$, yet some other edge $e''$ exists in $D$ that connects the skeleton of $D$ to some other internal vertex $v'$ in $D$. In that case, use the previous case and the transitivity of equality: (elimination starting with $e$)=(elimination starting with $e''$)=(elimination starting with $e'$).




**Case III.** Case III is what remains if neither Case I nor Case II hold. In that case, $D$ must have a schematic form as on the right, with the "blob" not connected to the skeleton other than via $e$ or $e'$, yet further arrows may exist outside of the blob. Let $f$ denote the edge connecting the blob to $e$ and $e'$. The "two in one out" rule for vertices implies that any part of a diagram must have an excess of incoming edges over outgoing edges, equal to the total number of vertices in that diagram part. Applying this principle to the blob, we find that it must contain exactly one vertex, and that $f$ and therefore $e$ and $e'$ must all be oriented upwards.

[19]"7" here is a symbol denoting an arbitrary natural number and "8" denotes $7 + 1$.

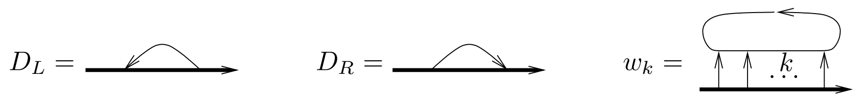
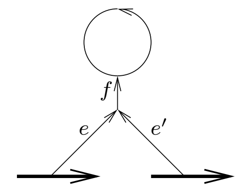



**Figure 13.** The left-arrow diagram $D_L$, the right-arrow diagram $D_R$ and the $k$-wheel $w_k$.




We leave it to the reader to verify that in this case the two ways of applying the elimination procedure, $e$ and then $f$ or $e'$ and then $f$, yield the same answer modulo $\overrightarrow{4T}$ (in fact, that answer is 0).

We also leave it to the reader to verify that $\overrightarrow{STU}_1$ implies $\overrightarrow{AS}$ and $\overrightarrow{IHX}$. Algebraically, these are restatements of the anti-symmetry of the bracket and of Jacobi's identity: if $[x,y] := xy - yx$, then $0 = [x,y] + [y,x]$ and $[x,[y,z]] = [[x,y],z] - [[x,z],y]$. □

Note that $\mathcal{A}^{wt}(\uparrow)$ inherits algebraic structure from $\mathcal{A}^w(\uparrow)$: it is an algebra by concatenation of diagrams, and a co-algebra with $\Delta(D)$, for $D \in \mathcal{D}^{wt}(\uparrow)$, being the sum of all ways of dividing $D$ between a "left co-factor" and a "right co-factor" so that connected components of $D-S$ are kept intact, where $S$ is the skeleton line of $D$ (compare with [BN1, Definition 3.7]).

As $\mathcal{A}^w(\uparrow)$ and $\mathcal{A}^{wt}(\uparrow)$ are canonically isomorphic, from this point on we will not keep the distinction between the two spaces.

**Theorem 3.16.** *The bi-algebra $\mathcal{A}^w(\uparrow)$ is the bi-algebra of polynomials in the diagrams $D_L$, $D_R$ and $w_k$ (for $k \geq 1$) shown in Figure 13, where* $\deg D_L = \deg D_R = 1$ *and* $\deg w_k = k$, *subject to the one relation $w_1 = D_L - D_R$. Thus $\mathcal{A}^w(\uparrow)$ has two generators in degree 1 and one generator in every degree greater than 1, as stated in Section 3.3.*

*Proof.* (sketch). Readers familiar with the diagrammatic PBW theorem [BN1, Theorem 8] will note that it has an obvious analogue for the $\mathcal{A}^w(\uparrow)$ case, and that the proof in [BN1] carries through almost verbatim. Namely, the space $\mathcal{A}^w(\uparrow)$ is isomorphic to a space $\mathcal{B}^w(\star)$ of "unitrivalent diagrams" with symmetrized univalent ends modulo $\overrightarrow{AS}$ and $\overrightarrow{IHX}$. Given the "two in one out" rule for arrow diagrams in $\mathcal{A}^w(\uparrow)$ (and hence in $\mathcal{B}^w(\star)$) the connected components of diagrams in $\mathcal{B}^w(\star)$ can only be trees or wheels. Trees vanish if they have more than one leaf, as their leafs are symmetric while their internal vertices are anti-symmetric, so $\mathcal{B}^w(\star)$ is generated by wheels (which become the $w_k$'s in $\mathcal{A}^w(\uparrow)$) and by the one-leaf-one-root tree, which is simply a single arrow, and which becomes the average of $D_L$ and $D_R$. The relation $w_1 = D_L - D_R$ is then easily verified using $\overrightarrow{STU}_2$.

One may also argue directly, without using sophisticated tools. In short, let $D$ be a diagram in $\mathcal{A}^w(\uparrow)$ and $S$ is its skeleton. Then $D - S$ may have several connected components, whose "legs" are intermingled along $S$. Using the $\overrightarrow{STU}$ relations these legs can be sorted (at a cost of diagrams with fewer connected components, which could have been treated earlier in an inductive proof). At the end of the sorting procedure one can see that the only diagrams that remain are our declared generators. It remains to show that our generators are linearly independent (apart for the relation $w_1 = D_L - D_R$). For the generators in degree 1, simply write everything out explicitly in the spirit of Section 7.5.2. In higher degrees there is only one primitive diagram in each degree, so it is enough to show that $w_k \neq 0$ for every $k$. This can be done "by hand", but it is more easily done using Lie algebraic tools. See Section 3.6. □

*Exercise* 3.17. Show that the bi-algebra $\mathcal{A}^{rw}(\uparrow)$ (see Section 3.3) is the bi-algebra of polynomials in the wheel diagrams $w_k$ $(k \geq 2)$.

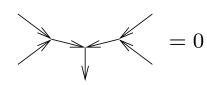
**Theorem 3.18.** *In $\mathcal{A}^w(\bigcirc)$ all wheels vanish and hence the bi-algebra $\mathcal{A}^w(\bigcirc)$ is the bi-algebra of polynomials in a single variable $D_L = D_R$.*

*Proof.* This is Lemma 2.7 of [Na]. In short, a wheel in $\mathcal{A}^w(\bigcirc)$ can be reduced using $\overrightarrow{STU}_2$ to a difference of trees. On of these trees has two adjoining leafs and hence is 0 by TC and $\overrightarrow{AS}$. In the other two of the leafs can be commuted "around the circle" using TC until they are adjoining and hence vanish by TC and $\overrightarrow{AS}$. A picture is worth a thousand words, but sometimes it takes up more space. □

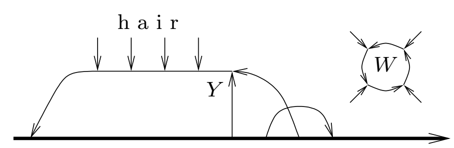
*Exercise* 3.19. Show that $\mathcal{A}^{rw}(\bigcirc)$ vanishes except in degree 0.

The following two exercises may help the reader to develop a better "feel" for $\mathcal{A}^w(\uparrow)$ and will be needed, within the discussion of the Alexander polynomial (especially within Definition 3.32).

*Exercise* 3.20. Show that the "Commutators Commute" (CC) relation, shown on the right, holds in $\mathcal{A}^w(\uparrow)$. (Interpreted in Lie algebras as in the next section, this relation becomes $[[x,y],[z,w]] = 0$, and hence the name "Commutators Commute"). Note that the proof of CC depends on the skeleton having a single component; later, when we will work with $\mathcal{A}^w$-spaces with more complicated skeleta, the CC relation will not hold.




*Exercise* 3.21. Show that "detached wheels" and "hairy $Y$'s" make sense in $\mathcal{A}^w(\uparrow)$. As on the right, a detached wheel is a wheel with a number of spokes, and a hairy $Y$ is a combinatorial $Y$ shape with further "hair" on its trunk (its outgoing arrow). It is specified where the trunk and the leafs of the $Y$ connect to the skeleton, but it is not specified where the spokes of the wheel and where the hair on the $Y$ connect to the skeleton. The content of the exercise is to show that modulo the relations of $\mathcal{A}^w(\uparrow)$, it is not necessary to specify this further information: all ways of connecting the spokes and the hair to the skeleton are equivalent. Like the previous exercise, this result depends on the skeleton having a single component.




*Remark* 3.22. On some level, the results of this section remain incomplete. In the case of classical knots and classical chord diagrams, Jacobi diagrams have a topological interpretation using the Goussarov-Habiro calculus of claspers [Gou2, Hab]. In the w case such interpretation is still missing, though it is possible that many of the necessary hints are present in [HKS, HS].

3.6. **The Relation with Lie Algebras.** The theory of finite type invariants of knots is related to the theory of metrized Lie algebras via the space $\mathcal{A}$ of chord diagrams, as explained in [BN1, Theorem 4, Exercise 5.1]. In a similar manner the theory of finite type invariants of w-knots is related to inhomogenized arbitrary finite-dimensional Lie algebras (or equivalently, to doubles of co-commutative Lie bialgebra) via the space $\mathcal{A}^w(\uparrow)$ of arrow diagrams.

3.6.1. *Preliminaries.* Given a finite dimensional Lie algebra $\mathfrak{g}$ let $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$ be the semi-direct product of the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ with $\mathfrak{g}$, with $\mathfrak{g}^*$ taken as an Abelian algebra and with $\mathfrak{g}$ acting on $\mathfrak{g}^*$ by the usual coadjoint action. In formulas,

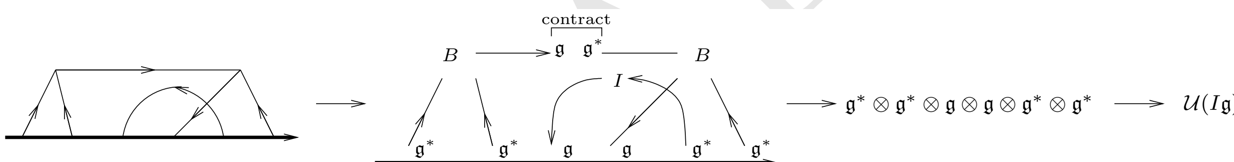
$$I\mathfrak{g} = \{(\varphi, x) : \varphi \in \mathfrak{g}^a st,\ x \in \mathfrak{g}\},$$

$$[(\varphi_1, x_1), (\varphi_2, x_2)] = (x_1\varphi_2 - x_2\varphi_1, [x_1, x_2]).$$

In the case where $\mathfrak{g}$ is the algebra $so(3)$ of infinitesimal symmetries of $\mathbb{R}^3$, its dual $\mathfrak{g}^*$ is itself $\mathbb{R}^3$ with the usual action of $so(3)$ on it, and $I\mathfrak{g}$ is the algebra $\mathbb{R}^3 \rtimes so(3)$ of infinitesimal affine isometries of $\mathbb{R}^3$. This is the Lie algebra of the Euclidean group of isometries of $\mathbb{R}^3$, which is often denoted $ISO(3)$. This explains our choice of the name $I\mathfrak{g}$.

Note that if $\mathfrak{g}$ is a co-commutative Lie bialgebra then $I\mathfrak{g}$ is the "double" of $\mathfrak{g}$ [Dr1]. This is a significant observation, for it is a part of the relationship between this paper and the Etingof-Kazhdan theory of quantization of Lie bialgebras [EK]. Yet we will make no explicit use of this observation below.

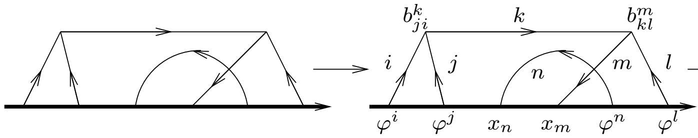
3.6.2. *The Construction.* Fixing a finite dimensional Lie algebra $\mathfrak{g}$ we construct a map $\mathcal{T}^w_\mathfrak{g} : \mathcal{A}^w \to \mathcal{U}(I\mathfrak{g})$ which assigns to every arrow diagram $D$ an element of the universal enveloping algebra $U(I\mathfrak{g})$. As is often the case in our subject, a picture of a typical example is worth more than a formal definition:




In short, we break up the diagram $D$ into its constituent pieces and assign a copy of the structure constants tensor $B \in \mathfrak{g}^* \otimes \mathfrak{g}^a st \otimes \mathfrak{g}$ to each internal vertex $v$ of $D$ (keeping an association between the tensor factors in $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ and the edges emanating from $v$, as dictated by the orientations of the edges and of the vertex $v$ itself). We assign the identity tensor in $\mathfrak{g}^* \otimes \mathfrak{g}$ to every arrow in $D$ that is not connected to an internal vertex, and contract any pair of factors connected by a fully internal arrow. The remaining tensor factors ($\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}^* \otimes \mathfrak{g}^*$ in our examples) are all along the skeleton and can thus be ordered by the skeleton. We then multiply these factors to get an output $\mathcal{T}^w_\mathfrak{g}(D)$ in $\mathcal{U}(I\mathfrak{g})$.

It is also useful to restate this construction given a choice of a basis. Let $(x_j)$ be a basis of $\mathfrak{g}$ and let $\varphi^i$ be the dual basis of $\mathfrak{g}^*$, so that $\varphi^i(x_j) = \delta^i_j$, and let $b^k_{ij}$ denote the structure constants of $\mathfrak{g}$ in the chosen basis: $[x_i, x_j] = \sum b^k_{ij} x_k$. Mark every arrow in $D$ with lower case Latin letter from within $\{i, j, k, \dots\}$[20]. Form a product $P_D$ by taking one $b^\gamma_{\alpha\beta}$ factor for each internal vertex $v$ of $D$ using the letters marking the edges around $v$ for $\alpha$, $\beta$ and $\gamma$ and by taking one $x_\alpha$ or $\varphi^\beta$ factor for each skeleton vertex of $D$, taken in the order that they appear along the skeleton, with the indices $\alpha$ and $\beta$ dictated by the edge markings and with the choice between factors in $\mathfrak{g}$ and factors in $\mathfrak{g}^*$ dictated by the orientations of the edges. Finally let $\mathcal{T}^w_\mathfrak{g}(D)$ be the sum of $P_D$ over the indices $i, j, k, \dots$ running from 1 to $\dim \mathfrak{g}$:

[20]The supply of these can be made inexhaustible by the addition of numerical subscripts.




$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim \mathfrak{g}} b^k_{ij} b^m_{kl} \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g}) \qquad (19)$$

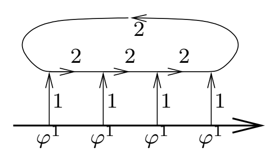
The following is easy to verify (compare with [BN1, Theorem 4, Exercise 5.1]):

**Proposition 3.23.** *The above two definitions of $T^w_{\mathfrak{g}}$ agree, are independent of the choices made within them, and respect all the relations defining $\mathcal{A}^w$.* □

While we do not provide a proof of this proposition here, it is worthwhile to state the correspondence between the relations defining $\mathcal{A}^w$ and the Lie algebraic information in $\mathcal{U}(I\mathfrak{g})$: $\overrightarrow{AS}$ is the antisymmetry of the bracket of $\mathfrak{g}$, $\overrightarrow{IHX}$ is the Jacobi identity of $\mathfrak{g}$, $\overrightarrow{STU}_1$ and $\overrightarrow{STU}_2$ are the relations $[x_i, x_j] = x_i x_j - x_j x_i$ and $[\varphi^i, x_j] = \varphi^i x_j - x_j \varphi^i$ in $\mathcal{U}(I\mathfrak{g})$, $TC$ is the fact that $\mathfrak{g}^\star$ is taken as an Abelian algebra, and $\overrightarrow{4T}$ is the fact that the identity tensor in $\mathfrak{g}^* \otimes \mathfrak{g}$ is $\mathfrak{g}$-invariant.

3.6.3. *Example: The 2 Dimensional Non-Abelian Lie Algebra.* Let $\mathfrak{g}$ be the Lie algebra with two generators $x_{1,2}$ satisfying $[x_1, x_2] = x_2$, so that the only non-vanishing structure constants $b^k_{ij}$ of $\mathfrak{g}$ are $b^2_{12} = -b^2_{21} = 1$. Let $\varphi^i \in \mathfrak{g}^*$ be the dual basis of $x_i$; by an easy calculation, we find that in $I\mathfrak{g}$ the element $\varphi^1$ is central, while $[x_1, \varphi^2] = -\varphi^2$ and $[x_2, \varphi^2] = \varphi^1$. We calculate $\mathcal{T}^w_{\mathfrak{g}}(D_L)$, $\mathcal{T}^w_{\mathfrak{g}}(D_R)$ and $\mathcal{T}^w_{\mathfrak{g}}(w_k)$ using the "in basis" technique of Equation (19). The outputs of these calculations lie in $\mathcal{U}(I\mathfrak{g})$; we display these results in a PBW basis in which the elements of $\mathfrak{g}^*$ precede the elements of $\mathfrak{g}$:

$$\begin{aligned} \mathcal{T}^w_{\mathfrak{g}}(D_L) &= x_1\varphi^1 + x_2\varphi^2 = \varphi^1 x_1 + \varphi^2 x_2 + [x_2, \varphi^2] = \varphi^1 x_1 + \varphi^2 x_2 + \varphi_1, \\ \mathcal{T}^w_{\mathfrak{g}}(D_R) &= \varphi^1 x_1 + \varphi^2 x_2, \\ \mathcal{T}^w_{\mathfrak{g}}(w_k) &= (\varphi^1)^k. \end{aligned} \qquad (20)$$




For the last assertion above, note that all non-vanishing structure constants $b^k_{ij}$ in our case have $k = 2$, and therefore all indices corresponding to edges that exit an internal vertex must be set equal to 2. This forces the "hub" of $w_k$ to be marked 2 and therefore the legs to be marked 1, and therefore $w_k$ is mapped to $(\varphi^1)^k$.

Note that the calculations in (20) are consistent with the relation $D_L - D_R = w_1$ of Theorem 3.16 and that they show that other than that relation, the generators of $\mathcal{A}^w$ are linearly independent.

3.7. **The Alexander Polynomial.** Let $K$ be a long w-knot, let $Z(K)$ be the invariant of Theorem 3.11. Theorem 3.27 below asserts that apart from framing issues, $Z(K)$ contains precisely the same information as the Alexander polynomial $A(K)$ of $K$ (defined below). But we have to start with some definitions as well as with an embarrassing acknowledgment (Conjecture 3.26).

**Definition 3.24.** Enumerate the crossings of $K$ from 1 to $n$ in some arbitrary order. For $1 \leq j \leq n$, the "span" of crossing $\#i$ is the connected open interval along the line parametrizing

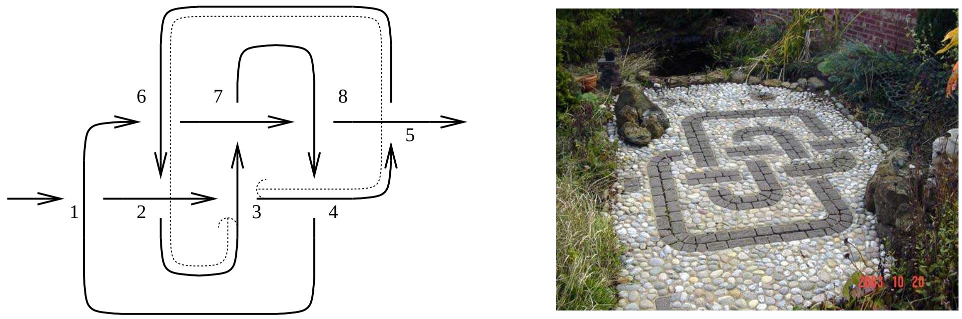



**Figure 14.** A long $8_{17}$, with the span of crossing $\#3$ marked. The projection is as in Brian Sanderson's garden. See [BND]/`SandersonsGarden.html`.

$K$ between the two times $K$ "visits" crossing $\#i$ (see Figure 14). Form a matrix $T = T(K)$ with $T_{ij}$ the indicator function of "the lower strand of crossing $\#j$ is within the span of crossing $\#i$" (so $T_{ij}$ is 1 if for a given $i, j$ the quoted statement is true, and 0 otherwise). Let $s_i$ be the sign of crossing $\#i$ ($(-,-,-,-,+,+,+,+)$ for Figure 14), let $d_i$ be $+1$ if $K$ visits the "over" strand of crossing $\#i$ before visiting the "under" strand of that crossing, and let $d_i = -1$ otherwise ($(-,+,-,+,-,+,-,+)$ for Figure 14). Let $S = S(K)$ be the diagonal matrix with $S_{ii} = s_i d_i$, and for an indeterminate $X$, let $X^{-S}$ denote the diagonal matrix with diagonal entries $X^{-s_i d_i}$. Finally, let $A(K)$ be the Laurent polynomial in $\mathbb{Z}[X, X^{-1}]$ given by

$$A(K)(X) := \det\left(I + T(I - X^{-S})\right). \tag{21}$$

*Example* 3.25. For the knot diagram in Figure 14,

$$T=\begin{pmatrix} 0&1&1&1&1&0&1&0\\ 0&0&1&0&1&0&0&0\\ 0&1&0&0&1&0&0&0\\ 0&1&0&0&1&0&1&0\\ 0&1&0&1&0&1&1&1\\ 0&1&0&1&0&0&1&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&1&0&1&0&0 \end{pmatrix},\quad S=\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix},\quad\text{and}\quad A=\begin{vmatrix} 1&1-X&1-X^{-1}&1-X&1-X&0&1-X&0\\ 0&1&1-X^{-1}&0&1-X&0&0&0\\ 0&1-X&1&0&1-X&0&0&0\\ 0&1-X&0&1&1-X&0&1-X&0\\ 0&1-X&0&1-X&1&1-X^{-1}&1-X&1-X^{-1}\\ 0&1-X&0&1-X&0&1&1-X&0\\ 0&0&0&1-X&0&1-X^{-1}&1&0\\ 0&0&0&1-X&0&1-X^{-1}&0&1 \end{vmatrix}.$$

The last determinant equals $-X^3 + 4X^2 - 8X + 11 - 8X^{-1} + 4X^{-2} - X^{-3}$, the Alexander polynomial of the knot $8_{17}$ (e.g. [Rol]).

**Conjecture 3.26.** *For any (classical) knot $K$, $A(K)$ is equal to the normalized Alexander polynomial* [Rol] *of $K$.*

The overall shape of the definition of $A(K)$, a determinant of a matrix constructed by reading out the crossings of $K$ in a certain manner, is very similar to several of the known definitions of the Alexander polynomial. The Mathematica notebook [BND, "wA"] verifies that Conjecture 3.26 for all prime knots with up to 11 crossings. Hence I have no doubt that Conjecture 3.26 is true. Yet I am embarrassed to acknowledge that so far I was not able to prove it by finding an appropriate Seifert surface for $K$ and using the linking matrix formula for the Alexander polynomial, or by finding an appropriate presentation for the fundamental group of the complement of $K$ and using the free differential (Fox) calculus formula for the Alexander polynomial[21].



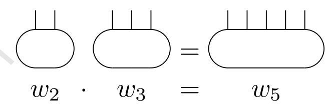
[21] In fact, Conjecture 3.26 probably follows from the work below relating $A(K)$ and $Z(K)$, from the known fact that the weight system of the Alexander polynomial is supported on wheels [Vai, Ch], and from some

The following theorem asserts that $Z(K)$ can be computed from $A(K)$ (22) and that modulo a certain additional relation and with the appropriate identifications in place, $Z(K)$ *is* $A(K)$ (23).

**Theorem 3.27.** *(Proof in Section 3.8). Let $x$ be an indeterminate, let $sl$ be as in Exercise 3.6, let $D_L$, $D_R$ and $w_k$ be as in Figure 13, and let $w : \mathbb{Q}[\![x]\!] \to \mathcal{A}^w$ be the linear map defined by $x^k \mapsto w_k$. Then for a w-knot $K$,*

$$Z(K) = \underbrace{\exp_{\mathcal{A}^w}\left(sl_L(K)D_L\right) \cdot \exp_{\mathcal{A}^w}\left(sl_R(K)D_R\right)}_{\text{minor part: self linking coded in arrows}} \cdot \underbrace{\exp_{\mathcal{A}^w}\left(-w\left(\log_{\mathbb{Q}[\![x]\!]} A(K)(e^x)\right)\right)}_{\text{main part: Alexander coded in wheels}}, \qquad (22)$$

*where the logarithm and inner exponentiation are computed by formal power series in $\mathbb{Q}[\![x]\!]$ and the outer exponentiations are likewise computed in $\mathcal{A}^w$.*




Let $\mathcal{A}^{\text{reduced}}$ be $\mathcal{A}^w$ modulo the additional relations $D_L = D_R = w_1 = 0$ and $w_k w_l = w_{k+l}$ for $k, l \neq 1$. The quotient $\mathcal{A}^{\text{reduced}}$ can be identified with vector space of (infinite) linear combinations of $w_k$'s (with $k \neq 1$). Identifying the $k$-wheel $w_k$ with $x^k$, we see that $\mathcal{A}^{\text{reduced}}$ is the space of power series in $x$ having no linear terms. Note by inspecting (21) that $A(K)(e^x)$ never has a term linear in $x$, and that modulo $w_k w_l = w_{k+l}$, the exponential and the logarithm in (22) cancel each other out. Hence within $\mathcal{A}^{\text{reduced}}$,

$$Z(K) = A^{-1}(K)(e^x). \qquad (23)$$

*Remark* 3.28. In [HKS] K. Habiro, T. Kanenobu, and A. Shima show that all coefficients of the Alexander polynomial are finite type invariants of w-knots, and in [HS] K. Habiro and A. Shima show that all finite type invariants of w-knots are polynomials in the coefficients of the Alexander polynomial. Thus Theorem 3.27 is merely an "explicit form" of these earlier results.

3.8. **Proof of Theorem 3.27.** We start with a sketch. The proof of Theorem 3.27 can be divided in three parts: differentiation, bulk management, and computation.

**Differentiation.** Both sides of our goal, Equation (22), are exponential in nature. When seeking to show an equality of exponentials it is often beneficial to compare their derivatives[22]. In our case the useful "derivatives" to use are the "Euler operator" $E$ ("multiply every term by its degree", an analogue of $f \mapsto xf'$, defined in Section 3.8.1), and the "normalized Euler operator" $Z \mapsto \tilde{E}Z := Z^{-1}EZ$, which is a variant of the logarithmic derivative $f \mapsto x(\log f)' = xf'/f$. Since $\tilde{E}$ is one to one (Section 3.8.1) and since we know how to apply $\tilde{E}$ to the right hand side of Equation (22) (Section 3.8.1), it is enough to show that with $B := T(\exp(-xS) - I)$ and suppressing the fixed w-knot $K$ from the notation,

$$EZ = Z \cdot \left(sl_L D_L + sl_R D_R - w\left[x \operatorname{tr}\left((I-B)^{-1} TS \exp(-xS)\right)\right]\right) \qquad \text{in } \mathcal{A}^w. \qquad (24)$$

**wClip 120418**

has further background on $E$, the differential of exp, and the BCH formula.

**Bulk Management.** Next we seek to understand the left hand side of (24). $Z$ is made up of "quantities in bulk": arrows that come in exponential "reservoirs". As it turns out, $EZ$ is made up of the same bulk quantities, but also allowing for a single non-bulk "red excitation"



minor further work to fix the normalizations. But this proof would be so indirect and ugly I would rather disown it.

[22]Thanks, Dylan.

(compare with $Ee^x = x \cdot e^x$; the "bulk" $e^x$ remains, and single "excited red" $x$ gets created). We wish manipulate and simplify that red excitation. This is best done by introducing a certain module, $IAM_K$, the "Infinitesimal Alexander Module" of $K$ (see Section 3.8.2). The elements of $IAM_K$ can be thought of as names for "bulk objects with a red excitation", and hence there is an "interpretation map" $\iota : IAM_K \to \mathcal{A}^w$, which maps every "name" into the object it represents. There are four special elements in $IAM_K$: an element $\lambda$, which is the name of $EZ$ (that is, $\iota(\lambda) = EZ$), two elements $\delta_L$ and $\delta_R$ which are the names of $D_L \cdot Z$ and $D_R \cdot Z$ (so $\iota(\delta_{L,R}) = D_{L,R} \cdot Z$), and an element $\omega_1$ which is the name of a "detached" 1-wheel that is appended to $Z$. The latter can take a coefficient which is a power of $x$, with $\iota(x^k\omega_1) = w(x^{k+1}) \cdot Z = (Z$ times a $(k+1)$-wheel). Thus it is enough to show that in $IAM_K$,

$$\lambda = sl_L\delta_L + sl_R\delta_R - \operatorname{tr}\left((I-B)^{-1}TSX^{-S}\right)\omega_1, \quad \text{with} \quad X = e^x. \tag{25}$$

Indeed, applying $\iota$ to both sides of the above equation, we get Equation (24) back again.
**Computation.** Last, we show in Section 3.8.3 that (25) holds true. This is a computation that happens entirely in $IAM_K$ and does not mention finite type invariants, expansions or arrow diagrams in any way.

3.8.1. *The Euler Operator.* Let $A$ be a completed graded algebra with unit, in which all degrees are $\geq 0$. Define a continuous linear operator $E : A \to A$ by setting $Ea = (\deg a)a$ for homogeneous $a \in A$. In the case $A = \mathbb{Q}[\![x]\!]$, we have $Ef = xf'$, the standard "Euler operator", and hence we adopt this name for $E$ in general.

We say that $Z \in A$ is a "perturbation of the identity" if its degree 0 piece is 1. Such a $Z$ is always invertible. For such a $Z$, set $\tilde{E}Z := Z^{-1} \cdot EZ$, and call the thus (partially) defined operator $\tilde{E} : A \to A$ the "normalized Euler operator". From this point on when we write $\tilde{E}Z$ for some $Z \in A$, we automatically assume that $Z$ is a perturbation of the identity or that it is trivial to show that $Z$ is a perturbation of the identity. Note that for $f \in \mathbb{Q}[\![x]\!]$, we have $\tilde{E}f = x(\log f)'$, so $\tilde{E}$ is a variant of the logarithmic derivative.

**Claim 3.29.** *$\tilde{E}$ is one to one.*

*Proof.* Assume $Z_1 \neq Z_2$ and let $d$ be the smallest degree in which they differ. Then $d > 0$ and in degree $d$ the difference $\tilde{E}Z_1 - \tilde{E}Z_2$ is $d$ times the difference $Z_1 - Z_2$, and hence $\tilde{E}Z_1 \neq \tilde{E}Z_2$. □

Thus in order to prove our goal, Equation (22), it is enough to compute $\tilde{E}$ of both sides and to show the equality then. We start with the right hand side of (22); but first, we need some simple properties of $E$ and $\tilde{E}$. The proofs of these properties are routine and hence they are omitted.

**Proposition 3.30.** *The following hold true:*

1. *$E$ is a derivation: $E(fg) = (Ef)g + f(Eg)$.*
2. *If $Z_1$ commutes with $Z_2$, then $\tilde{E}(Z_1Z_2) = \tilde{E}Z_1 + \tilde{E}Z_2$.*
3. *If $z$ commutes with $Ez$, then $Ee^z = e^z(Ez)$ and $\tilde{E}e^z = Ez$.*
4. *If $w : A \to \mathcal{A}$ is a morphism of graded algebras, then it commutes with $E$ and $\tilde{E}$.* □

Let us denote the right hand side of (22) by $Z_1(K)$. Then by the above proposition, remembering (Theorem 3.16) that $\mathcal{A}^w$ is commutative and that $\deg D_L = \deg D_R = 1$, we

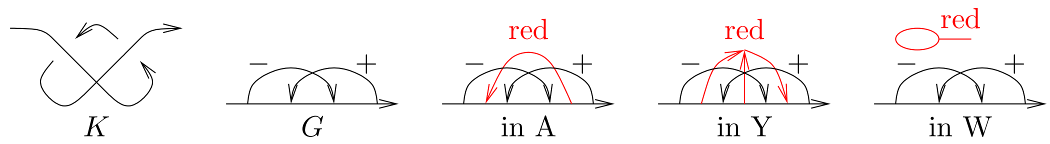



**Figure 15.** A sample w-knot $K$, it's Gauss diagram $G$, and one generator from each of the A, Y, and W sectors of $IAM^0_K$. Red parts are marked with the word "red".

have

$$\tilde{E}Z_1(K) = sl_L D_L + sl_R D_R - w(E \log A(K)(e^x)) = SL - w\left(x\frac{d}{dx}\log A(K)(e^x)\right),$$

with $SL := sl_L D_L + sl_R D_R$. The rest is an exercise in matrices and differentiation. $A(K)$ is a determinant (21), and in general, $\frac{d}{dx}\log\det(M) = \operatorname{tr}\left(M^{-1}\frac{d}{dx}M\right)$. So with $B = T(e^{-xS} - I)$ (so $M = I - B$), we have

$$\tilde{E}Z_1(K) = SL + w\left(x\operatorname{tr}\left((I-B)^{-1}\frac{d}{dx}B\right)\right) = SL - w\left(x\operatorname{tr}\left((I-B)^{-1}TSe^{-xS}\right)\right),$$

as promised in Equation (24).

3.8.2. *The Infinitesimal Alexander Module.* Let $K$ be a w-knot diagram. The Infinitesimal Alexander Module $IAM_K$ of $K$ is a certain module made from a certain space $IAM^0_K$ of pictures "annotating" $K$ with "red excitations" modulo some pictorial relations that indicate how the red excitations can be moved around. The space $IAM^0_K$ in itself is made of three pieces, or "sectors". The "A sector" in which the excitations are red arrows, the "Y sector" in which the excitations are "red hairy Y-diagrams", and a rank 1 "W sector" for "red hairy wheels". There is an "interpretation map" $\iota : IAM^0_K \to \mathcal{A}^w$ which descends to a well defined (and homonymous) $\iota : IAM_K \to \mathcal{A}^w$. Finally, there are some special elements $\lambda$, $\delta_L$, and $\delta_R$ that live in the A sector of $IAM^0_K$ and $\omega_1$ that lives in the W sector.

In principle, the description of $IAM^0_K$ and of $IAM_K$ can be given independently of the interpretation map $\iota$, and there are some good questions to ask about $IAM_K$ (and the special elements in it) that are completely independent of the interpretation of the elements of $IAM_K$ as "perturbed bulk quantities" within $\mathcal{A}^w$. Yet $IAM_K$ is a complicated object and I fear its definition will appear completely artificial without its interpretation. Hence below the two definitions will be woven together.

$IAM_K$ and $\iota$ may equally well be described in terms of $K$ or in terms of the Gauss diagram of $K$ (Remark 3.4). For pictorial simplicity, we choose to use the latter; so let $G = G(K)$ be the Gauss diagram of $K$. It is best to read the following definition while at the same time studying Figure 15.

**Definition 3.31.** Let $R$ be the ring $\mathbb{Z}[X, X^{-1}]$ of Laurent polynomials in $X$, and let $R_1$ be the subring of polynomials that vanish at $X = 1$ (i.e., whose sum of coefficients is 0)[23]. Let $IAM^0_K$ be the direct sum of the following three modules (which for the purpose of taking the direct sum, are all regarded as $\mathbb{Z}$-modules):

[23] $R_1$ is only very lightly needed, and only within Definition 3.32. In particluar, all that we say about $IAM_K$ that does not concern the interpretation map $\iota$ is equally valid with $R$ replacing $R_1$.

(1) The "A sector" is the free $\mathbb{Z}$-module generated by all diagrams made from $G$ by the addition of a single unmarked "red excitation" arrow, whose endpoints are on the skeleton of $G$ and are distinct from each other and from all other endpoints of arrows in $G$. Such diagrams are considered combinatorially — so two are equivalent iff they differ only by an orientation preserving diffeomorphism of the skeleton. Let us count: if $K$ has $n$ crossings, then $G$ has $n$ arrows and the skeleton of $G$ get subdivided into $m := 2n+1$ arcs. An A sector diagram is specified by the choice of an arc for the tail of the red arrow and an arc for the head ($m^2$ choices), except if the head and the tail fall within the same arc, their relative ordering has to be specified as well ($m$ further choices). So the rank of the A sector over $\mathbb{Z}$ is $m(m+1)$.
(2) The "Y sector" is the free $R_1$-module generated by all diagrams made from $G$ by the addition of a single "red excitation" $Y$-shape single-vertex graph, with two incoming edges ("tails") and one outgoing ("head"), modulo anti-symmetry for the two incoming edges (again, considered combinatorially). Counting is more elaborate: when the three edges of the $Y$ end in distinct arcs in the skeleton of $G$, we have $\frac{1}{2}m(m-1)(m-2)$ possibilities ($\frac{1}{2}$ for the antisymmetry). When the two tails of the $Y$ lie on the same arc, we get 0 by anti-symmetry. The remaining possibility is to have the head and one tail on one arc (order matters!) and the other tail on another, at $2m(m-1)$ possibilities. So the rank of the Y sector over $R_1$ is $m(m-1)(\frac{1}{2}m+1)$.
(3) The "W sector" is the rank 1 free $R$-module with a single generator $w_1$. It is not necessary for $w_1$ to have a pictorial representation, yet one, involving a single "red" 1-wheel, is shown in Figure 15.

**Definition 3.32.** The "interpretation map" $\iota : IAM^0_K \to \mathcal{A}^w$ is defined by sending the arrows (marked $+$ or $-$) of a diagram in $IAM^0_K$ to $e^{\pm a}$-exponential reservoirs of arrows, as in the definition of $Z$ (see Remark 3.12). In addition, the red excitations of diagrams in $IAM^0_K$ are interpreted as follows:

(1) In the A sector, the red arrow is simply mapped to itself, with the colour red suppressed.
(2) In the $Y$ sector diagrams have red $Y$'s and coefficients $f \in R_1$. Substitute $X = e^x$ in $f$, expand in powers of $x$, and interpret $x^kY$ as a "hairy $Y$ with $k-1$ hairs" as in Exercise 3.21. Note that $f(1) = 0$, so only positive powers of $x$ occur, so we never need to worry about "$Y$'s with $-1$ hairs". This is the only point where the condition $f \in R_1$ (as opposed to $f \in R$) is needed.
(3) In the $W$ sector treat the coefficients as above, but interpret $x^k w_1$ as a detached $w_{k+1}$. I.e., as a detached wheel with $k+1$ spokes, as in Exercise 3.21.

As stated above, $IAM^0_K$ is the quotient of $IAM_K$ by some set of relations. The best way to think of this set of relations is as "everything that's obviously annihilated by $\iota$". Here's the same thing, in a more formal language:

**Definition 3.33.** Let $IAM_K := IAM^0_K/\mathcal{R}$, where $\mathcal{R}$ is the set of relations depicted in Figure 16. The top 8 relations are about moving a leg of the red excitation across an arrow head or an arrow tail in $G$. Since the red excitation may be either an arrow ($A$) or a $Y$, its leg in motion may be either a tail or a head, and it may be moving either past a tail or past a head, there are 8 relations of that type. The last two relations indicate the "price" (always a red $w_1$), of commuting a red head across a red tail. As per custom, in each case

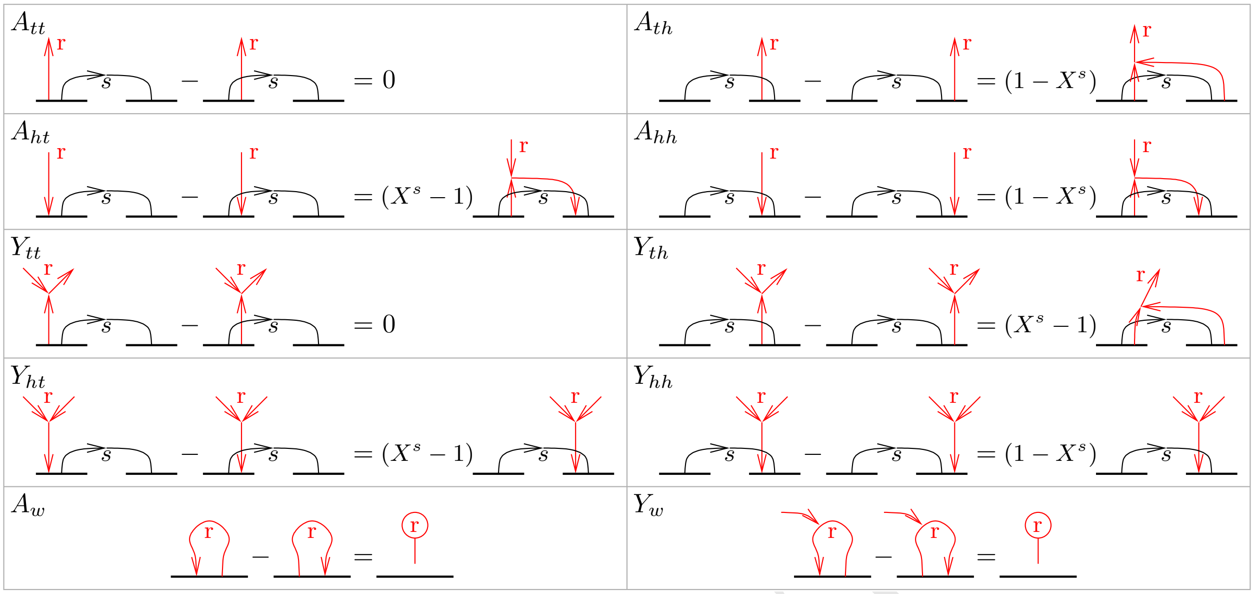



**Figure 16.** The relations $\mathcal{R}$ making $IAM_K$.

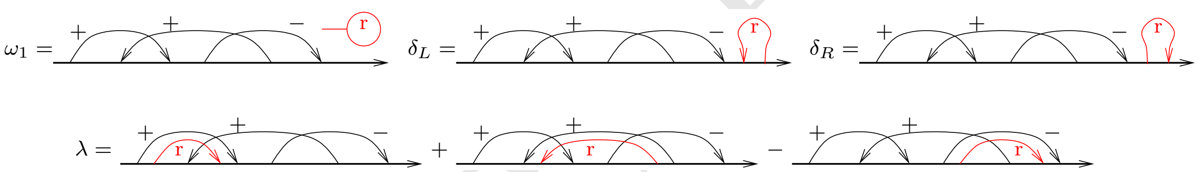



**Figure 17.** The special elements $\omega_1$, $\delta_L$, $\delta_R$, and $\lambda$ in $IAM_G$, for a sample 3-arrow Gauss diagram $G$.

only the changing part of the diagrams involved is shown. Further, the red excitations are marked with the letter "r" and the sign of an arrow in $G$ is marked $s$; so always $s \in \{\pm 1\}$.

**Proposition 3.34.** *The interpretation map $\iota$ indeed annihilates all the relations in $\mathcal{R}$.*

*Proof.* $\iota A_{tt}$ and $\iota Y_{tt}$ follow immediately from "Tails Commute". The formal identity $e^{\operatorname{ad} b}(a) = e^b a e^{-b}$ implies $e^{\operatorname{ad} b}(a) e^b = e^b a$ and hence $ae^b - e^b a = (1 - e^{\operatorname{ad} b})(a)e^b$. With $a$ interpreted as "red head", $b$ as "black head", and $\operatorname{ad} b$ as "hair" (justified by the $\iota$-meaning of hair and by the $\overrightarrow{STU}_1$ relation, Figure 11), the last equality becomes a proof of $\iota Y_{hh}$. Further pushing that same equality, we get $ae^b - e^b a = \frac{1-e^{\operatorname{ad} b}}{\operatorname{ad} b}([b, a])$, where $\frac{1-e^{\operatorname{ad} b}}{\operatorname{ad} b}$ is first interpreted as a power series $\frac{1-e^y}{y}$ involving only non-negative powers of $y$, and then the substitution $y = \operatorname{ad} b$ is made. But that's $\iota A_{hh}$, when one remembers that $\iota$ on the Y sector automatically contains a single "$\frac{1}{\text{hair}}$" factor. Similar arguments, though using $\overrightarrow{STU}_2$ instead of $\overrightarrow{STU}_1$, prove that $Y_{ht}$, $Y_{th}$, $A_{ht}$, and $A_{th}$ are all in $\ker \iota$. Finally, $\iota A_w$ and $\iota Y_w$ are direct consequences of $\overrightarrow{STU}_2$. In fact, $\iota A_w$ was encountered once before, as the relation $D_L - D_R = w_1$ of Theorem 3.16. □

Finally, we come to the special elements $\lambda$, $\delta_L$, $\delta_R$ and $\omega_1$.

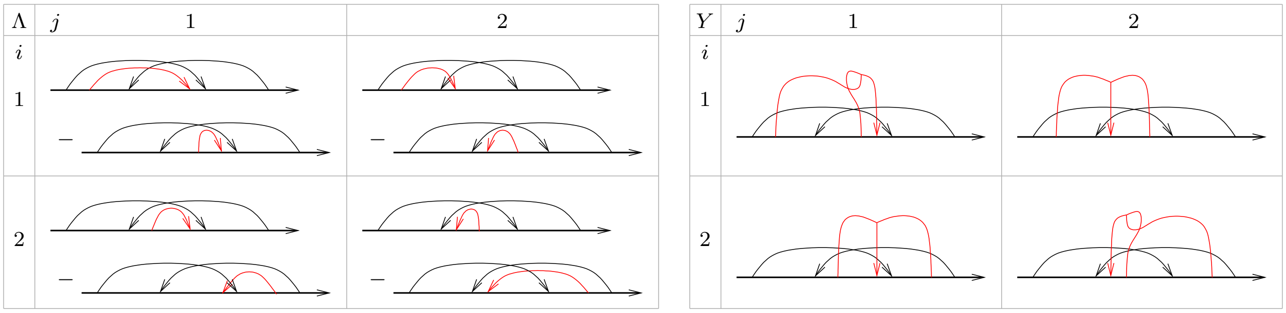



**Figure 18.** The matrices $\Lambda$ and $Y$ for a sample 2-arrow Gauss diagram (the signs on $a_1$ and $a_2$ are suppressed, and so are the $r$ marks). The twists in $y_{11}$ and $y_{22}$ may be replaced by minus signs.

**Definition 3.35.** Within $IAM_G$, let $\omega_1$ be, as before, the generator of the W sector. Let $\delta_L$ and $\delta_R$ be "short" red arrows, as on the left hand side of the $A_w$ relation (exercise: modulo $\mathcal{R}$, this is independent of the placement of these short arrows within $G$). Finally, let $\lambda$ be the signed sum of exciting each of the (black) arrows in $G$ in turn. The picture says all, and it is Figure 17.

**Proposition 3.36.** *In $\mathcal{A}^w(\uparrow)$, the special elements of $IAM_G$ are interpreted as follows: $\iota(\omega_1) = Zw_1$, $\iota(\delta_{L,R}) = ZD_{L,R}$, and most interesting, $\iota(\lambda) = EZ$. Therefore, Equation* (25) *(if true) implies Equation* (24) *and hence it implies our goal, Theorem 3.27.*

*Proof.* For the proof of this proposition, the only thing that isn't done yet and isn't trivial is the assertion $\iota(\lambda) = EZ$. But this assertion is a conseqeuence of $Ee^{\pm a} = \pm ae^{\pm a}$ and of a Leibnitz law for the derivation $E$, appropiately generalized to a context where $Z$ can be thought of as a "product" of "arrow reservoirs". The details are left to the reader. □

3.8.3. *The Computation of $\lambda$.* Naturally, our next task is to prove Equation (25). This is done entirely algebraically within the finite rank module $IAM_G$. To read this section one need not know about $\mathcal{A}^w(\uparrow)$, or $\iota$, or $Z$, but we do need to lay out some notation. Start by marking the arrows of $G$ with $a_1$ through $a_n$ in some order.

Let $\epsilon$ stand for the informal yet useful quantity "a little". Let $\lambda_{ij}$ denote the difference $\lambda'_{ij} - \lambda''_{ij}$ of red excitations in the A sector of $IAM_G$, where $\lambda'_{ij}$ is the diagram with a red arrow whose tail is $\epsilon$ to the right of the left end of $a_i$ and whose head is $\frac{1}{2}\epsilon$ away from head of $a_j$ in the direction of the tail of $a_j$, and where $\lambda''_{ij}$ has a red arrow whose tail is $\epsilon$ to the left of the right end of $a_i$ and whose head is as before, $\frac{1}{2}\epsilon$ away from head of $a_j$ in the direction of the tail of $a_j$. Let $\Lambda = (\lambda_{ij})$ be the matrix whose entries are the $\lambda_{ij}$'s, as shown in Figure 18.

Similarly, let $y_{ij}$ denote the element in the Y sector of $IAM_G$ whose red Y has its head $\frac{1}{2}\epsilon$ away from head of $a_j$ in the direction of the tail of $a_j$, its right tail (as seen from the head) $\epsilon$ to the left of the right end of $a_i$ and its left tail $\epsilon$ to the right of the left end of $a_i$. Let $Y = (y_{ij})$ be the matrix whose entries are the $y_{ij}$'s, as shown in Figure 18.

**Proposition 3.37.** *With $S$ and $T$ as in Definition 3.24, and with $B = T(X^{-S} - I)$ and $\lambda$ and $SL$ as above, the following identities between elements of $IAM_G$ and matrices with* *entries in $IAM_G$ hold true:*

$$\lambda - SL = \operatorname{tr} S\Lambda \tag{26}$$
$$\Lambda = -BY - TX^{-S}w_1 \tag{27}$$
$$Y = BY + TX^{-S}w_1 \tag{28}$$

*Proof of Equation* (25) *given Proposition 3.37.* The last of the equalities above implies that $Y = (I-B)^{-1}TX^{-S}w_1$. Thus

$$\lambda - SL = \operatorname{tr} S\Lambda = -\operatorname{tr} S(BY + TX^{-S}w_1) = -\operatorname{tr} S(B(I-B)^{-1}TX^{-S} + TX^{-S})w_1$$
$$= -\operatorname{tr}\left((I-B)^{-1}TSX^{-S}\right)w_1.$$

This is exactly Equation (25). □

*Proof of Proposition 3.37.* Equation (26) is trivial. The proofs of Equations (27) and (28) both have the same simple cores, that have to be supplamented by highly unpleasant tracking of signs and conventions and powers of $X$. Let us start from the cores.

To prove Equation (27) we wish to "compute" $\lambda_{ik} = \lambda'_{ik} - \lambda''_{ik}$. As $\lambda'_{ik}$ and $\lambda''_{ik}$ have their heads in the same place, we can compute their difference by gradually sliding the tail of $\lambda'_{ik}$ from its original position near the left end of $a_i$ towards the right end of $a_i$, where it would be cancelled by $\lambda''_{ik}$. As the tail slides we pick up a $y_{jk}$ term each time it crosses a head of an $a_j$ (relation $A_{th}$), we pick up a vanishing term each time it crosses a tail (relation $A_{tt}$), and we pick up a $w_1$ term if the tail needs to cross over its own head (relation $A_w$). Ignoring signs and $(X^{\pm 1} - 1)$ factors, the sum of the $y_{jk}$-terms should be proportional to $TY$, for indeed, the matrix $T$ has non-zero entries precisely when the head of an $a_j$ falls within the span of an $a_i$. Unignoring these signs and factors, we get $-BY$ (recall that $B = T(X^{-S} - I)$ is just $T$ with added $(X^{\pm 1} - 1)$ factors). Similarly, a $w_1$ term arises in this process when a tail has to cross over its own head, that is, when the head of $a_k$ is within the span of $a_i$. Thus the $w_1$ term should be proportional to $Tw_1$, and we claim it is $-TX^{-S}w_1$.

The core of the proof of Equation (28) is more or less the same. We wish to "compute" $y_{ik}$ by sliding its left leg, starting near the left end of $a_i$, towards its right leg, which is stationary near the right end of $a_i$. When the two legs come together, we get 0 because of the anti-symmetry of Y excitations. Along the way we pick up further Y terms from the $Y_{th}$ relations, and sometimes a $w_1$ term from the $Y_w$ relation. When all signs and $(X^{\pm 1} - 1)$ factors are accounted for, we get Equation (28).

I leave it to the reader to complete the details in the above proofs. It is a major headache, and I would not have trusted myself had I not written a computer program to manipulate quantities in $IAM_G$ by a brute force application of the relations in $\mathcal{R}$. Everything checks; see [BND, "The Infinitesimal Alexander Module"]. □

This concludes the proof of Theorem 3.27. □

*Remark* 3.38. I chose the name "Infinitesimal Alexander Module" as in my mind there is some similarity between $IAM_K$ and the "Alexander Module" of $K$. Yet beyond the above, I did not embark on any serious study of $IAM_K$. In particular, I do not know if $IAM_K$ in itself is an invariant of $K$ (though I suspect it wouldn't be hard to show that it is), I do not know if $IAM_K$ contains any further information beyond $SL$ and the Alexander polynomial, and I do not know if there is any formal relationship between $IAM_K$ and the Alexander module of $K$.

*Remark* 3.39. The logarithmic derivative of the Alexander polynomial also appears in Lescop's [Les1, Les2]. I don't know if its appearances there are related to its appearance here.

3.9. **The Relationship with u-Knots.** Unlike in the case of braids, there is a canonical universal finite type invariant of $u$-knots: the Kontsevich integral $Z^u$. So it makes sense to ask how it is related to the expansion $Z^w$.

$$\begin{array}{ccc} \mathcal{K}^u(\uparrow) & \xrightarrow{Z^u} & \mathcal{A}^u(\uparrow) \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \alpha} \\ \mathcal{K}^w(\uparrow) & \xrightarrow{Z^w} & \mathcal{A}^w(\uparrow) \end{array}$$

We claim that the square on the left commutes, where $\mathcal{K}^u(\uparrow)$ stands for long $u$-knots (knottings of an oriented line), and similarly $\mathcal{K}^w(\uparrow)$ denotes long $w$-knots. As before, $a$ is the composition of the maps $u$-knots $\to$ $v$-knots $\to$ $w$-knots, and $\alpha$ is the induced map on the projectivizations, mapping each chord to the sum of the two ways to direct it.

Recall that $\alpha$ kills everything but wheels and arrows. We are going to use the formula for the "wheel part" of the Kontsevich integral as stated in [Kr]. Let $K$ be a 0-framed long knot, and let $A(K)$ denote the Alexander polynomial. Then by [Kr],

$$Z^u(K) = \exp_{\mathcal{A}^u}\left(-\frac{1}{2}\log A(K)(e^h)|_{h^{2n}\to w^u_{2n}}\right) + \text{ "loopy terms"},$$

where $w^u_{2n}$ stands for the unoriented wheel with $2n$ spokes; and "loopy terms" means terms that contain diagrams with more than one loop, which are killed by $\alpha$. Note that by the symmetry $A(z) = A(z^{-1})$ of the Alexander polynomial, $A(K)(e^h)$ contains only even powers of $h$, as suggested by the formula.

We need to understand how $\alpha$ acts on wheels. Due to the two-in-one-out rule, a wheel is zero unless all the "spokes" are oriented inward, and the cycle oriented in one direction. In other words, there are two ways to orient an unoriented wheel: clockwise or counterclockwise. Due to the anti-symmetry of chord vertices, we get that for odd wheels $\alpha(w^u_{2h+1}) = 0$ and for even wheels $\alpha(w^u_{2h}) = 2w^w_{2h}$. As a result,

$$\alpha Z^u(K) = \exp_{\mathcal{A}^w}\left(-\frac{1}{2}\log A(K)(e^h)|_{h^{2n}\to 2w_{2n}}\right) = \exp_{\mathcal{A}^w}\left(-\log A(K)(e^h)|_{h^{2n}\to w_{2n}}\right)$$

which agrees with the formula (22) of Theorem 3.27. Note that since $K$ is 0-framed, the first part ("self-linking coded in arrows") of (22) is trivial.



## 4. Algebraic Structures, Projectivizations, Expansions, Circuit Algebras

**Section Summary.** In this section we define the "projectivization" (see 4.2) of an arbitrary algebraic structure (4.1) and introduce the notions of "expansions" and "homomorphic expansions" (4.3) for such projectivizations. Everything is so general that practically anything is an example. The baby-example of quandles is built in into the section; the braid groups and w-braid groups appeared already in Section 2, yet our main goal is to set the language for the examples of w-tangles and w-tangled foams, which appear later in this paper. Both of these examples are types of "circuit algebras", and hence we end this section with a general discussion of circuit algebras (see 4.4).

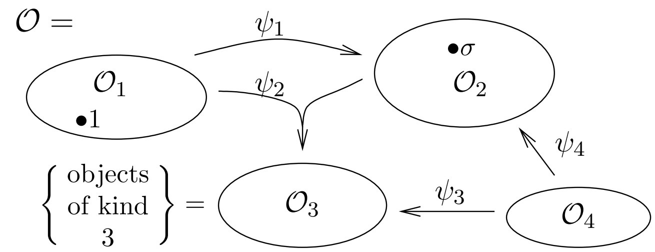



**Figure 19.** An algebraic structure $\mathcal{O}$ with 4 kinds of objects and one binary, 3 unary and two 0-nary operations (the constants 1 and $\sigma$).

4.1. **Algebraic Structures.** An "algebraic structure" $\mathcal{O}$ is some collection $(\mathcal{O}_\alpha)$ of sets of objects of different kinds, where the subscript $\alpha$ denotes the "kind" of the objects in $\mathcal{O}_\alpha$, along with some collection of "operations" $\psi_\beta$, where each $\psi_\beta$ is an arbitrary map with domain some product $\mathcal{O}_{\alpha_1} \times \cdots \times \mathcal{O}_{\alpha_k}$ of sets of objects, and range a single set $\mathcal{O}_{\alpha_0}$ (so operations may be unary or binary or multinary, but they always return a value of some fixed kind). We also allow some named "constants" within some $\mathcal{O}_\alpha$'s (or equivalently, allow some 0-nary operations).[24] The operations may or may not be subject to axioms — an "axiom" is an identity asserting that some composition of operations is equal to some other composition of operations.

Figure 19 illustrates the general notion of an algebraic structure. Here are a few specific examples:

- Groups: one kind of objects, one binary "multiplication", one unary "inverse", one constant "the identity", and some axioms.
- Group homomorphisms: Two kinds of objects, one for each group. 7 operations — 3 for each of the two groups and the homomorphism itself, going between the two groups. Many axioms.
- A group acting on a set, a group extension, a split group extension and many other examples from group theory.
- A quandle. It is worthwhile to quote the abstract of the paper that introduced the definition (Joyce, [Joy]):

  *The two operations of conjugation in a group, $x \triangleright y = y^{-1}xy$ and $x \triangleright^{-1} y = yxy^{-1}$ satisfy certain identities. A set with two operations satisfying these identities is called a quandle. The Wirtinger presentation of the knot group involves only relations of the form $y^{-1}xy = z$ and so may be construed as presenting a quandle rather than a group. This quandle, called the knot quandle, is not only an invariant of the knot, but in fact a classifying invariant of the knot.*

  Also see Definition 4.2.
- Planar algebras as in [Jon] and circuit algebras as in Section 4.4.
- The algebra of knotted trivalent graphs as in [BN8, Da].
- Let $\varsigma : B \to S$ be an arbitrary homomorphism of groups (though our notation suggests what we have in mind — $B$ may well be braids, and $S$ may well be permutations). We can consider an algebraic structure $\mathcal{O}$ whose kinds are the elements of



[24]One may alternatively define "algebraic structures" using the theory of "multicategories" [Lei]. Using this language, an algebraic structure is simply a functor from some "structure" multicategory $\mathcal{C}$ into the multicategory **Set** (or into **Vect**, if all $\mathcal{O}_i$ are vector spaces and all operations are multilinear). A "morphism" between two algebraic structures over the same multicategory $\mathcal{C}$ is a natural transformation between the two functors representing those structures.

$S$, for which the objects of kind $s \in S$ are the elements of $\mathcal{O}_s := \varsigma^{-1}(s)$, and with the product in $B$ defining operations $\mathcal{O}_{s_1} \times \mathcal{O}_{s_2} \to \mathcal{O}_{s_1 s_2}$.
- Clearly, many more examples appear throughout mathematics.

4.2. **Projectivization.** Any algebraic structure $\mathcal{O}$ has a projectivization. First extend $\mathcal{O}$ to allow formal linear combinations of objects of the same kind (extending the operations in a linear or multi-linear manner), then let $\mathcal{I}$, the "augmentation ideal", be the sub-structure made out of all such combinations in which the sum of coefficients is 0, then let $\mathcal{I}^m$ be the set of all outputs of algebraic expressions (that is, arbitrary compositions of the operations in $\mathcal{O}$) that have at least $m$ inputs in $\mathcal{I}$ (and possibly, further inputs in $\mathcal{O}$), and finally, set

$$\operatorname{proj}\mathcal{O} := \bigoplus_{m\geq 0} \mathcal{I}^m/\mathcal{I}^{m+1}. \tag{29}$$

Clearly, with the operation inherited from $\mathcal{O}$, the projectivization $\operatorname{proj}\mathcal{O}$ is again algebraic structure with the same multi-graph of spaces and operations, but with new objects and with new operations that may or may not satisfy the axioms satisfied by the operations of $\mathcal{O}$. The main new feature in $\operatorname{proj}\mathcal{O}$ is that it is a "graded" structure; we denote the degree $m$ piece $\mathcal{I}^m/\mathcal{I}^{m+1}$ of $\operatorname{proj}\mathcal{O}$ by $\operatorname{proj}_m\mathcal{O}$.

I believe that many of the most interesting graded structures that appear in mathematics are the result of this construction, and that many of the interesting graded equations that appear in mathematics arise when one tries to find "expansions", or "universal finite type invariants", which are also morphisms[25] $Z : \mathcal{O} \to \operatorname{proj}\mathcal{O}$ (see Section 4.3) or when one studies "automorphisms" of such expansions[26] Indeed, the paper you are reading now is really the study of the projectivizations of various algebraic structures associated with w-knotted objects. I would like to believe that much of the theory of quantum groups (at "generic" $\hbar$) will eventually be shown to be a study of the projectivizations of various algebraic structures associated with v-knotted objects.

Thus I believe that the operation described in Equation (29) is truly fundamental and therefore worthy of a catchy name. So why "projectivization"? Well, it reminds me of graded spaces, but really, that's all. I simply found no better name. I'm open to suggestions.

Let us end this section with two examples.

**Proposition 4.1.** *If $G$ is a group,* $\operatorname{proj} G$ *is a graded associative algebra with unit.* □

**Definition 4.2.** A quandle is a set $Q$ with a binary operation $\uparrow : Q \times Q \to Q$ satisfying the following axioms:

(1) $\forall x \in Q$, $x{\uparrow}x = x$.
(2) For any fixed $y \in Q$, the map $x \mapsto x{\uparrow}y$ is invertible[27].
(3) Self-distributivity: $\forall x, y, x \in Q$, $(x{\uparrow}y){\uparrow}z = (x{\uparrow}z){\uparrow}(y{\uparrow}z)$.

[25] Indeed, if $\mathcal{O}$ is finitely presented then finding such a morphism $Z : \mathcal{O} \to \operatorname{proj}\mathcal{O}$ amounts to finding its values on the generators of $\mathcal{O}$, subject to the relations of $\mathcal{O}$. Thus it is equivalent to solving a system of equations written in some graded spaces.

[26] The Drinfel'd graded Grothendieck-Teichmuller group $GRT$ is an example of such an automorphism group. See [Dr3, BN6].

[27] This can alternatively be stated as "there exists a second binary operation $\uparrow^{-1}$ so that $\forall x$, $x = (x{\uparrow}y){\uparrow}^{-1}y = (x{\uparrow}^{-1}y){\uparrow}y$", so this axiom can still be phrased within the language of "algebraic structures". Yet note that below we do not use this axiom at all.

We say that a quandle $Q$ has a unit, or is unital, if there is a distiguished element $1 \in Q$ satisfying the further axiom:

(4) $\forall x \in Q$, $x\uparrow 1 = x$ and $1\uparrow x = 1$.

If $G$ is a group, it is also a (unital) quandle by setting $x\uparrow y := y^{-1}xy$, yet there are many quandles that do not arise from groups in this way.

**Proposition 4.3.** *If $Q$ is a unital quandle,* $\mathrm{proj}_0 Q$ *is one-dimensional and* $\mathrm{proj}_{>0} Q$ *is a graded right Leibniz algebra*[28] *generated by* $\mathrm{proj}_1 Q$.

*Proof.* For any algebraic structure $A$ with just one kind of objects, $\mathrm{proj}_0 A$ is one-dimensional, generated by the equivalence class $[x]$ of any single object $x$. In particular, $\mathrm{proj}_0 Q$ is one-dimensional and generated by $[1]$. Let $\mathcal{I} \subset \mathbb{Q}Q$ be the augmentation ideal of $Q$. For any $x \in Q$ set $\bar{x} := x - 1 \in \mathcal{I}$. Then $\mathcal{I}$ is generated by the $\bar{x}$'s, and therefore $\mathcal{I}^m$ is generated by expressions involving the operation $\uparrow$ applied to some $m$ elements of $\bar{Q} := \{\bar{x} : x \in Q\}$ and possibly some further elements $y_i \in Q$. When regarded in $\mathcal{I}^m/\mathcal{I}^{m+1}$, any $y_i$ in such a generating expression can be replaced by 1, for the difference would be the same expression with $y_i$ replaced by $\bar{y}_i$, and this is now a member of $\mathcal{I}^{m+1}$. But for any element $z \in \mathcal{I}$ we have $z\uparrow 1 = z$ and $1\uparrow z = 0$, so all the 1's can be eliminated from the expressions generating $\mathcal{I}^m$. Thus $\mathrm{proj}_{>0} Q$ is generated by $\bar{Q}$ and hence by $\mathrm{proj}_1 Q$.

Let $\Delta : \mathbb{Q}Q \to \mathbb{Q}Q \otimes \mathbb{Q}Q$ be the linear extension of the operation $x \mapsto x \otimes x$ defined on $x \in Q$, and extend $\uparrow$ to a binary operator $\uparrow_2 : (\mathbb{Q}Q \otimes \mathbb{Q}Q) \otimes (\mathbb{Q}Q \otimes \mathbb{Q}Q) \to \mathbb{Q}Q \otimes \mathbb{Q}Q$ by using $\uparrow$ twice, to pair the first and third tensor factors and then to pair the second and the fourth tensor factors. With this language in place, the self-distributivity axiom becomes the following *linear* statement, which holds for every $x, y, z \in \mathbb{Q}Q$:

$$(x\uparrow y)\uparrow z = \uparrow \circ \uparrow_2 (x \otimes y \otimes \Delta z). \tag{30}$$

Clearly, we need to understand $\Delta$ better. By direct computation, if $x \in Q$ then $\Delta\bar{x} = \bar{x} \otimes 1 + 1 \otimes \bar{x} + \bar{x} \otimes \bar{x}$. We claim that in general, if $z$ is a generating expression of $\mathcal{I}^m$ (that is, a formula made of $m$ elements of $\bar{Q}$ and $m-1$ applications of $\uparrow$), then

$$\Delta z = z \otimes 1 + 1 \otimes z + \sum z'_i \otimes z''_i, \qquad \text{with} \qquad \sum z'_i \otimes z''_i \in \sum_{\substack{m'+m''=m+1,\\ m',m''>0}} \mathcal{I}^{m'} \otimes \mathcal{I}^{m''}. \tag{31}$$

Indeed, for the generators of $\mathcal{I}^1$ this had just been shown, and if $z = z_1\uparrow z_2$ is a generator of $\mathcal{I}^m$, with $z_1$ and $z_2$ generators of $\mathcal{I}^{m_1}$ and $\mathcal{I}^{m_2}$ with $1 \le m_1, m_2 < m$ and $m_1 + m_2 = m$, then (using $w\uparrow 1 = w$ and $1\uparrow w = 0$ for $w \in \mathcal{I}$),

$$\begin{aligned}
\Delta z &= \Delta(z_1\uparrow z_2) = (\Delta z_1)\uparrow_2(\Delta z_2) \\
&= (z_1 \otimes 1 + 1 \otimes z_1 + \sum z'_{1j} \otimes z''_{1j})\uparrow_2(z_2 \otimes 1 + 1 \otimes z_2 + \sum z'_{2k} \otimes z''_{2k}) \\
&= (z_1\uparrow z_2) \otimes 1 + 1 \otimes (z_1\uparrow z_2) \\
&\quad + \sum_j \left( (z'_{1j}\uparrow z_2) \otimes z''_{1j} + z'_{1j} \otimes (z''_{1j}\uparrow z_2) + \sum_k (z'_{1j}\uparrow z'_{2k}) \otimes (z''_{1j}\uparrow z''_{2k}) \right),
\end{aligned}$$

and it is easy to see that the last line agrees with (31).

[28]A Leibniz algebra is a Lie algebra without anticommutativity, as defined by Loday in [Lod].

We can now combine Equations (30) and (31) to get that for any $x, y, z \in \mathbb{Q}Q$,

$$(x\uparrow y)\uparrow z = (x\uparrow z)\uparrow y + x\uparrow(y\uparrow z) + \sum (x\uparrow z_i')\uparrow(y\uparrow z_i'').$$

If $x \in \mathcal{I}^{m_1}$, $y \in \mathcal{I}^{m_2}$, and $z \in \mathcal{I}^{m_3}$, then by (31) the last term above is in $\mathcal{I}^{m_1+m_2+m_3+1}$, and so the above identity becomes the Jacobi identity $(x\uparrow y)\uparrow z = (x\uparrow z)\uparrow y + x\uparrow(y\uparrow z)$ in $\operatorname{proj}_{m_1+m_2+m_3} Q$.

Note that in the above proof neither axiom (1) nor axiom (2) of Definition 4.2 was used.

*Exercise* 4.4. Show that axiom (1) implies the antisymmetry of $\uparrow$ on $\mathcal{I}^1$.

4.3. **Expansions and Homomorphic Expansions.** We start with the definition. Given an algebraic structure $\mathcal{O}$ let fil $\mathcal{O}$ denote the filtered structure of linear combinations of objects in $\mathcal{O}$ (respecting kinds), filtered by the powers $(\mathcal{I}^m)$ of the augmentation ideal $\mathcal{I}$. Recall also that any graded space $G = \bigoplus_m G_m$ is automatically filtered, by $\left(\bigoplus_{n\geq m} G_n\right)_{m=0}^{\infty}$.

**Definition 4.5.** An "expansion" $Z$ for $\mathcal{O}$ is a map $Z : \mathcal{O} \to \operatorname{proj} \mathcal{O}$ that preserves the kinds of objects and whose linear extention (also called $Z$) to fil $\mathcal{O}$ respects the filtration of both sides, and for which $(\operatorname{gr} Z) : (\operatorname{gr} \operatorname{fil} \mathcal{O} = \operatorname{proj} \mathcal{O}) \to (\operatorname{gr} \operatorname{proj} \mathcal{O} = \operatorname{proj} \mathcal{O})$ is the identity map of $\operatorname{proj} \mathcal{O}$.

In practical terms, this is equivalent to saying that $Z$ is a map $\mathcal{O} \to \operatorname{proj} \mathcal{O}$ whose restriction to $\mathcal{I}^m$ vanishes in degrees less than $m$ (in $\operatorname{proj} \mathcal{O}$) and whose degree $m$ piece is the projection $\mathcal{I}^m \to \mathcal{I}^m/\mathcal{I}^{m+1}$.

We come now to what is perhaps the most crucial definition in this paper.

**Definition 4.6.** A "homomorphic expansion" is an expansion which also commutes with all the algebraic operations defined on the algebraic structure $\mathcal{O}$.

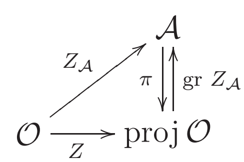
**Why Bother with Homomorphic Expansions?** Primarily, for two reasons:

- Often times $\operatorname{proj} \mathcal{O}$ is simpler to work with than $\mathcal{O}$; for one, it is graded and so it allows for finite "degree by degree" computations, whereas often times, such as in many topological examples, anything in $\mathcal{O}$ is inherently infinite. Thus it can be beneficial to translate questions about $\mathcal{O}$ to questions about $\operatorname{proj} \mathcal{O}$. A simplistic example would be, "is some element $a \in \mathcal{O}$ the square (relative to some fixed operation) of an element $b \in \mathcal{O}$?". Well, if $Z$ is a homomorphic expansion and by a finite computation it can be shown that $Z(a)$ is not a square already in degree 7 in $\operatorname{proj} \mathcal{O}$, then we've given a conclusive negative answer to the example question. Some less simplistic and more relevant examples appear in [BN8].
- Often times $\operatorname{proj} \mathcal{O}$ is "finitely presented", meaning that it is generated by some finitely many elements $g_1, \ldots, g_k \in \mathcal{O}$, subject to some relations $R_1 \ldots R_n$ that can be written in terms of $g_1, \ldots, g_k$ and the operations of $\mathcal{O}$. In this case, finding a homomorphic expansion $Z$ is essentially equivalent to guessing the values of $Z$ on $g_1, \ldots, g_k$, in such a manner that these values $Z(g_1), \ldots, Z(g_k)$ would satisfy the $\operatorname{proj} \mathcal{O}$ versions of the relations $R_1 \ldots R_n$. So finding $Z$ amounts to solving equations in graded spaces. It is often the case (as will be demonstrated in this paper; see also [BN3, BN6]) that these equations are very interesting for their own algebraic sake, and that viewing such equations as arising from an attempt to solve a problem about $\mathcal{O}$ sheds further light on their meaning.

In practice, often times the first difficulty in searching for an expansion (or a homomorphic expansion) $Z : \mathcal{O} \to \operatorname{proj}\mathcal{O}$ is that its would-be target space $\operatorname{proj}\mathcal{O}$ is hard to identify. It is typically easy to make a suggestion $\mathcal{A}$ for what $\operatorname{proj}\mathcal{O}$ could be. It is typically easy to come up with a reasonable generating set $\mathcal{D}_m$ for $\mathcal{I}^m$ (keep some knot theoretic examples in mind, or the case of quandles as in Proposition 4.3). It is a bit harder but not exceedingly difficult to discover some relations $\mathcal{R}$ satisfied by the elements of the image of $\mathcal{D}$ in $\mathcal{I}^m/\mathcal{I}^{m+1}$ (4T, $\overrightarrow{4T}$, and more in knot theory, the Jacobi relation in Proposition 4.3). Thus we set $\mathcal{A} := \mathcal{D}/\mathcal{R}$; but it is often very hard to be sure that we found everything that ought to go in $\mathcal{R}$; so perhaps our suggestion $\mathcal{A}$ is still too big? Finding 4T, or Jacobi in Proposition 4.3 was actually not *that* easy. Perhaps we missed some further relations that are hiding in $\operatorname{proj} Q$, for example?

The notion of an $\mathcal{A}$-expansion, defined below, solves two problems are once. Once we find an $\mathcal{A}$-expansion we know that we've identified $\operatorname{proj}\mathcal{O}$ correctly, and we automatically get what we really wanted, a $(\operatorname{proj}\mathcal{O})$-valued expansion.

**Definition 4.7.** A "candidate projectivization" for an algebraic structure $\mathcal{O}$ is a graded structure $\mathcal{A}$ with the same operations as $\mathcal{O}$ along with a homomorphic surjective graded map $\pi : \mathcal{A} \to \operatorname{proj}\mathcal{O}$. An "$\mathcal{A}$-expansion" is a kind and filtration respecting map $Z_\mathcal{A} : \mathcal{O} \to \mathcal{A}$ for which $(\operatorname{gr} Z_\mathcal{A}) \circ \pi : \mathcal{A} \to \mathcal{A}$ is the identity. There's no need to define "homomorphic $\mathcal{A}$-expansions".




**Proposition 4.8.** *If $\mathcal{A}$ is a candidate projectivization of $\mathcal{O}$ and $Z_\mathcal{A} : \mathcal{O} \to \mathcal{A}$ is a homomorphic $\mathcal{A}$-expansion, then $\pi : \mathcal{A} \to \operatorname{proj}\mathcal{O}$ is an isomorphism and $Z := \pi \circ Z_\mathcal{A}$ is a homomorphic expansion. (Often in this case, $\mathcal{A}$ is identified with $\operatorname{proj}\mathcal{O}$ and $Z_\mathcal{A}$ is identified with $Z$).*

*Proof.* $\pi$ is surjective by birth. Since $(\operatorname{gr} Z_\mathcal{A}) \circ \pi$ is the identity, $\pi$ it is also injective and hence it is an isomorphism. The rest is immediate. □

MORE: A bit on the general theory of expansions and their indeterminacy, expansions for free groups and free quandles.

4.4. **Circuit Algebras.** "Circuit algebras" are so common and everyday, and they make such a useful language (definitely for the purposes of this paper, but also elsewhere), I find it hard to believe they haven't made it into the standard mathematical vocabulary[29]. People familiar with planar algebras [Jon] may note that circuit algebras are just the same as planar algebras, except with the planarity requirement dropped from the "connection diagrams" (and all colourings dropped as well). For the rest, I'll start with an image and then move on to the dry definition.

**Image 4.9.** Electronic circuits are made of "components" that can be wired together in many ways. On a logical level, we only care to know which pin of which component is connected with which other pin of the same or other component. On a logical level, we don't really need to know how the wires between those pins are embedded in space (see Figures 20 and 21). "Printed Circuit Boards" (PCBs) are operators that make smaller components ("chips") into bigger ones ("circuits") — logically speaking, a PCB is simply a set of "wiring instructions", telling us which pins on which components are made to connect (and again, we never care

[29]Or have they, and I've been looking the wrong way?

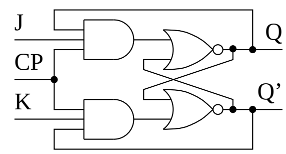



**Figure 20.** The J-K flip flop, a very basic memory cell, is an electronic circuit that can be realized using 9 components — two triple-input "and" gates, two standard "nor" gates, and 5 "junctions" in which 3 wires connect (many engineers would not consider the junctions to be real components, but we do). Note that the "crossing" in the middle of the figure is merely a projection artifact and does not indicate an electrical connection, and that electronically speaking, we need not specify how this crossing may be implemented in $\mathbb{R}^3$. The J-K flip flop has 5 external connections (labeled J, K, CP, Q, and Q') and hence in the circuit algebra of computer parts, it lives in $C_5$. In the directed circuit algebra of computer parts it would be in $C_{3,2}$ as it has 3 incoming wires (J, CP, and K) and two outgoing wires (Q and Q').

**Figure 21.** The circuit algebra product of 4 big black components and 1 small black component carried out using a green wiring diagram, is an even bigger component that has many golden connections (at bottom). When plugged into a yet bigger circuit, the CPU board of a laptop, our circuit functions as 4,294,967,296 binary memory cells.

precisely how the wires are routed provided they reach their intended destinations, and ever since the invention of multi-layered PCBs, all conceivable topologies for wiring are actually realizable). PCBs can be composed (think "plugging a graphics card onto a motherboard"); the result of a composition of PCBs, logically speaking, is simply a larger PCB which takes a larger number of components as inputs and outputs a larger circuit. Finally, it doesn't matter if several PCB are connected together and then the chips are placed on them, or if the chips are placed first and the PCBs are connected later; the resulting overall circuit remains the same.



We start process of drying (formalizing) this image by defining "wiring diagrams", the abstract analogs of printed circuit boards. Let $\mathbb{N}$ denote the set of natural numbers including 0, and for $n \in \mathbb{N}$ let $\underline{n}$ denote some fixed set with $n$ elements, say $\{1, 2, \ldots, n\}$.

**Definition 4.10.** Let $k, n, n_1, \ldots, n_k \in \mathbb{N}$ be natural numbers. A "wiring diagram" $D$ with inputs $\underline{n_1}, \ldots \underline{n_k}$ and outputs $\underline{n}$ is an unoriented compact 1-manifold whose boundary is $\underline{n} \amalg \underline{n_1} \amalg \cdots \amalg \underline{n_k}$, regarded up to homeomorphism. In strictly combinatorial terms, it is a pairing of the elements of the set $\underline{n} \amalg \underline{n_1} \amalg \cdots \amalg \underline{n_k}$ along with a single further natural number that counts closed circles. If $D_1; \ldots; D_m$ are wiring diagrams with inputs $\underline{n_{11}}, \ldots, \underline{n_{1k_1}}; \ldots; \underline{n_{m1}}, \ldots, \underline{n_{mk_m}}$ and outputs $\underline{n_1}; \ldots; \underline{n_m}$ and $D$ is a wiring diagram with inputs $\underline{n_1}; \ldots; \underline{n_m}$ and outputs $\underline{n}$, there is an obvious "composition" $D(D_1, \ldots, D_m)$ (obtained by gluing the corresponding 1-manifolds, and also describable in completely combinatorial terms) which is a wiring diagram with inputs $(\underline{n_{ij}})_{1 \le i \le k_j, 1 \le j \le m}$ and outputs $\underline{n}$ (note that closed circles may be created in $D(D_1, \ldots, D_m)$ even if none existed in $D$ and in $D_1; \ldots; D_m$).

A circuit algebra is an algebraic structure (in the sense of Section 4.2) whose operations are parametrized by wiring diagrams. Here's a formal definition:

**Definition 4.11.** A circuit algebra consists of the following data:

- For every natural number $n \geq 0$ a set (or a $\mathbb{Z}$-module) $C_n$ "of circuits with $n$ legs".
- For any wiring diagram $D$ with inputs $\underline{n_1}, \dots \underline{n_k}$ and outputs $\underline{n}$, an operation (denoted by the same letter) $D : C_{n_1} \times \cdots \times C_{n_k} \to C_n$ (or linear $D : C_{n_1} \otimes \cdots \otimes C_{n_k} \to C_n$ if we work with $\mathbb{Z}$-modules).

We insist that the obvious "identity" wiring diagrams with $\underline{n}$ inputs and $\underline{n}$ outputs act as the identity of $C_n$, and that the actions of wiring diagrams be compatible in the obvious sense with the composition operation on wiring diagrams.

A silly but useful example of a circuit algebra is the circuit algebra $\mathcal{S}$ of empty circuits, or in our context, of "skeletons". The circuits with $n$ legs for $\mathcal{S}$ are wiring diagrams with $n$ outputs and no inputs; namely, they are 1-manifolds with boundary $\underline{n}$ (so $n$ must be even).

More generally one may pick some collection of "basic components" (perhaps some logic gates and junctions for electronic circuits as in Figure 20) and speak of the "free circuit algebra" generated by these components. Even more generally we can speak of circuit algebras given in terms of "generators and relations"; in the case of electronics, our relations may include the likes of De Morgan's law $\neg(p \vee q) = (\neg p) \wedge (\neg q)$ and the laws governing the placement of resistors in parallel or in series. We feel there is no need to present the details here, yet many examples of circuit algebras given in terms of generators and relations appear in this paper, starting with the next section. We will use the notation $C = \mathrm{CA}\langle G \mid R \rangle$ to denote the circuit algebra generated by a collection of elements $G$ subject to some collection $R$ of relations.

People familiar with electric circuits know very well that connectors sometimes come in "male" and "female" versions, and that you can't plug a USB cable into a headphone jack and expect your system to cooperate. Thus one may define "directed circuit algebras" in which the wiring diagrams are oriented, the circuit sets $C_n$ get replaced by $C_{n_1 n_2}$ for "circuits with $n_1$ incoming wires and $n_2$ outgoing wires" and only orientation preserving connections are ever allowed. Likewise there is a "coloured" version of everything, in which the wires may be coloured by the elements of some given set $X$ which may include among its members the elements "USB" and "audio" and in which connections are allowed only if the colour coding is respected. We will not give formal definitions of directed and/or coloured circuit algebras here, yet we will allow ourselves to freely use these notions. Likewise for the obvious analogues of the skeletons algebra $\mathcal{S}$ and for algebras given in terms of generators and relations.

Note that there is an obvious notion of "a morphism between two circuit algebras" and that circuit algebras (directed or not, coloured or not) form a category. We feel that a precise definition is not needed. Yet a lovely example is the "implementation morphism" of logic circuits in the style of Figure 20 into more basic circuits made of transistors and resistors.

Perhaps the prime mathematical example of a circuit algebra is tensor algebra. If $t_1$ is an element (a "circuit") in some tensor product of vector spaces and their duals, and $t_2$ is the same except in a possibly different tensor product of vector spaces and their duals, then once an appropriate pairing $D$ (a "wiring diagram") of the relevant vector spaces is chosen, $t_1$ and $t_2$ can be contracted ("wired together") to make a new tensor $D(t_1, t_2)$. The pairing $D$ must pair a vector space with its own dual, and so this circuit algebra is coloured by the set of vector spaces involved, and directed, by declaring (say) that some vector spaces are of one gender and their duals are of the other. We have in fact encountered this circuit algebra already, in Section 3.6.

Let $G$ be a group. A $G$-graded algebra $A$ is a collection $\{A_g : g \in G\}$ of vector spaces, along with products $A_g \otimes A_h \to A_{gh}$ that induce an overall structure of an algebra on $A := \bigoplus_{g \in G} A_g$. In a similar vein, we define the notion of an $\mathcal{S}$-graded circuit algebra:

**Definition 4.12.** An $\mathcal{S}$-graded circuit algebra, or a "circuit algebra with skeletons", is an algebraic structure $C$ with spaces $C_\beta$, one for each element $\beta$ of the circuit algebra of skeletons $\mathcal{S}$, along with composition operations $D_{\beta_1,\dots,\beta_k} : C_{\beta_1} \times \cdots \times C_{\beta_k} \to C_\beta$, defined whenever $D$ is a wiring diagram and $\beta = D(\beta_1, \dots, \beta_k)$, so that with the obvious induced structure, $\coprod_\beta C_\beta$ is a circuit algebra. A similar definition can be made if/when the skeletons are taken to be directed or coloured.

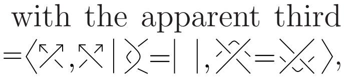
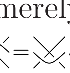
Loosely speaking, a circuit algebra with skeletons is a circuit algebra in which every element $T$ has a well-defined skeleton $\varsigma(T) \in \mathcal{S}$. Yet note that as an algebraic structure a circuit algebra with skeletons has more "spaces" than an ordinary circuit algebra, for its spaces are enumerated by skeleta and not merely by integers. The prime examples for circuit algebras with skeletons appear in the next section.

## 5. w-Tangles

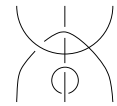
**Section Summary.** In 5.1 we introduce v-tangles and w-tangles, the obvious v- and w- counterparts of the standard knot-theoretic notion of "tangles", and briefly discuss their finite type invariants and their associated spaces of "arrow diagrams", $\mathcal{A}^v(\uparrow_n)$ and $\mathcal{A}^w(\uparrow_n)$. We then construct a homomorphic expansion, or a "well-behaved" universal finite type invariant for w-tangles. Once again, the only algebraic tool we need to use is $\exp(a) := \sum a^n/n!$, and indeed, Section 5.1 is but a routine extension of parts of Section 3. We break away in 5.2 and show that $\mathcal{A}^w(\uparrow_n) \cong \mathcal{U}(\mathfrak{a}_n \oplus \mathfrak{tder}_n \ltimes \mathfrak{tr}_n)$, where $\mathfrak{a}_n$ is an Abelian algebra of rank $n$ and where $\mathfrak{tder}_n$ and $\mathfrak{tr}_n$, two of the primary spaces used by Alekseev and Torossian [AT], have simple descriptions in terms of words and free Lie algebras. In 5.3 we discuss a subclass of w-tangles called "special" w-tangles, and relate them by similar means to Alekseev and Torossian's $\mathfrak{sder}_n$ and to "tree level" ordinary Vassiliev theory.

### 5.1. v-Tangles and w-Tangles.

With the (surprisingly pleasant) task of defining circuit algebras completed in Section 4.4, the definition of v-tangles and w-tangles is simple.

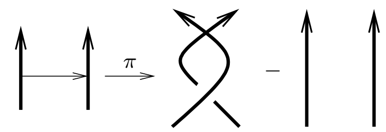
**Definition 5.1.** The ($\mathcal{S}$-graded) circuit algebra $\mathit{vT}$ of v-tangles is the $\mathcal{S}$-graded directed circuit algebra generated by two generators in $C_{2,2}$ called the "positive crossing" and the "negative crossing", modulo the usual R2 and R3 moves as depicted in Figure 6 (these relations clearly make sense as circuit algebra relations between our two generators), with the obvious meaning for their skeleta. The circuit algebra $\mathit{wT}$ of w-tangles is the same, except we also mod out by the OC relation of Figure 6 (note that each side in that relation involves only two generators, with the apparent third crossing being merely a projection artifact). With less words, $vT :=$ , and $wT := vT/$ .

*Remark* 5.2. One may also define v-tangles and w-tangles using the language of planar algebras, except then another generator is required (the "virtual crossing") and also a few further relations (VR1–VR3, M), and some of the operations (non-planar wirings) become less elegant to define.



Our next task is to study the projectivizations proj $\mathit{vT}$ and proj $\mathit{wT}$ of $\mathit{vT}$ and $\mathit{wT}$. Again, the language of circuit algebras makes it exceedingly simple.

**Definition 5.3.** The ($\mathcal{S}$-graded) circuit algebra $\mathcal{D}^v = \mathcal{D}^w$ of arrow diagrams is the graded and $\mathcal{S}$-graded directed circuit algebra generated by a single degree 1 generator $a$ in $C_{2,2}$ called "the arrow" as shown on the right, with the obvious meaning for its skeleton. There are morphisms $\pi : \mathcal{D}^v \to \mathit{vT}$ and $\pi : \mathcal{D}^w \to \mathit{wT}$ defined by mapping the arrow to an overcrossing minus a no-crossing. (On the right some virtual crossings were added to make the skeleta match). Let $\mathcal{A}^v$ be $\mathcal{D}^v/6T$ and let $\mathcal{A}^w := \mathcal{A}^v/TC = \mathcal{D}^w/\overrightarrow{4T}, TC$, with $6T$, $\overrightarrow{4T}$, and $TC$ being the same relation as in Figures 8 and 9 (allowing skeleta parts that are not explcitly connected to really lie on separate skeleton components).




**Proposition 5.4.** *The maps $\pi$ above induce surjections $\pi : \mathcal{A}^v \to$ proj $\mathit{vT}$ and $\pi : \mathcal{A}^w \to$ proj $\mathit{wT}$. Hence in the language of Definition 4.7, $\mathcal{A}^v$ and $\mathcal{A}^w$ are candidate projectivizations of $\mathit{vT}$ and $\mathit{wT}$.*

*Proof.* MORE

We do not know if $\mathcal{A}^v$ is indeed the projectivizations of $\mathit{vT}$ (also see [BHLR]). Yet in the w case, the picture is simple:

**Theorem 5.5.** *The assignment $\overcrossing \mapsto e^a$ (with an obvious interpretation for $e^a$) extends to a well defined $Z : \mathit{wT} \to \mathcal{A}^w$. The resulting map $Z$ is a homomorphic $\mathcal{A}^w$-expansion, and in particular, $\mathcal{A}^w \cong$ proj $\mathit{wT}$ and $Z$ is a homomorphic expansion.*

*Proof.* There is nothing new here. $Z$ is satisfies the Reidemeister moves for the same reasons as in Theorem 2.15 and Theorem 3.11 and as there it also satisfies the universallity property. The rest follows from Proposition 4.8. □

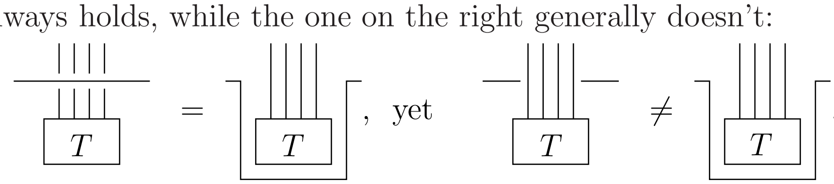
In a similar spirit to Definition 3.13, one may define a "w-Jacobi diagram" (often shorts to "arrow diagram") on an arbitrary skeleton. Denote the circuit algebra of formal linear combinations of arrow diagrams by $\mathcal{A}^{wt}$. We have the following bracket-rise theorem:

**Theorem 5.6.** *The obvious inclusion of diagrams induces a circuit algebra isomorphism $\mathcal{A}^w \cong \mathcal{A}^{wt}$. Furthermore, the $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations of Figure 12 hold in $\mathcal{A}^{wt}$.*

*Proof.* The proof of Theorem 3.15 can be repeated verbatim. Note that that proof does not make use of the connectivity of the skeleton.

Given the above theorem, we no longer keep the distinction between $\mathcal{A}^w$ and $\mathcal{A}^{wt}$.

*Remark* 5.7. Note that if $T$ is an arbitrary $w$ tangle, then the equality on the left side of the figure below always holds, while the one on the right generally doesn't:



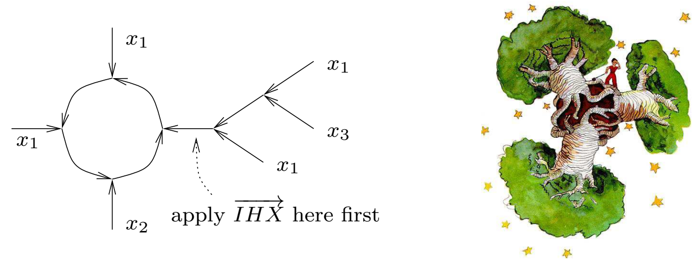
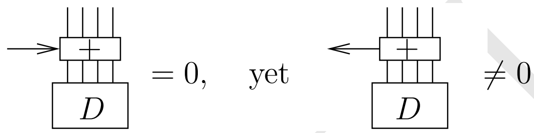



**Figure 22.** A wheel of trees can be reduced to a combination of wheels, and a wheel of trees with a Little Prince.

The arrow diagram version of this statement is that if $D$ is an arbitrary arrow diagram in $\mathcal{A}^w$, then the left side equality in the figure below always holds (we will sometimes refer to this as the "head-invariance" of arrow diagrams), while the right side equality generally fails.




We skip the proof of this fact; see an analogous one in [BN3], Lemma 3.4.

5.2. $\mathcal{A}^w(\uparrow_n)$ **and the Alekseev-Torossian Spaces.**

**Definition 5.8.** Let $v\mathcal{T}(\uparrow_n)$ (likewise $w\mathcal{T}(\uparrow_n)$) be the set of v-tangles (w-tangles) whose skeleton is the disjoint union of $n$ directed lines. Likewise let $\mathcal{A}^v(\uparrow_n)$ and $\mathcal{A}^w(\uparrow_n)$ be the parts of $\mathcal{A}^v$ and $\mathcal{A}^w$ in which the skeleton is the disjoint union of $n$ directed lines.

In the same manner as in the case of knots (Theorem 3.16), $\mathcal{A}^w(\uparrow_n)$ is a bi-algebra isomorphic (via a diagrammatic PBW theorem, applied independently on each component of the skeleton) to a space $\mathcal{B}^w(\star_n)$ of unitravelent diagrams with symmetrized ends coloured with colours in some $n$-element set (say $\{x_1, \ldots, x_n\}$), modulo $\overrightarrow{AS}$ and $\overrightarrow{IHX}$. The primitives $\mathcal{P}_n$ of $\mathcal{B}^w(\star_n)$ are the connected diagrams (and hence the primitives of $\mathcal{A}^w(\uparrow_n)$ are the diagrams that remain connected even when the skeleton is removed). Given the "two in one out" rule for internal vertices, the diagrams in $\mathcal{P}_n$ can only be trees or wheels ("wheels of trees" can be reduced to simple wheels by repeatedly using $\overrightarrow{IHX}$, as in Figure 22).

Thus $\mathcal{P}_n$ is easy to identify. It is a direct sum $\mathcal{P}_n = \langle\text{trees}\rangle \oplus \langle\text{wheels}\rangle$. The wheels part is simply the vector space generated by all cyclic words in the letters $x_1, \ldots, x_n$. Alekseev and Torossian [AT] denote this space $\mathfrak{tr}_n$, and so shall we. The trees in $\mathcal{P}_n$ have leafs coloured $x_1, \ldots, x_n$. Modulo $\overrightarrow{AS}$ and $\overrightarrow{IHX}$, they correspond to elements of the free Lie algebra $\mathfrak{lie}_n$ on the generators $x_1, \ldots, x_n$. But the root of each such tree also carries a label in $\{x_1, \ldots, x_n\}$, hence there are $n$ types of such trees as separated by their roots, and so $\mathcal{P}_n$ is isomorphic to the direct sum $\mathfrak{tr}_n \oplus \bigoplus_{i=1}^n \mathfrak{lie}_n$ of $\mathfrak{tr}_n$ and $n$ copies of $\mathfrak{lie}_n$.

By the Milnor-Moore theorem [MM], $\mathcal{A}^w(\uparrow_n)$ is isomrphic to the universal enveloping algebra $\mathcal{U}(\mathcal{P}_n)$, with $\mathcal{P}_n$ identified as a subspace of $\mathcal{A}^w(\uparrow_n)$ using the PBW symmetrization map $\chi : \mathcal{B}(\star_n) \to \mathcal{A}^w(\uparrow_n)$. Thus in order to understand $\mathcal{A}^w(\uparrow_n)$ as an associative algebra, it is enough to understand the Lie algebra structure induced on $\mathcal{P}_n$ via the commutator bracket of $\mathcal{A}^w(\uparrow_n)$.

We now wish to identify $\mathcal{P}(\uparrow_n)$ as the Lie algebra $\mathfrak{tr}_n \rtimes (\mathfrak{a}_n \oplus \mathfrak{tder}_n)$, which in itself is a combination of the Lie algebras $\mathfrak{a}_n$, $\mathfrak{tder}_n$ and $\mathfrak{tr}_n$ studied by Alekseev and Torossian [AT]. Here are the relevant definitions:

**Definition 5.9.** Let $\mathfrak{a}_n$ denote the vector space with basis $x_1, \ldots, x_n$, also regarded as an Abelian Lie algebra of dimension $n$. As before, let $\mathfrak{lie}_n = \mathfrak{lie}(\mathfrak{a}_n)$ denote the free Lie algebra on $n$ generators, now identified as the basis elements of $\mathfrak{a}_n$. Let $\mathfrak{der}_n = \mathfrak{der}(\mathfrak{lie}_n)$ be the (graded) Lie algebra of derivations acting on $\mathfrak{lie}_n$, and let

$$\mathfrak{tder}_n = \{D \in \mathfrak{der}_n : \forall i \ \exists a_i \text{ s.t. } D(x_i) = [x_i, a_i]\}$$

denote the subalgebra of "tangential derivations". A tangential derivation $D$ is determined by the $a_i$'s for which $D(x_i) = [x_i, a_i]$, and determines them up to the ambiguity $a_i \mapsto a_i + \alpha_i x_i$, where the $\alpha_i$'s are scalars. Thus as vector spaces, $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_{i=1}^n \mathfrak{lie}_n$.

**Definition 5.10.** Let $\mathrm{Ass}_n = \mathcal{U}(\mathfrak{lie}_n)$ be the free associative algebra "of words", and let $\mathrm{Ass}_n^+$ be the degree $> 0$ part of $\mathrm{Ass}_n$. As before, we let $\mathfrak{tr}_n = \mathrm{Ass}_n^+/(x_{i_1}x_{i_2}\cdots x_{i_m} = x_{i_2}\cdots x_{i_m}x_{i_1})$ denote "cyclic words" or "(coloured) wheels". $\mathrm{Ass}_n$, $\mathrm{Ass}_n^+$, and $\mathfrak{tr}_n$ are $\mathfrak{tder}_n$-modules and there is an obvious equivariant "trace" $\mathrm{tr} : \mathrm{Ass}_n^+ \to \mathfrak{tr}_n$.

**Proposition 5.11.** *There is a short exact sequence of Lie algebras*

$$0 \longrightarrow \mathfrak{tr}_n \xrightarrow{\iota} \mathcal{P}(\uparrow_n) \xrightarrow{\pi} \mathfrak{a}_n \oplus \mathfrak{tder}_n \longrightarrow 0.$$

*Proof.* The inclusion $\iota$ is defined the natural way: $\mathfrak{tr}_n$ is spanned by coloured "floating" wheels, and such a wheel is mapped into $\mathcal{P}_n$ by attaching its legs to their assigned strands in arbitrary order. Note that this is well-defined: wheels have only tails, and tails commute.

As vector spaces, the statement is already proven: $\mathcal{P}(\uparrow_n)$ is generated by trees and wheels (with the legs fixed on $n$ strands). When factoring out by the wheels, only trees remain. Trees have one head and many tails. All the tails commute with each other, and commuting a tail with a head on a strands costs a wheel (by $\overrightarrow{STU}$), thus in the quotient the head also commutes with the tails. Therefore, the quotient is the space of floating (coloured) trees, which we have previously identified with $\bigoplus_{i=1}^n \mathfrak{lie}_n \cong \mathfrak{a}_n \oplus \mathfrak{tder}_n$.

It remains to show that the maps are Lie algebra maps as well. For $\iota$ this is easy: the Lie algebra $\mathfrak{tr}_n$ is commutative, and is mapped to the commutative (due to $TC$) subalgebra of $\mathcal{P}(\uparrow_n)$ generated by wheels.

To show that $\pi$ is a map of Lie algebras, we give two proofs, first a "hands-on" one, then a "conceptual" one.

**Hands-on argument.** $\mathfrak{a}_n$ is the image of single arrows on one strand. These commute with everything in $\mathcal{P}(\uparrow_n)$, and so does $\mathfrak{a}_n$ in the direct sum.

It remains to show that the bracket of $\mathfrak{tder}_n$ works the same way as commuting trees in $\mathcal{P}(\uparrow_n)$. Let $D$ and $D'$ be elements of $\mathfrak{tder}_n$ represented by $(a_1, ..., a_n)$ and $(a_1', ..., a_n')$, meaning that $D(x_i) = [x_i, a_i]$ and $D'(x_i) = [x_i, a_i']$ for $i = 1, ..., n$. Let us compute the commutator of these elements:

$$[D, D'](x_i) = (DD' - D'D)(x_i) = D[x_i, a_i'] - D'[x_i, a_i] =$$

$$= [[x_i, a_i], a_i'] + [x_i, Da_i'] - [[x_i, a_i'], a_i] - [x_i, D'a_i] =$$

$$= [x_i, Da_i'] - [x_i, D'a_i] + [x_i, [a_i, a_i']] = [x_i, Da_i' - D'a_i + [a_i, a_i']].$$

Here the third equality is due to the Leibnitz rule of drivations, while the fourth is a Jacobi identity.

Now let $T$ and $T'$ be two trees in $\mathcal{P}(\uparrow_n)/\mathfrak{tr}_n$, their heads on strands $i$ and $j$, respectively ($i$ may or may not equal $j$). Let us call the element in $\mathfrak{lie}_n$ given by forming the appropriate commutator of the colors of $T$'s leaves $a_i$, and similarly $a'_j$ for $T'$. In $\mathfrak{tder}_n$, let $D = \pi(T)$ and $D' = \pi(T')$. $D$ and $D'$ are determined by $(0, ..., a_i, ..., 0)$, and $(0, ..., a'_j, ...0)$, respectively. (In each case, the $i-th$ or the $j-th$ is the only non-zero component.) The commutator of these elements is given by $[D, D'](x_i) = [Da'_i - D'a_i + [a_i, a'_i], x_i]$, and $[D, D'](x_j) = [Da'_j - D'a_j + [a_j, a'_j], x_j]$. Note that unless $i = j$, $a_j = a'_i = 0$.

In $\mathcal{P}(\uparrow_n)/\mathfrak{tr}_n$, all tails commute, as well as a head of a tree with its own tails. Therefore, commuting two trees only incurs a cost when commuting a head of one tree over the tails of the other on the same strand, and the two heads over each other, if they are on the same strand.

If $i \neq j$, then commuting the head of $T$ over the tails of $T'$ by $\overrightarrow{STU}$ costs a sum of trees given by $Da'_j$, with heads on strand $j$, while moving the head of $T'$ over the tails of $T$ costs exactly $-D'a_i$, with heads on strand $i$, as needed.

If $i = j$, then everything happens on starnd $i$, and the cost is $(Da'_i - D'a_i + [a_i, a'_i])$, where the last term is what happens when the two heads cross each other.

**Conceptual argument.** There is an action of $\mathcal{P}(\uparrow_n)$ on $\mathfrak{lie}_n$, the following way: introduce and extra strand on the right. An element of $\mathfrak{lie}_n$ corresponds to a tree with its head on the extra strand. Its commutator with an element of $\mathcal{P}(\uparrow_n)$ (considered as an element of $\mathcal{P}(\uparrow_{n+1})$ by the obvious inclusion) is again a tree with head on strand $(n+1)$, defined to be the result of the action.

The tree we are acting on has only tails on the first $n$ strands, so elements of $\mathfrak{tr}_n$, which also only have tails, act trivially. So do single (local) arrows on one strand ($\mathfrak{a}_n$). It remains to show that trees act as $\mathfrak{tder}_n$, and it's enough to check this on the generators of $\mathfrak{lie}_n$ (as the Leibnitz rule is obviously satisfied). The generators of $\mathfrak{lie}_n$ are arrows pointing from one of the first $n$ strands, say strand $i$, to strand $(n+1)$. A tree with head on strand $i$ acts on this element, according $\overrightarrow{STU}$, by forming the commutator, which is exactly the action of $\mathfrak{tder}_n$.

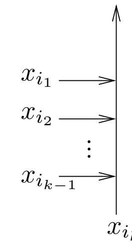
To identify $\mathcal{P}(\uparrow_n)$ as the semidirect product $\mathfrak{tr}_n \rtimes (\mathfrak{a}_n \oplus \mathfrak{tder}_n)$, it remains to show that the short exact sequence above splits. This is indeed the case, although not canonically. Two — of the many— splitting maps $u, l : \mathfrak{tder}_n \oplus \mathfrak{a}_n \to \mathcal{P}(\uparrow_n)$ are described as follows: $\mathfrak{tder}_n \oplus \mathfrak{a}_n$ is identified with $\bigoplus_{i=1}^n \mathfrak{lie}_n$, which in turn is identified with floating (coloured) trees (including arrows). A map to $\mathcal{P}(\uparrow_n)$ can be given by specifying how to place the legs on thir specified strands. A tree may have many tails but has only one head, and due to $TC$, only the positioning of the head matters. Let $u$ (for *upper*) be the map placing the head of each tree above all its tails on the same strand, while $l$ (for *lower*) places the head below all the tails. It is obvious that these are both Lie algebra maps and that $\pi \circ u$ and $\pi \circ l$ are both the identity of $\mathfrak{tder}_n \oplus \mathfrak{a}_n$. This makes $\mathcal{P}(\uparrow_n)$ a semidirect product. □

**Definition 5.12.** For any $D \in \mathfrak{tder}_n$, $(l-u)D$ is in the kernel of $\pi$, therefore is in the image of $\iota$, so $\iota^{-1}(l-u)D$ makes sense. We call this element $\mathrm{div}D$.

**Definition 5.13.** [AT] define div the following way: $\mathrm{div}(a_1, ..., a_n) := \sum_{k=1}^n \mathrm{tr}((\partial_k a_k)x_k)$, where $\partial_k$ picks out the words of a sum which end in $x_k$ and deletes their last letter $x_k$, and deletes all other words (the ones which do not end in $x_k$).

**Proposition 5.14.** *The div of Definition 5.12 the div of* [AT] *are the same.*

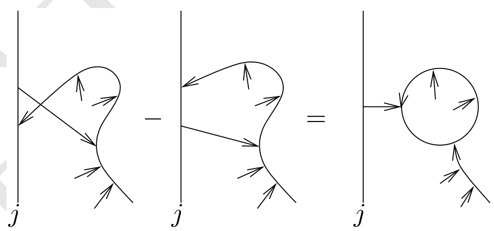
*Proof.* It is enough to verify the claim for the linear generators of $\mathfrak{tder}_n$, namely, elements of the form $(0, ..., a_j, ..., 0)$, where $a_j \in \mathfrak{lie}_n$ or equivalently, single (floating, colored) trees, where the color of the head is $j$. By the Jacobi identity, each $a_j$ can be written in a form $a_j = [x_{i_1}, [x_{i_2}, [..., x_{i_k}]...]$. Equivalently, by $\overrightarrow{IHX}$, each tree has a standard "comb" form, as shown on the picture on the right.




For an associative word $Y = y_1 y_2 ... y_l \in \mathrm{Ass}_n^+$, we introduce the notation $[Y] := [y_1, [y_2, [..., y_l]...]$. The div of [AT] picks out the words that end in $x_j$, forgets the rest, and considers these as cyclic words. Therefore, by interpreting the Lie brackets as commutators, one can easily check that for $a_j$ written as above,

$$\mathrm{div}((0, ..., a_j, ..., 0)) = \sum_{\alpha : i_\alpha = x_j} -x_{i_1} ... x_{i_{\alpha-1}} [x_{i_{\alpha+1}} ... x_{i_k}] x_j. \tag{32}$$

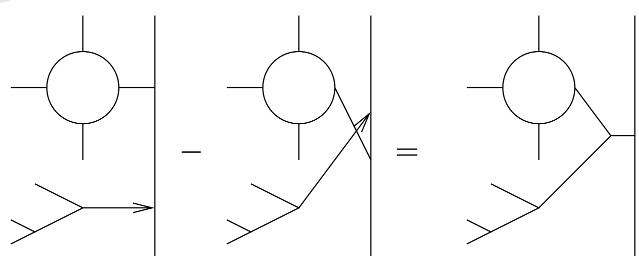
In Definition 5.12, div of a tree is the difference between attaching its head on the appropriate strand (here, strand $j$) below all of its tails and above. As shown in the figure on the right, moving the head across each of the tails on strand $j$ requires an $\overrightarrow{STU}$ relation, which "costs" a wheel (of trees, which is equivalent to a sum of honest wheels), namely, the head gets connected to the tail in question. So div of the tree represented by $a_j$ is given by




$$\sum_{\alpha : x_{i_\alpha} = j} \text{"connect the head to the } \alpha \text{ leaf"}.$$

This obviously gets mapped to the formula above via the correspondence between wheels and cyclic words. □

There is an action of $\mathfrak{tder}_n$ on $\mathfrak{tr}_n$ as follows. Represent a cyclic word $w \in \mathfrak{tr}_n$ as a wheel in $\mathcal{P}(\uparrow_n)$ via the map $\iota$. Given an element $D \in \mathfrak{tder}_n$, $u(D)$, as defined above, is a tree in $\mathcal{P}(\uparrow_n)$ whose head is above all of its tails. We define $D \cdot w := \iota^{-1}(u(D)\iota(w) - \iota(w)u(D))$. Note that $u(D)\iota(w) - \iota(w)u(D)$ is in the image of $\iota$, i.e., a linear combination of wheels: the wheel $\iota(w)$ has only tails. As we commute the tree $u(D)$ across the wheel, the head of the tree is commuted across tails of the wheel on the same strand. Each time this happens the cost, by the $\overrightarrow{STU}$ relation, is a wheel with the tree attached to it, as shown on the right, which in turn (by $\overrightarrow{IHX}$ relations, as Figure 22 shows) is a sum of wheels. Once the head of the tree has been moved to the top, the tails of the tree commute up for free by $TC$. Note that the alternative definition, $D \cdot w := \iota^{-1}(l(D)\iota(w) - \iota(w)l(D))$ is in fact equal to the definition above.




**Definition 5.15.** In [AT], the group $\mathrm{TAut}_n$ is defined as $\exp(\mathfrak{tder}_n)$. Note that $\mathfrak{tder}_n$ is positively graded, hence integrates to a group. Note also that $\mathrm{TAut}_n$ is the group of "basis-conjugating" automorphisms of $\mathfrak{lie}_n$, i.e., for $g \in \mathrm{TAut}_n$, and any $x_i$, $i = 1, ..., n$ generator of $\mathfrak{lie}_n$, there exists an element $g_i \in \exp(\mathfrak{lie}_n)$ such that $g(x_i) = g_i^{-1} x_i g_i$.

The action of $\mathfrak{tder}_n$ on $\mathfrak{tr}_n$ lifts to an action of $\mathrm{TAut}_n$ on $\mathfrak{tr}_n$, by interpreting exponentials formally, in other words $e^D$ acts as $\sum_{n=0}^{\infty} \frac{D^n}{n!}$. The lifted action is by conjugation: for $w \in \mathfrak{tr}_n$ and $e^D \in \mathrm{TAut}_n$, $e^D \cdot w = \iota^{-1}(e^{uD}\iota(w)e^{-uD})$.

Recall that in Proposition 5.1 of [AT] Alekseev and Torossian construct a map $j : \mathrm{TAut}_n \to \mathfrak{tr}_n$ which is characterised by two properties: the cocycle property

$$j(gh) = j(g) + g \cdot j(h), \tag{33}$$

where in the second term multiplication by $g$ denotes the action described above; and the condition

$$\frac{d}{ds} j(\exp(sD))|_{s=0} = \mathrm{div}(D). \tag{34}$$

Now let us interpret $j$ in our context.

**Definition 5.16.** The adjoint map $* : \mathcal{A}^w(\uparrow_n) \to \mathcal{A}^w(\uparrow_n)$ acts by "flipping over diagrams and negating arrow heads on the skeleton". In other words, for an arrow diagram $D$,

$$D^* := (-1)^{\#\{\text{tails on skeleton}\}} S(D),$$

where $S$ denotes the map which switches the orientation of the skeleton strands (i.e. flips the diagram over), and multiplies by $(-1)^{\#\text{skeleton vertices}}$.

**Proposition 5.17.** *For $D \in \mathfrak{tder}_n$, define a map $J : \mathrm{TAut}_n \to \exp(\mathfrak{tr}_n)$ by $J(e^D) := e^{uD}(e^{uD})^*$. Then*

$$\exp(j(e^D)) = J(e^D).$$

*Proof.* Note that $(e^{uD})^* = e^{-lD}$, due to "Tails Commute" and the fact that a tree has only one head.

Let us check that $\log J$ satisfies proerties 33 and 34. Namely, with $g = e^{D_1}$ and $h = e^{D_2}$, and using that $\mathfrak{tr}_n$ is commutative, we need to show that

$$J(e^{D_1}e^{D_2}) = J(e^{D_1})\big(e^{uD_1} \cdot J(e^{D_2})\big), \tag{35}$$

where $\cdot$ denotes the action of $\mathfrak{tder}_n$ on $\mathfrak{tr}_n$; and that

$$\frac{d}{ds} J(e^{sD})|_{s=0} = \mathrm{div}\, D. \tag{36}$$

Indeed, with $\mathrm{BCH}(D_1, D_2) = \log e^{D_1}e^{D_2}$ being the standard Baker–Campbell–Haussdorff formula,

$$J(e^{D_1}e^{D_2}) = J(e^{\mathrm{BCH}(D_1,D_2)}) == e^{u(\mathrm{BCH}(D_1,D_2)}e^{-l(\mathrm{BCH}(D_1,D_2)} = e^{\mathrm{BCH}(uD_1,uD_2)}e^{-\mathrm{BCH}(lD_1,lD_2)} =$$

$$= e^{uD_1}e^{uD_2}e^{-lD_2}e^{-lD_1} = e^{uD_1}(e^{uD_2}e^{-lD_2})e^{-uD_1}e^{uD_1}e^{lD_1} = (e^{uD_1} \cdot J(D_2))J(D_1),$$

which is what we needed.

As for condition 34, a direct computation of the derivative yields

$$\frac{d}{ds} J(e^{sD})|_{s=0} = uD - lD = \mathrm{div}\, D,$$

as desired.

5.3. **The Relationship with u-Tangles.** Let $u\!T$ be the planar algebra of There is an obvious map $a : u\!T \to w\!T$ of "usual" $u$-tangles into $w$-tangles. It induces a map $\alpha : \mathcal{A}^u \to \mathcal{A}^w$, which maps an ordinary Jacobi diagram (i.e., unoriented chords with internal trivalent vertices modulo the usual $AS$, $IHX$ and $STU$ relations) to the sum of all possible orientations of its chords (many of which are zero due to the "two in one out" rule).

$$\begin{array}{ccc} u\!T & \xrightarrow{Z^u} & \mathcal{A}^u \\ \downarrow a & & \downarrow \alpha \\ w\!T & \xrightarrow{Z^w} & \mathcal{A}^w \end{array}$$

It is tempting to ask whether the square on the left commutes. Unfortunately, this question hardly makes sense, as there is no canonical choice for the dotted line in it. Similarly to the braid case, the definition of the Kontsevich integral for $u$-tangles typically depends on various choices of "parenthetizations".

Yet we can recover something from that diagram: an interpretation of the Alekseev–Torossian space of special derivations, $\mathfrak{sder}_n$. Recall that according to [AT], $\mathfrak{sder}_n$ is the Lie algebra of elements $D \in \mathfrak{tder}_n$ for which $D(\sum_{i=1}^n x_i) = 0$.

Let $\mathcal{P}^u(\uparrow_n)$ denote the primitives of $\mathcal{A}^u(\uparrow_n)$, that is, Jacobi diagrams that remain connected when the skeleton is removed. Remember that $\mathcal{P}^w(\uparrow_n)$ stands for the primitives of $\mathcal{A}^w(\uparrow_n)$.

**Theorem 5.18.** *The image of the composition* $\mathcal{P}^u(\uparrow_n) \xrightarrow{\alpha} \mathcal{P}^w(\uparrow_n) \xrightarrow{\pi} \mathfrak{a}_n \oplus \mathfrak{tder}_n$ *is* $\mathfrak{a}_n \oplus \mathfrak{sder}_n$.

This theorem was first proven by Drinfel'd (Lemma after Proposition 6.1 in [Dr3]), but the proof we give here is due to Levine [Lev].

*Proof.* Let $\mathfrak{lie}_n^d$ denote the degree $d$ piece of $\mathfrak{lie}_n$. Let $V_n$ be the vector space with basis $x_1, x_2, ..., x_n$. Note that

$$V_n \otimes \mathfrak{lie}_n^d \cong \bigoplus_{i=1}^n \mathfrak{lie}_n^d \cong (\mathfrak{tder}_n \oplus \mathfrak{a}_n)^d,$$

where $\mathfrak{tder}_n$ is graded by the number of tails of a tree, and $\mathfrak{a}_n$ is contained in degree 1.

The bracket defines a map $\beta : V_n \otimes \mathfrak{lie}_n^d \to \mathfrak{lie}_n^{d+1}$: for $a_i \in \mathfrak{lie}_n^d$ where $i = 1, ..., n$, the "tree" $D = (a_1, a_2, ..., a_n) \in (\mathfrak{tder}_n \oplus \mathfrak{a}_n)^d$ is mapped to

$$\beta(D) = \sum_{i=1}^n [x_i, a_i] = D\left(\sum_{i=1}^n x_i\right),$$

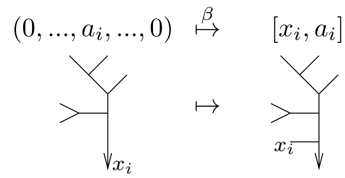
where the first equality is by the definition of tensor product and the bracket, and the second is by the definition of the action of $\mathfrak{tder}_n$ on $\mathfrak{lie}_n$.

Since $\mathfrak{a}_n$ is contained in degree 1, by definition $\mathfrak{sder}_n^d = \ker\beta$ for $d \geq 2$. In degree 1, $\mathfrak{a}_n$ is obviously in the kernel, hence $\ker\beta = \mathfrak{a}_n \oplus \mathfrak{sder}_n^1$. So overall, $\ker\beta = \mathfrak{a}_n \oplus \mathfrak{sder}_n$.

We want to study the image of the map $\mathcal{P}^u(\uparrow^n) \xrightarrow{\pi\alpha} \mathfrak{a}_n \oplus \mathfrak{tder}_n$. Under $\alpha$, all connected Jacobi diagrams that are not trees or wheels go to zero, and under $\pi$ so do all wheels. Furthermore, $\pi$ maps trees that live on $n$ strands to "floating" trees with univalent vertices coloured by the strand they used to end on. So for determining the image, we may replace $\mathcal{P}^u(\uparrow^n)$ by the space $\mathcal{T}_n$ of connected unoriented floating trees (uni-trivalent graphs), the leaves of which are coloured by the $x_i$, $i = 1, .., n$. We denote the degree $d$ piece of $\mathcal{T}_n$, i.e., the space of trees with $d+1$ leaves, by $\mathcal{T}_n^d$. Abusing notation, we shall denote the map induced by $\pi\alpha$ on $\mathcal{T}_n$ by $\alpha : \mathcal{T}_n \to \mathfrak{a}_n \oplus \mathfrak{tder}_n$. Since choosing a "head" determines the entire orientation of a tree by the two-in-one-out rule, $\alpha$ maps a tree in $\mathcal{T}_n^d$ to the sum of $d+1$ ways of choosing one of the leaves to be the head.

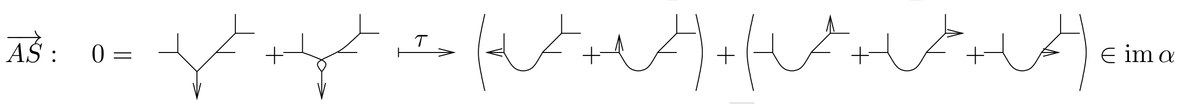
We want to show that $\ker\beta = \operatorname{im}\alpha$. This is equivalent to saying that $\bar{\beta}$ is injective, where $\bar{\beta} : \mathfrak{lie}_n \otimes \mathfrak{a}_n / \operatorname{im}\alpha \to \mathfrak{lie}_n$ is map induced by $\beta$ on the quotient by $\operatorname{im}\alpha$.

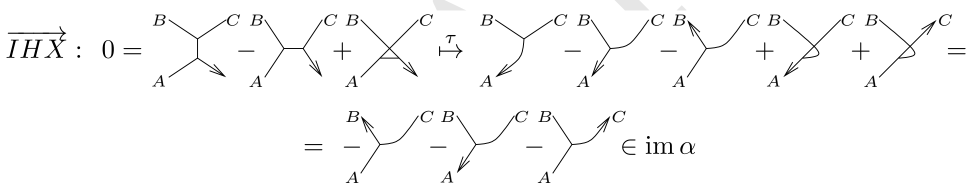
The degree $d$ piece of $V_n \otimes \mathfrak{lie}_n$, in the pictorial description, is generated by floating trees with $d$ tails and one head, all colored by $x_i$, $i = 1, ..., n$. This is mapped to $\mathfrak{lie}_n^{d+1}$, which is isomorphic to the space of floating trees with $n+1$ tails and one head, where only the tails are colored by the $x_i$. The map $\beta$ acts as shown on the picture on the right.




We show that $\bar{\beta}$ is injective by exhibiting a map $\tau : \mathfrak{lie}_n^{d+1} \to \mathfrak{lie}_n^{d} \otimes V_n / \operatorname{im}\alpha$ so that $\tau\bar{\beta} = I$. $\tau$ is defined as follows: given a tree with one head and $d+1$ tails $\tau$ acts by deleting the head and summing over all ways of chosing a head to the left of the original. As long as we show that $\tau$ is well-defined, it follows from the definition and the pictorial description of $\beta$ that $\tau\bar{\beta} = I$.

For well-definedness we need to check that the images of $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations under $\tau$ are in the image of $\alpha$. This we do in the picture below. In both cases it is enough to check the case when the "head" of the relation is the head of the tree itself, as otherwise an $\overrightarrow{AS}$ or $\overrightarrow{IHX}$ relation in the domain is mapped to an $\overrightarrow{AS}$ or $\overrightarrow{IHX}$ relation, thus zero, in the image.

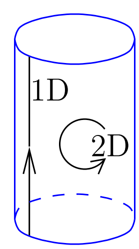



In the $\overrightarrow{IHX}$ picture, in higher degrees $A$, $B$ and $C$ may denote an entire tree. In this case, the arrow at $A$ (for example) means the sum of all head choices from the tree $A$. □

5.4. **The local topology of w-tangles.** So far throughout this section we have presented $w$-tanles as a Reidemeister theory: a circuit algebra given by generators and relations. It is not known whether this presentation represents tangled tubes in $\mathbb{R}^4$, however, this intuition explains the local relations (Reidemeister moves). The purpose of this subsection is to explain the local topology of crossings and understand orientations, signs and orientation reversals.

Our tubes are endowed with two orientations, we will call these the 1- and 2-dimensional orientations. The one dimensional orientation is the direction of the tube as a "strand" of the tangle. In other words, each tube has a "core"[30]: a distinguished line along the tube, which we require to be oriented as a 1-dimensional manifold. Furthermore, the tube as a 2-dimensional surface is oriented. An example is shown on the right.

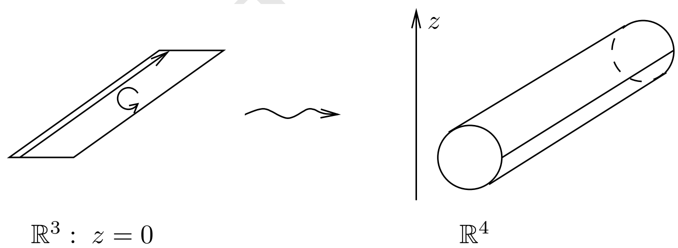



To understand crossings, note that an tube in $\mathbb{R}^4$ has a "filling": a solid (3 dimensional) cylinder embedded in $\mathbb{R}^4$, with boundary the tube, and the 2D orientation of the tube induces an orientation of its filling as a 3 dimensional manifold. A (non-virtual) crossing is when the core of one tube intersects the filling of another transversely. Due to the

[30]The core of Lord Voldemort's wand was made of a phoenix feather.

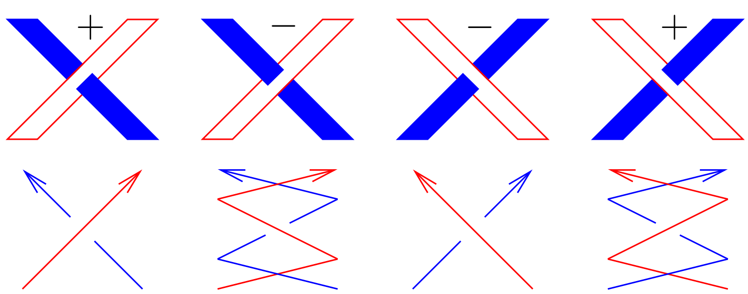
complementary dimensions, the intersection is a single point, and the 1D orientation of the core along with the 3D orientation of the filling it passes through determines an orientation of the ambient space. We say that the crossing is positive if this agrees with the standard orientation of $\mathbb{R}^4$, and negative otherwise. Hence, there are four types of crossings, given by whenther the core of tube A intersects the filling of B or vice versa, and two possible signs in each case.

As discussed in Section 2.2, braided tubes in $\mathbb{R}^4$ can be thought of as movies of flying rings in $\mathbb{R}^3$, and in particular a crossing represents a ring flying through another ring. In this interpretation, the 1D orientation of the tube is given by time moving forward. The 2D and 1D orientations of the tube tugether induce an orientation of the flying ring which is a cross-section of the tube at each moment. Hence, saying "below" and "above" the ring makes sense, and as mentioned in Exercise 2.7 there are four types of crossings: ring A flies through ring B from below or from above; and ring B flies through ring A from below or from above. A crossing is positive if the inner ring comes from below, and negative otherwise.




In Section 2.2 we have discussed how to translate from braid diagrams (of one dimensional strands with over- and undercrossing information) to braided tubes in $\mathbb{R}^4$: "under" represents "through". Locally, this works the same way for tangles. We take the opportunity here to introduce another notation, to be called the "band notation", which is more suggestive of the 4D topology. We represent an tube in $\mathbb{R}^4$ by a picture of an oriented band in $\mathbb{R}^3$. By "oriented band" we mean that it has two orientations: a 1D direction (for example an orientation of one of the edges), and a 2D orientation as a surface. To interpret the 3D picture of a band as an tube in $\mathbb{R}^4$, we add an extra coordinate. Let us refer to the $\mathbb{R}^3$ coordinates as $x, y$ and $t$, and to this extra one as $z$. Think of $\mathbb{R}^3$ as being embedded in $\mathbb{R}^4$ as the hyperplane $z = 0$, and think of the band as being made of a thin double membrane. Push the membrane up and down in the $z$ direction at each point as far as the distance of that point from the boundary of the band, as shown on the right. Furthermore, keep the 2D orientation of the top membrane (the one being pushed up), but reverse it on the bottom. This produces an oriented tube embedded in $\mathbb{R}^4$.

In band notation, the four possible crossings appear as follows, where underneath each crossing we indicate the corresponding strand picture, as mentioned in Exercise 2.7:




The signs for each type of crossing are shown in the figure above. Note that switching only the direction (1D orientation) of a strand changes the sign of the crossing if and only if the strand of changing direction is the through strand. However, fully changing the orientation (both 1D and 2D) always switches the sign of the crossing.

5.5. **Good properties and uniqueness.** In much the same way as in Section 2.5.1, $Z$ has a number of good properties with respect to various tangle operations: it is group-like; commutes with adding an inert strand; commutes with deleting a strand and with strand orientation reversals. All but the last of these were explained in the context of braids and the explanations still hold. Orientation reversal $S_k : w\!T \to w\!T$ is the operation which switches the full (1D and 2D) orientation of the $k$-th component. The induced diagrammatic operation $S_k : \mathcal{A}(T) \to \mathcal{A}(S_k(T))$ acts by multiplying each arrow diagram by $(-1)$ raised to the power the number of chord endings on the $k$-th strand. "$Z$ commutes with $S_k$" means that the appropriate square commutes.

The following theorem asserts that a well-behaved homomorphic expansion of $w$-tangles is unique:

**Theorem 5.19.** *The only homomorphic expansion satisfying the good properties described above is the $Z$ defined in Section 5.1.*

*Proof.* We first prove the following claim: Assume, by contradiction, that $Z'$ is a different homomorphic expansion of $w$-tangles with the good properties described above. Let $R' = Z'(\nearrow\!\!\!\!\nwarrow)$ and $R = Z(\nearrow\!\!\!\!\nwarrow)$, and denote by $\rho$ the lowest degree homogeneous non-vanishing term of $R' - R$. (Note that $R'$ determines $Z'$, so if $Z' \neq Z$, then $R' \neq R$.) Suppose $\rho$ is of degree $k$. Then we claim that $\rho = \alpha_1 w_k^1 + \alpha_2 w_k^2$ is a linear combination of $w_k^1$ and $w_k^2$, where $w_k^i$ denotes a $k$-wheel living on strand $i$.

Before proving the claim, note that it leads to a contradiction: up to degree $k$ $d_1 R' = \alpha_2 w_k^1$ and $d_2 R' = \alpha_1 w_k^2$, where $d_i$ denotes the operation "deleting strand $i$". But $Z'$ is compatible with strand deletions, so $\alpha_1 = \alpha_2 = 0$. Hence $Z$ is unique, as stated.

On to the proof of the claim, note that $Z'$ being an expansion determines the degree 1 term of $R'$ (namely, the single arrow $a$ with coefficient 1), so we can assume that $k \geq 2$. Note also that since both $R'$ and $R$ are group-like, $\rho$ is primitive. Hence $\rho$ is a linear combination of connected diagrams, namely trees and wheels.

Both $R$ and $R'$ satisfy the Reidemeister 3 relation:

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}, \qquad R'^{12}R'^{13}R'^{23} = R'^{23}R'^{13}R'^{12}$$

where the superscripts denote the strands on which $R$ is placed (compare with Remark 2.16). We focus our attention on the degree $k+1$ part of the equation for $R'$, and use that up to degree $k+1$, $R' = R + \rho + \mu$, where $\mu$ denotes the degree $k+1$ homogeneous part of $R' - R$. Thus, up to degree $k+1$, we have

$$(R^{12}+\rho^{12}+\mu^{12})(R^{13}+\rho^{13}+\mu^{13})(R^{23}+\rho^{23}+\mu^{23}) = (R^{23}+\rho^{23}+\mu^{23})(R^{13}+\rho^{13}+\mu^{13})(R^{12}+\rho^{12}+\mu^{12}).$$

The homogeneous degree $k+1$ part of this equation is a sum of some terms which contain $\rho$ and some which don't. The diligent reader can check that those which don't involve $\rho$ cancel on both sides, either due to the fact that $R$ satisfies the Reidemeister 3 relation, or by simple degree counting. Rearranging all the terms which do involve $\rho$ to the left side, we get the following equation:

$$[a^{12}, \rho^{13}] + [\rho^{12}, a^{13}] + [a^{12}, \rho^{23}] + [\rho^{12}, a^{23}] + [a^{13}, \rho^{23}] + [\rho^{13}, a^{23}] = 0. \tag{37}$$

Here $a^{ij}$ denotes an arrow pointing from strand $i$ to strand $j$.

The third and fifth terms sum to $[a^{12}+a^{13},\rho^{23}]$, which is zero due to the "head-invariance" of diagrams, as in Remark 5.7.

We treat the tree and wheel parts of $\rho$ separately. Let us first assume that $\rho$ is a linear combination of trees. Recall that the space of trees on two strands is isomorphic to $\mathfrak{lie}_2 \oplus \mathfrak{lie}_2$, the first component given by trees whose head is on the first srtand, and the second component by trees with their head on the second strand. Let $\rho = \rho_1 + \rho_2$, where $\rho_i$ is the projection to the $i$-th component for $i = 1, 2$.

Note that due to $TC$, $[a^{12}, \rho_2^{13}] = [\rho_2^{12}, a^{13}] = [\rho_1^{12}, a^{23}] = 0$. So Equation (37) reduces to

$$[a^{12}, \rho_1^{13}] + [\rho_1^{12}, a^{13}] + [\rho_2^{12}, a^{23}] + [\rho_1^{13}, a^{23}] + [\rho_2^{13}, a^{23}] = 0$$

The left side of this equation lives in $\bigoplus_{i=1}^{3} \mathfrak{lie}_3$. Notice that only the first term lies in the second direct sum component, while the second, third and last terms live in the third one, and the fourth term lives in the first. This means that the first term is itself zero. By $\overrightarrow{STU}$, this implies

$$0 = [a^{12}, \rho_1^{13}] = -[\rho_1, x_1]_2^{13},$$

where by $[\rho_1, x_1]_2^{13}$ we mean the tree defined by the element $[\rho_1, x_1] \in \mathfrak{lie}_2$, with its tails on strands 1 and 3, and head on strand 2. Hence, $[\rho_1, x_1] = 0$, so $\rho_1$ is a multiple of $x_1$. The tree $\rho_1 = x_1$ is a degree 1 element, a possibility we have eliminated, so $\rho_1 = 0$.

Equation (37) is now reduced to

$$[\rho_2^{12}, a^{23}] + [\rho_2^{13}, a^{23}] = 0.$$

Both terms are words in $\mathfrak{lie}_3$, but notice that the first term does not involve the letter $x_3$. This means that if the second term involves $x_3$ at all, i.e., if $\rho_2$ has tails on the second strand, then both terms have to be zero individually. Looking at the first term, we view $\rho_2^{12}$ as a Lie word in $x_1$ and $x_2$, which does involve $x_2$ by our assumption above. We have $[\rho_2^{12}, a^{23}] = [x_2, \rho_2^{12}] = 0$, which implies $\rho_2^{12}$ is a multiple of $x_2$, in other words, $\rho$ is a single arrow on the second strand. This is ruled out by the assumption that $k \geq 2$.

If the second term does not involve $x_3$ at all, then $\rho_2$ has no tails on the second strand, hence it is of degree 1, but again $k \geq 2$. We have proven that the "tree part" of $\rho$ is zero.

Now assume that $\rho$ is a linear combination of wheels. Wheels have only tails, so the first, second and fourth terms of (37) are zero due to the tails commute relation. What remains is $[\rho^{13}, a^{23}] = 0$. We assert that this is true if and only if each linear component of $\rho$ has all of its tails on one strand.

To prove this, recall each wheel of $\rho^{13}$ represents a cyclic word in letters $x_1$ and $x_3$. The map $r : \rho^{13} \mapsto [\rho^{13}, a^{23}]$ is a map $\mathfrak{tr}_2 \to \mathfrak{tr}_3$, which sends each cyclic word in letters $x_1$ and $x_3$ to the sum of all ways of substituting $[x_2, x_3]$ for one of the $x_3$'s in the word. Note that if we write out the commutators, then all terms that have $x_2$ between two $x_3$'s cancel. Hence all remaining terms will be cyclic words in $x_1$ and $x_3$ with one occurrence of $x_2$ in between an $x_1$ and an $x_3$.

We construct an almost-inverse $r'$ to $r$: for a cyclic word $w$ in $\mathfrak{tr}_3$ with one occurrence of $x_2$, let $r'$ be the map that deletes $x_2$ from $w$ and maps it to the resulting word in $\mathfrak{tr}_2$ if $x_2$ is followed by $x_3$ in $w$, and maps it to 0 otherwise. On the rest of $\mathfrak{tr}_3$ the map $r'$ may be defined to be 0.

The composition $r'r$ takes a cyclic word in $x_1$ and $x_3$ to itself multiplied by the number of times a letter $x_3$ follows a letter $x_1$ in it. The kernel of this map can consist only of cyclic words that do not contain the sub-word $x_3x_1$; namely, these are the words of the form $x_3^k$ or

$x_1^k$. Such words are indeed in the kernel of $r$, so these make up exactly the kernel of $r$. This is what we wanted to prove: all wheels in the "wheel part" have all their tails on one strand.

This concludes the proof of the claim, and the proof of the theorem.

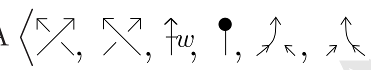
MORE.

## 6. w-Tangled Foams

If you have come this far, you must have noticed the approximate Bolero spirit of this article. In every chapter a new instrument comes to play; the overall theme remains the same, but the composition is more and more intricate. In this chapter we add "foam vertices" to our w-tangles (and a few lesser things as well) and ask the same questions we asked before; primarily, "is there a homomorphic expansion?". As we shall see, in the current context this question is more or less equivalent (details to come) to the Alekseev-Torossian [AT] version of the Kashiwara-Vergne [KV] problem.

6.1. **The Circuit Algebra of w-Tangled Foams.** For reasons we will reluctantly acknowledge at the end of this section (see Comment 6.2), we will present the circuit algebra of w-tangled foams via its Reidemeister-style diagrammatic description (accompanied by a local topological interpretation) rather than as an entirely topological construct.

**Definition 6.1.** Let *wTF* be the algebraic structure

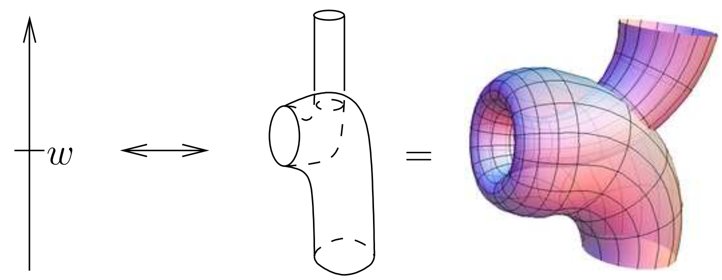
$wTF = \mathrm{CA}\Big\langle$  $\Big|$ w-relations as in Section 6.1.2 $\Big|$ w-operations as in Section 6.1.3 $\Big\rangle$.

Hence *wTF* is the circuit algebra generated by the generators listed above and described below, modulo the relations described in Section 6.1.2, and augmented with several "auxiliary operations", which are a part of the algebraic structure of *wTF* but are not a part of its structure as a circuit algebra, as described in Section 6.1.3.

6.1.1. *The generators of wTF.* There is topological meaning to each of the generators of *wTF*: they each stand for a certain local feature of (framed) knotted ribbon tubes in $\mathbb{R}^4$. As in Section 5.4, we require the tubes to be oriented as 2-dimensional surfaces, and also to have a distinguished core with a 1-dimensional orientation (direction).

The crossings are as explained in Section 2.2.2 and Section 5.4: the under-strand denotes the ring flying through, or the "thin" tube. Remember that there really are four kinds of crossing, but the other two are generated by these two in the circuit algebra.

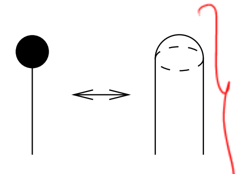
The third generator denotes the *wen*, which was introduced in 2.5.4, amounts to changing the 2D orientation of the tube, and is shown in the picture below:



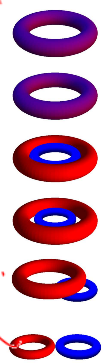

~~NEW.~~

The dotted end denotes a cap on the tube, as in the figure on the right.

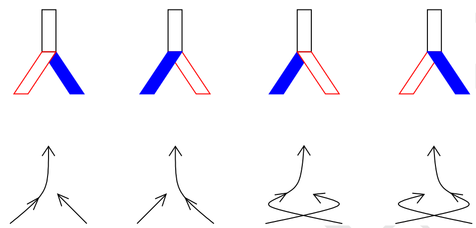
\raisebox. . .



We should lower the resolution.



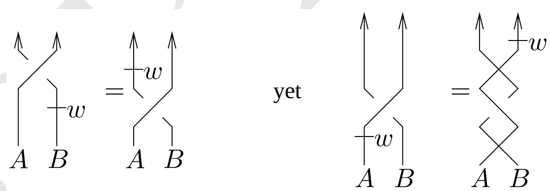
The last two generators denote singular vertices. As the notation suggests, a vertex can be thought of as "half of a crossing". To make this precise using the flying rings interpretation, the first singular vertex represents the movie shown on the left: the ring corresponding to the right strand approaches the ring represented by the left strand from below, flies inside it, and then the two rings merge (as opposed to a crossing where the ring coming from the right would continue to fly out to above and to the left of the other one). The second vertex is the movie where the left ring approaches the right ring from above, flies inside it, and they merge. The reader might object that there really are four types of vertices (the same way as there are four types of crossings). However, two of these are the vertices above composed with virtual crossings, which are circuit algebra artifacts. The picture below illustrates the four types of vertices in band and strand notation. The band notation here is the same as it is for verices: the fully colored band stands for the thin (inner) ring.





6.1.2. *The relations of wTF.* In addition to the usual R2, R3, and OC moves of Figure 6, we need more rlations to describe the behavior of the additional features. These are as follows:

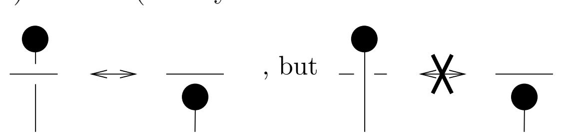
The interaction of a wen and a crossing has already been mentioned in Section 2.5.4, and is described by Equation (16), which we repeat hear for convenience:




Recall that in flying ring language, a wen is a ring flipping over. It does not matter whether ring B flips first and then flies through ring A or vice versa. However, the movies in which ring A first flips and then ring B flies through it, or B flies through A first and then A flips differ in the fly-through direction, which is canceled by virtual crossings, as in the picture. We will refer to these relations as the Flip Relations, and abbreviate them by *FR*.

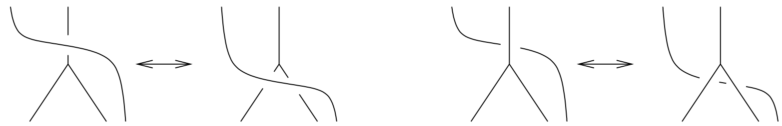
A double flip is homotopic to no flip, in other words the square of a wen is trivial, $w^2 = 1$. We will call this the wen relation and denote it by $W$. Note that this relation also implies that there are no left and right wens.

A cap means capping the tube represented by a strand or shrinking a flying ring to a point. Hence, a cap on the through strand can be "pulled out" from a crossing, but not a cap on the thick (or over) strand (in any orientation of the strands):



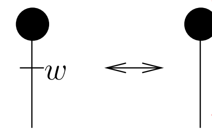

band picture

The Reidemeister 4 relations assert that a strand can be moved under or over a crossing, as shown in the picture below. The symmetrically drawn vertex in the picture denotes a vertex of any kind (as described in Section 6.1.1), and the starnds can be oriented arbitrarily. The local topological (flying ring) interpretations can be read from the pictures the same way as in the cases above and are left to the reader. These relations will be denoted $R4$.

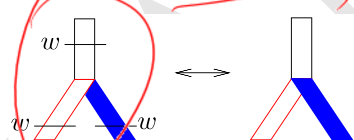


band pictures.

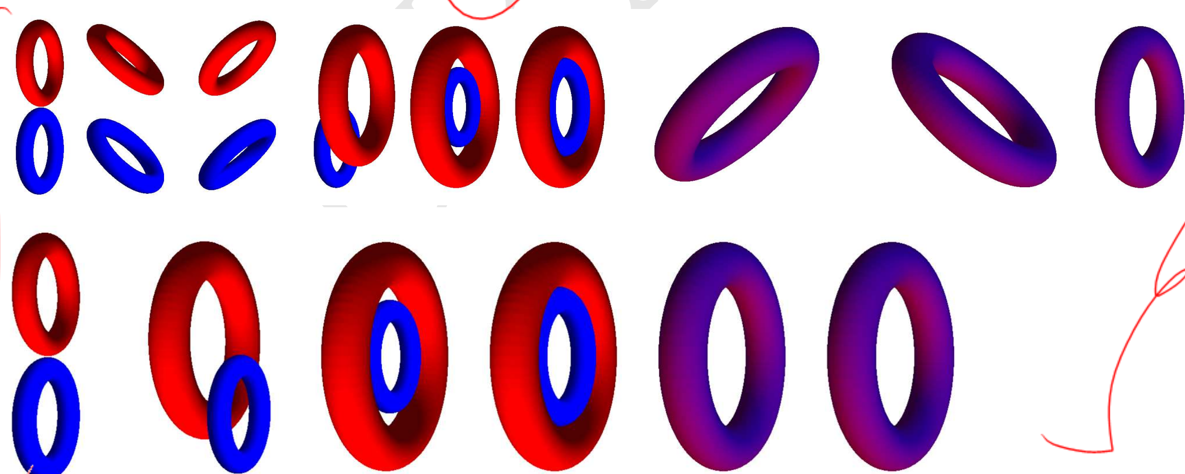
A cap can slide through a wen, hence a capped wen disappears, as shown on the right.




The last relation describes the interaction of wens and vertives. Recall that there are four types of verices: among the bottom two bands (in the pictures so far and below) there is a non-filled and a filled band (corresponding to over/under in the line pictures), meaning the "large" ring and the "small" one which flies into it before they merge. Furthermore, there is a top and a bottom band (among the bottom two, with apologies for the ambiguity in using the word bottom): this denotes the fly-in direction (flying in from below or from above). Conjugating a vertex by three wens switches the top and bottom bands, as shown in the figure below: if both rings flip, then merge, and then the merged ring flips again, this is homotopic to no flips, except the fly-in direction (from below or from above) has changed.




The wen is actually a band-twist.



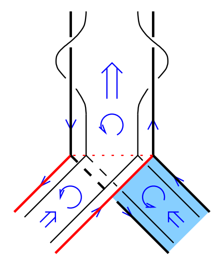
Way too large. If smaller, I think it will fit in the vertical direction after all.

lower res.

We have presented the space $wTF$ as a circuit algebra generated by certain pictures and factored out by some relations. We have given local topological meaning to the pictures and explained how the topological intuition justifies the relations, but we only conjecture that the generators and relations above provide a reidemeister theory for knotted ribbon tubes in $\mathbb{R}^4$.

As a result, framing has not played a role here, except to explain the lack of a Reidemeister 1 relation. In the following section though, we will need a notion of framing to define the unzip (tube doubling) operation. Framing is a continuous up-to-homotopy choice of unit normal vector at every point of the tube. We do not allow any such choice, however. Recall that the knotted tubes we consider have a "filling", whith only "ribbon" self-intersections. When we double a tube by pushing it off itself slightly in the direction of the framing, we want this ribbon property to be preserved. This is eqvivalent to saying that the ring obtained by pushing off any given girth of the tube in the framing direction is not linked with the tube. In the flying ring language, the framing translates to a "companion ring" to each ring, which can fly parallel inside, outside above and below it and change these positions, but is never linked with it.

Without loss of generality, we restrict ourselves to framing choices as follows: fixing a $t$ coordinate cuts out a ring ($S^1$) from each tube, choices of unit normal vectors along this ring are continuous maps $S^1 \to S^1$. We require that each of these maps be constant, in other words it is enough to specify the framing along the core of the tube. Hence the blackboard framing of a line diagram gives rise to a well-defined framing of the tube. We require framings to match at the vertices, with the normal vectors pointing either directly towards or away from the center of the singular ring; while the orientations of the three tubes may or may not match. An example of a vertex with the orientations and framings shown is on the right. Note that the framings on the two sides of each band are mirror images of each other, as they should be.

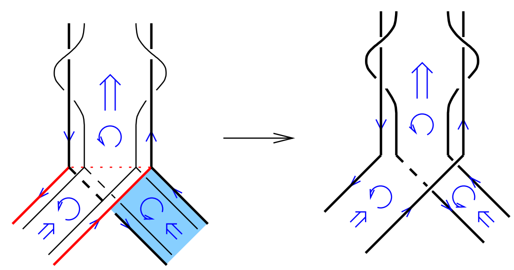


MORE.

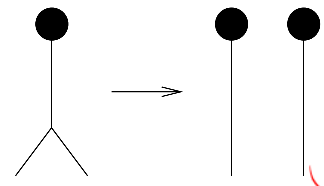
6.1.3. *The auxiliary operations of wTF.* *wTF* is, by definition, a circuit algebra. In addition it is equipped with several extra operations.

The first one of these is the familiar orientation switch. We will, as mentioned in Section 5.4, distinguish between switching both the 2D amd 1D orientations, or just the strand direction. *Orientation switch* will denote the total switch (of both orientations), while we call the operation of only reversing the strand direction *"antipode"*.

Perhaps the most interesting of the auxiliary operations is unzip, or tube doubling. This is done by pushing the tube off itself slightly in the framing direction. At each of the vertices at the two ends of the doubled there are two tubes to be attached to the doubled tube. At each end, the normal vectors pointed either directly towards or away from the center, so there is an "inside" and an "outside" ending ring. The two tubes to be attached also come as an "inside" and an "outside" one, which defines which one to attach to which. An example (in band notation) is shown on the right. Unzip can only be done if the strand orientations match at both ends, as shown in the example.



The last operation, *disk unzip*, is unzip done on a capped strand, pushing the tube off in the direction of the framing, as before. An example in the line notation (with the framing suppressed) is shown on the right.



MORE.

*Comment* 6.2. Acknowledging that we don't know if the CA presented here is a Reidemeister theory for wTFs.

MORE

END NEW.

## 7. Odds and Ends

7.1. **What means "closed form"?** As stated earlier, one of my hopes for this paper is that it will lead to closed-form formulas for tree-level associators. The notion "closed-form" in itself requires an explanation (see footnote 3). Is $e^x$ a closed form expression for $\sum_{n=0}^{\infty}\frac{x^n}{n!}$, or is it just an artificial name given for a transcendental function we cannot otherwise reduce? Likewise, why not call some tree-level associator $\Phi^{\text{tree}}$ and now it is "in closed form"?

For us, "closed-form" should mean "useful for computations". More precisely, it means that the quantity in question is an element of some space $\mathcal{A}^{cf}$ of "useful closed-form thingies" whose elements have finite descriptions (hopefully, finite and short) and on which some operations are defined by algorithms which terminate in finite time (hopefully, finite and short). Furthermore, there should be a finite-time algorithm to decide whether two descriptions of elements of $\mathcal{A}^{cf}$ describe the same element[31]. It is even better if the said decision algorithm takes the form "bring each of the two elements in question to a canonical form by means of some finite (and hopefully short) procedure, and then compare the canonical forms verbatim"; if this is the case, many algorithms that involve managing a large number of elements become simpler and faster.

Thus for example, polynomials in a variable $x$ are always of closed form, for they are simply described by finite sequences of integers (which in themselves are finite sequences of digits), the standard operations on polynomials ($+$, $\times$, and, say, $\frac{d}{dx}$) are algorithmically computable, and it is easy to write the "polynomial equality" computer program. Likewise for rational functions and even for rational functions of $x$ and $e^x$.

On the other hand, general elements $\Phi$ of the space $\mathcal{A}^{\text{tree}}(\uparrow_3)$ of potential tree-level associators are not closed-form, for they are determined by infinitely many coefficients. Thus iterative constructions of associators, such as the one in [BN3] are computationally useful only within bounded-degree quotients of $\mathcal{A}^{\text{tree}}(\uparrow_3)$ and not as all-degree closed-form formulas. Likewise, "explicit" formulas for an associator $\Phi$ in terms of multiple $\zeta$-values (e.g. [LM1]) are not useful for computations as it is not clear how to apply tangle-theoretic operations to $\Phi$ (such as $\Phi\mapsto\Phi^{1342}$ or $\Phi\mapsto(1\otimes\Delta\otimes 1)\Phi$) while staying within some space of "objects with finite description in terms of multiple $\zeta$-values". And even if a reasonable space of such objects could be defined, it remains an open problem to decide whether a given rational linear combination of multiple $\zeta$-values is equal to 0.



[31]In our context, if it is hard to decide within the target space of an invariant whether two elements are equal or not, the invariant is not too useful in deciding whether two knotted objects are equal or not.

7.2. **The Injectivity of $i_u : F_n \to wB_{n+1}$.** Just for completeness, we sketch here an algebraic proof of the injectivity of the map $i_u : F_n \to wB_{n+1}$ discussed in Section 2.2.3. There's some circularity in our argument — we need this injectivity in order to motivate the definition of the map $\Psi : wB_n \to \mathrm{Aut}(F_n)$, and in the proof below we use $\Psi$ to prove the injectivity of $i_u$. But $\Psi$ exists regardless of how its definition is motivated, and it can be shown to be well defined by explicitly verifying that it respects the relations defining $wB_n$. So our proof is logically valid.

**Claim 7.1.** *The map $i_u : F_n \to wB_{n+1}$ is injective.*

*Proof.* (sketch). Let $H$ be the subgroup of $wB_{n+1}$ MORE

7.3. **Finite Type Invariants of v-Braids and w-Braids, in some Detail.** As mentioned in Section 2.2, w-braids are v-braids modulo an additional relation. So we start with a discussion of finite type invariants of v-braids. For simplicity we take our base ring to be $\mathbb{Q}$; everywhere we could replace it by an arbitrary field of characteristic 0[32], and many definitions make sense also over $\mathbb{Z}$ or even with $\mathbb{Q}$ replaced by an arbitrary Abelian group.

7.3.1. *Basic Definitions.* Let $\mathbb{Q}vB_n$ denote group ring of $vB_n$, the algebra of formal linear combinations of elements of $vB_n$, and let $\mathbb{Q}S_n$ be the group ring of $S_n$. The skeleton homomorphism of Remark 2.1 extends to a homomorphism $\varsigma : \mathbb{Q}vB_n \to \mathbb{Q}S_n$. Let $\mathcal{I}$ (or $\mathcal{I}_n$ when we need to be more precise) denote the kernel of the skeleton homomorphism; it is the ideal in $\mathbb{Q}vB_n$ generated by formal differences of v-braids having the same skeleton. One may easily check that $\mathcal{I}$ is generated by differences of the form $\overcrossing - \virtualcrossing$ and $\virtualcrossing - \undercrossing$. Following [GPV] we call such differences "semi-virtual crossings" and denote them by $\semivirtualover$ and $\semivirtualunder$, respectively[33]. In a similar manner, for any natural number $m$ the $m$th power $\mathcal{I}^m$ of $\mathcal{I}$ is generated by "$m$-fold iterated differences" of v-braids, or equally well, by "$m$-singular v-braids", which are v-braids that are also have exactly $m$ semi-virtual crossings (subject to relations which we don't need to specify).

Let $V : vB_n \to A$ be an invariant of v-braids with values in some vector space $A$. We say that $V$ is "of type $m$" (for some $m \in \mathbb{Z}_{\geq 0}$) if its linear extension to $\mathbb{Q}vB_n$ vanishes on $\mathcal{I}^{m+1}$ (alternatively, on all $m+1$-singular v-braids, in clear analogy with the standard definition of finite type invariants). If $V$ is of type $m$ for some unspecified $m$, we say that $V$ is "of finite type". Given a type $m$ invariant $V$, we can restrict it to $\mathcal{I}^m$ and as it vanishes on $\mathcal{I}^{m+1}$, this restriction can be regarded as an element of $(\mathcal{I}^m/\mathcal{I}^{m+1})^\star$. If two type $m$ invariants define the same element of $(\mathcal{I}^m/\mathcal{I}^{m+1})^\star$ then their difference vanishes on $\mathcal{I}^m$, and so it is an invariant of type $m-1$. Thus it is clear that an understanding of $\mathcal{I}^m/\mathcal{I}^{m+1}$ will be instrumental to an inductive understanding of finite type invariants. Hence the following definition.

**Definition 7.2.** The projectivization[34] proj $vB_n$ is the direct sum

$$\operatorname{proj} vB_n := \bigoplus_{m\geq 0} \mathcal{I}^m/\mathcal{I}^{m+1}.$$

[32] Or using the variation of constants method, we can simply declare that $\mathbb{Q}$ is an arbitrary field of characteristic 0.

[33] The signs in $\semivirtualover \leftrightarrow \overcrossing - \virtualcrossing$ and $\semivirtualunder \leftrightarrow \virtualcrossing - \undercrossing$ are "crossings come with their sign and their virtual counterparts come with the opposite sign".

[34] Why "projectivization"? See Section 4.2.

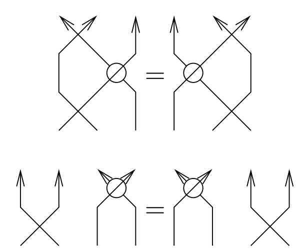
Note that throughout this paper, whenever we write an infinite direct sum, we automatically complete it. Therefore an element in proj $vB_n$ is an infinite sequence $D = (D_0, D_1, \dots)$, where $D_m \in \mathcal{I}^m/\mathcal{I}^{m+1}$. The projectivization proj $vB_n$ is a graded space, with the degree $m$ piece being $\mathcal{I}^m/\mathcal{I}^{m+1}$.

We proceed with the study of proj $vB_n$ (and thus of finite type invariants of v-braids) in three steps. In Section 7.3.2 we introduce a space $\mathcal{D}_n^v$ and a surjection $\rho_0 : \mathcal{D}_n^v \to \operatorname{proj} vB_n$. In Section 7.3.3 we find some relations in $\ker \rho_0$, most notably the $6T$ relation, and introduce the quotient $\mathcal{A}_n^v := \mathcal{D}_n^v/6T$. And then in Section 7.3.4 we introduce the notion of a "universal finite type invariant" and explain how the existence of such a gadget proves that proj $vB_n$ is isomorphic to $\mathcal{A}_n^v$ (in a more traditional language this is the statement that every weight system integrates to an invariant).

Unfortunately, we do not know if there is a universal finite type invariant of v-braids. Thus in Section 7.4 we return to the subject of w-braids and prove the weaker statement that there exists a universal finite type invariant of w-braids.

7.3.2. *Arrow Diagrams.* We are looking for a space that will surject on $\mathcal{I}^m/\mathcal{I}^{m+1}$. In other words, we are looking for a set of generators for $\mathcal{I}^m$, and we are willing to call two such generators the same if their difference is in $\mathcal{I}^{m+1}$. But that's easy. Left and right multiples of the formal differences $\semivirtualover = \overcrossing - \virtualcrossing$ and $\semivirtualunder = \undercrossing - \virtualcrossing$ generate $I$, so products of the schematic form

$$B_0(\semivirtualover|\semivirtualunder)B_1(\semivirtualover|\semivirtualunder)B_2\cdots B_{m-1}(\semivirtualover|\semivirtualunder)B_m \tag{38}$$



generate $\mathcal{I}^m$ (here $(\semivirtualover|\semivirtualunder)$ means "either a $\semivirtualover$ or a $\semivirtualunder$", and there are exactly $m$ of those in any product). Furthermore, inside such a product any $B_k$ can be replaced by any other v-braid $B_k'$ having the same skeleton (e.g., with $\varsigma(B_k)$), for then $B_k - B_k' \in \mathcal{I}$ and the whole product changes by something in $\mathcal{I}^{m+1}$. Also, the relations in (3) and in (5) imply the relations shown on the right for $\semivirtualover$, and similar relations for $\semivirtualunder$. With this freedom, a product as in (38) is determined by the strand-placements of the $\semivirtualover$'s and the $\semivirtualunder$'s. That is, for each semi-virtual crossing in such a product, we only need to know which strand number is the "over" strand, which strand number is the "under" strand, and a sign that determines whether it is the positive semi-virtual $\semivirtualover$ or the negative semi-virtual $\semivirtualunder$. This motivates the following definition.

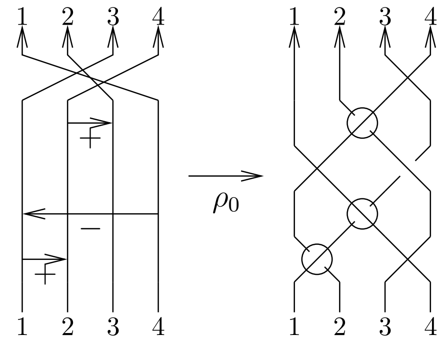
**Definition 7.3.** A "horizontal $m$-arrow diagrams" (analogues to the "chord diagrams" of, say, [BN1]) is an ordered pair $(D, \beta)$ in which $D$ is a word of length $m$ in the alphabet $\{a_{ij}^+, a_{ij}^- : i, j \in \{1, \dots, n\}, i \neq j\}$ and $\beta$ is a permutation in $S_n$. Let $\mathcal{D}_m^{vh}$ be the space of formal linear combinations of horizontal $m$-arrow diagrams. We usually use a pictorial notation for horizontal arrow diagram, as demonstrated in Figure 23.

There is a surjection $\rho_0 : \mathcal{D}_m^{vh} \to \mathcal{I}^m/\mathcal{I}^{m+1}$. The definition of $\rho_0$ is suggested by the first paragraph of this section and an example is shown in Figure 23; we will skip the formal definition here. We also skip the formal proof of the surjectivity of $\rho_0$.

Finally, consider the product $\semivirtualover \cdot \semivirtualunder$ and use the second Reidemeister move for both virtual and non-virtual crossings:

$$\semivirtualover\semivirtualunder = (\overcrossing - \virtualcrossing)(\undercrossing - \virtualcrossing) = \overcrossing\undercrossing + \virtualcrossing\virtualcrossing - \overcrossing\virtualcrossing - \virtualcrossing\undercrossing = (\overcrossing\virtualcrossing - 1) + (\virtualcrossing\undercrossing) = \semivirtualover\virtualcrossing - \virtualcrossing\semivirtualunder.$$

**Figure 23.** The horizontal 3-arrow diagram $(D,\beta) = (a^+_{12}a^-_{41}a^+_{23},\ 3421)$ and its image via $\rho_0$. The first arrow, $a^+_{12}$ starts at strand 1, ends at strand 2 and carries a $+$ sign, so it is mapped to a positive semi-virtual crossing of strand 1 over strand 2. Likewise the second arrow $a^-_{41}$ maps to a negative semi-virtual crossing of strand 4 over strand 1, and $a^+_{23}$ to a positive semi-virtual crossing of strand 2 over strand 3. We also show one possible choice for a representative of the image of $\rho_0(D,\beta)$ in $\mathcal{I}^m/\mathcal{I}^{m+1}$: it is a v-braid with semi-virtual crossings as specified by $D$ and whose overall skeleton is 3421.




If a total of $m-1$ further semi-virtual crossings are multiplied into this equality on the left and on the right, along with arbitrary further crossings and virtual crossings, the left hand side of the equality becomes a member of $\mathcal{I}^{m+1}$, and therefore, as a member of $\mathcal{I}^m/\mathcal{I}^{m+1}$, it is 0. Thus with "…" standing for extras added on the left and on the right, we have that in $\mathcal{I}^m/\mathcal{I}^{m+1}$,

$$0 = \dots(\boxtimes\times - \times\boxtimes)\dots = \rho_0(\dots ?? \dots)$$

MORE.

7.3.3. *The* $6T$ *Relations.* MORE.

7.3.4. *The Notion of a Universal Finite Type Invariant.* MORE.

7.4. **Finite type invariants of w-braids.** MORE.

7.5. **Arrow Diagrams to Degree 2.** Just as an example, in this section we study the spaces $\mathcal{A}^-(\uparrow)$, $\mathcal{A}^{r-}(\uparrow)$, $\mathcal{P}^-(\uparrow)$, $\mathcal{A}^-(\bigcirc)$ and $\mathcal{A}^{r-}(\bigcirc)$ in degrees $m \leq 2$ in detail, both in the "v" case and in the "w" case (the "u" case has been known since long).

7.5.1. *Arrow Diagrams in Degree 0.* There is only one degree 0 arrow diagram, the empty diagram $D_0$ (see Figure 24). There are no relations, and thus $\{D_0\}$ is the basis of $\mathcal{G}_0\mathcal{A}^-(\uparrow)$ and of $\mathcal{G}_0\mathcal{A}^{r-}(\uparrow)$ and its obvious closure, the empty circle, is the basis of $\mathcal{G}_0\mathcal{A}^-(\bigcirc)$ and of $\mathcal{G}_0\mathcal{A}^{r-}(\bigcirc)$. $D_0$ is the unit 1, yet $\Delta D_0 = D_0\otimes D_0 = 1\otimes 1 \neq D_0\otimes 1 + 1\otimes D_0$, so $D_0$ is not primitive and $\dim \mathcal{G}_0\mathcal{P}^-(\uparrow) = 0$.

7.5.2. *Arrow Diagrams in Degree 1.* There is only two degree 1 arrow diagrams, the "right arrow" diagram $D_R$ and the "left arrow" diagram $D_L$ (see Figure 24). There are no $6T$ relations, and thus $\{D_R, D_L\}$ is the basis of $\mathcal{G}_1\mathcal{A}^-(\uparrow)$. Both $D_R$ and $D_L$ vanish modulo FI, so $\dim \mathcal{G}_1\mathcal{A}^{r-}(\uparrow) = \dim \mathcal{G}_1\mathcal{A}^{r-}(\bigcirc) = 0$. Both $D_R$ and $D_L$ are primitive, so $\dim \mathcal{G}_1\mathcal{P}^-(\uparrow) = 2$. Finally, the closures of $D_R$ and $D_L$ are equal, so $\mathcal{G}_0\mathcal{A}^-(\bigcirc) = \langle D_R\rangle = \langle D_L\rangle$.

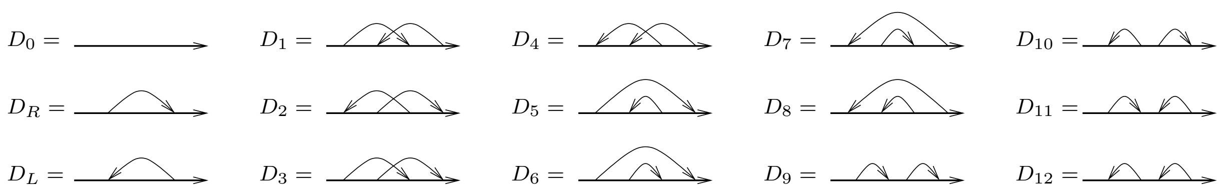



**Figure 24.** The 15 arrow diagrams of degree at most 2.

7.5.3. *Arrow Diagrams in Degree 2.* There are 12 degree 2 arrow diagrams, which we denote $D_1, \ldots, D_{12}$ (see Figure 24). There are six $6T$ relations, corresponding to the 6 ways of ordering the 3 vertical strands that appear in a $6T$ relation (see Figure 3) along a long line. The ordering $(ijk)$ becomes the relation $D_3 + D_9 + D_3 = D_6 + D_3 + D_6$. Likewise, $(ikj) \mapsto D_6 + D_1 + D_{11} = D_3 + D_5 + D_1$, $(jik) \mapsto D_{10} + D_2 + D_6 = D_2 + D_5 + D_3$, $(jki) \mapsto D_4 + D_7 + D_1 = D_8 + D_1 + D_{11}$, $(kij) \mapsto D_2 + D_7 + D_4 = D_{10} + D_2 + D_8$, and $(kji) \mapsto D_8 + D_4 + D_8 = D_4 + D_{12} + D_4$. After some linear algebra, we find that $\{D_1, D_2, D_6, D_8, D_9, D_{11}, D_{12}\}$ form a basis of $\mathcal{G}_2\mathcal{A}^v(\uparrow)$, and that the remaining diagrams reduce to the basis as follows: $D_3 = 2D_6 - D_9$, $D_4 = 2D_8 - D_{12}$, $D_5 = D_9 + D_{11} - D_6$, $D_7 = D_{11} + D_{12} - D_8$, and $D_{10} = D_{11}$. In $\mathcal{G}_2\mathcal{A}^{rv}(\uparrow)$ we have that $D_{5-12} = 0$, and in view of the above relations, we also get that $D_3 = D_4 = 0$. Thus $\{D_1, D_2\}$ is a basis of $\mathcal{G}_2\mathcal{A}^{rv}(\uparrow)$. There are 3 OC relations to write for $\mathcal{G}_2\mathcal{A}^w(\uparrow)$: $D_2 = D_{10}$, $D_3 = D_6$, and $D_4 = D_8$. Along with the $6T$ relations, we find that $\{D_1, D_3 = D_6 = D_9, D_2 = D_5 = D_7 = D_{10} = D_{11}, D_4 = D_8 = D_{12}\}$ is a basis of $\mathcal{G}_2\mathcal{A}^w(\uparrow)$ When also mod out by FI, only one diagram remains non-zero in $\mathcal{G}_2\mathcal{A}^{rw}(\uparrow)$ and it is $D_1$. We leave the determination of the primitives and the spaces with a circle skeleton as an exercise to the reader.



## 8. Glossary of notation

| | |
|---|---|
| $\Delta$ | Cloning, co-product, 2.5.1.2. |
| $\Psi$ | the map $\Psi : wB_n \to \mathrm{Aut}(F_n)$, 2.2.3. |
| $\sigma_i$ | a crossing between adjacent strands, 2.1.1. |
| $\sigma_{ij}$ | strand $i$ crosses over strand $j$, 2.1.2. |
| $\varsigma$ | the skeleton morphism, 2.1.1. |
| $\theta$ | inversion, antipode, 2.5.1.1. |
| $\xi_i$ | the generators of $F_n$, 2.2.3. |
| $B_n$ | the braid group, 2.1.1. |
| $F_n$ | the free group, 2.2.3. |
| $FA_n$ | the free associative algebra, 2.5.1.5. |
| OC | the Overcrossings Commute relation, 2.2. |
| $PvB_n$ | the group of pure v-braids, 2.1.1. |
| $PwB_n$ | the group of pure w-braids, 2.2. |
| $S_n$ | the symmetric group, 2.1.1. |
| $s_i$ | a virtual crossing between adjacent strands, 2.1.1. |
| UC | the Undercrossings Commute relation, 2.2. |
| $vB_n$ | the virtual braid group, 2.1.1. |
| $wB_n$ | the group of w-braids, 2.2. |
| $x_i$ | the generators of $FA_n$, 2.5.1.5. |
| $Z$ | expansions throughout. |

## References

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators,* arXiv:0802.4300.

[AS] Z. Amir-Khosravi and S. Sankaran, *VasCalc — A Vassiliev Invariants Calculator,* electronic document tree, `http://katlas.math.toronto.edu/drorbn/?title=VasCalc`.

[Ar] E. Artin, *Theory of Braids,* Ann. of Math. **48-1** (1947) 101–126.

[BWC] J. C. Baez, D. K. Wise, and A. S. Crans, *Exotic Statistics for Strings in 4d BF Theory,* Adv. Theor. Math. Phys. **11** (2007) 707–749, arXiv:gr-qc/0603085.

[Ba] V. G. Bardakov, *The Virtual and Universal Braids,* Fundamenta Mathematicae **184** (2004) 1–18, arXiv:math.GR/0407400.

[BB] V. G. Bardakov and P. Bellingeri, *Combinatorial Properties of Virtual Braids,* to appear in Topology and its Applications, arXiv:math.GR/0609563.

[BN1] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology **34** (1995) 423–472.

[BN2] D. Bar-Natan, *Vassiliev homotopy string link invariants,* Jour. of Knot Theory and its Ramifications **4** (1995) 13–32.

[BN3] D. Bar-Natan, *Non-associative tangles,* in *Geometric topology* (proceedings of the Georgia international topology conference), (W. H. Kazez, ed.), 139–183, Amer. Math. Soc. and International Press, Providence, 1997.

[BN4] D. Bar-Natan, *Some computations related to Vassiliev invariants,* electronic publication, `http://www.math.toronto.edu/~drorbn/LOP.html#Computations`.

[BN5] D. Bar-Natan, *Vassiliev and quantum invariants of braids,* in Proc. of Symp. in Appl. Math. **51** (1996) 129–144, *The interface of knots and physics,* (L. H. Kauffman, ed.), Amer. Math. Soc., Providence.

[BN6] D. Bar-Natan, *On Associators and the Grothendieck-Teichmuller Group I,* Selecta Mathematica, New Series **4** (1998) 183–212.

[BN7] D. Bar-Natan, *Finite Type Invariants,* in *Encyclopedia of Mathematical Physics,* (J.-P. Francoise, G. L. Naber and Tsou S. T., eds.) Elsevier, Oxford, 2006 (vol. 2 p. 340).

[BN8] D. Bar-Natan, *Algebraic Knot Theory — A Call for Action,* web document, 2006, `http://www.math.toronto.edu/~drorbn/papers/AKT-CFA.html`.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne* (self-reference), paper, videos (wClips) and related files at `http://www.math.toronto.edu/~drorbn/papers/WKO/`

[BGRT] D. Bar-Natan, S. Garoufalidis, L. Rozansky, and D. P. Thurston, *The Århus integral of rational homology 3-spheres I–III,* Selecta Mathematica, New Series **8** (2002) 315–339, arXiv:q-alg/9706004, **8** (2002) 341–371, arXiv:math.QA/9801049, **10** (2004) 305–324, arXiv:math.QA/9808013.

[BHLR] D. Bar-Natan, I. Halacheva, L. Leung, and F. Roukema, *Some Dimensions of Spaces of Finite Type Invariants of Virtual Knots,* submitted.

[BLT] D. Bar-Natan, T. Q. T. Le, and D. P. Thurston, *Two applications of elementary knot theory to Lie algebras and Vassiliev invariants,* Geometry and Topology **7-1** (2003) 1–31, arXiv:math.QA/0204311.

[BS] D. Bar-Natan and A. Stoimenow, *The fundamental theorem of Vassiliev invariants,* in Proc. of the Århus Conf. *Geometry and physics,* (J. E. Andersen, J. Dupont, H. Pedersen, and A. Swann, eds.), lecture notes in pure and applied mathematics **184** (1997) 101–134, Marcel Dekker, New-York. Also arXiv:q-alg/9702009.

[BP] B. Berceanu and S. Papadima, *Universal Representations of Braid and Braid-Permutation Groups,* arXiv:0708.0634.

[BT] R. Bott and C. Taubes, *On the self-linking of knots,* Jour. Math. Phys. **35** (1994).

[BH] T. Brendle and A. Hatcher, *Configuration Spaces of Rings and Wickets,* arXiv:0805.4354.

[CS] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams,* Mathematical Surveys and Monographs **55**, American Mathematical Society, Providence 1998.

[CCM] A. S. Cattaneo, P. Cotta-Ramusino, and M. Martellini, *Three-dimensional BF Theories and the Alexander-Conway Invariant of Knots,* Nucl. Phys. **B436** (1995) 355–384, arXiv:hep-th/9407070.

[CCFM] A. S. Cattaneo, P. Cotta-Ramusino, J. Froehlich, and M. Martellini, *Topological BF Theories in 3 and 4 Dimensions,* J. Math. Phys. **36** (1995) 6137–6160, arXiv:hep-th/9505027.

[Ch] S. V. Chmutov, *A proof of the Melvin-Morton conjecture and Feynman diagrams,* Jour. of Knot Theory and its Ramifications **7-1** (1998) 23–40.

[Da] Z. Dancso, *On a Kontsevich Integral for Knotted Trivalent Graphs,* arXiv:0811.4615.

[Dr1] V. G. Drinfel'd, *Quantum Groups,* in *Proceedings of the International Congress of Mathematicians,* 798–820, Berkeley, 1986.

[Dr2] V. G. Drinfel'd, *Quasi-Hopf Algebras,* Leningrad Math. J. **1** (1990) 1419–1457.

[Dr3] V. G. Drinfel'd, *On Quasitriangular Quasi-Hopf Algebras and a Group Closely Connected with* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$*,* Leningrad Math. J. **2** (1991) 829–860.

[Dye] H. A. Dye, *Virtual knots undetected by 1 and 2-strand bracket polynomials,* Topology and its Applications **153-1** (2005) 141–160, arXiv:math.GT/0402308.

[Ep] D. Epstein, *Word Processing in Groups,* AK Peters, 1992.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I,* Selecta Mathematica, New Series **2** (1996) 1–41, arXiv:q-alg/9506005.

[FRR] R. Fenn, R. Rimanyi, and C. Rourke, *The Braid-Permutation Group,* Topology **36** (1997) 123–135.

[Gol] D. L. Goldsmith, *The Theory of Motion Groups,* Mich. Math. J. **28-1** (1981) 3–17.

[Gou1] M. Goussarov, *On $n$-equivalence of knots and invariants of finite degree,* Zapiski nauch. sem. POMI **208** (1993) 152–173 (English translation in *Topology of manifolds and varieties* (O. Viro, editor), Amer. Math. Soc., Providence 1994, 173–192).

[Gou2] M. Goussarov, *Finite type invariants and n-equivalence of 3-manifolds,* C. R. Acad. Sci. Paris Sér I Math. **329-6** (1999) 517–522.

[GPV] M. Goussarov, M. Polyak, and O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

[GK] M. Gutiérrez and S. Krstić, *Normal Forms for Basis-Conjugating Automorphisms of a Free Group,* Int. Jour. of Algebra and Computation **8-6** (1998) 631–669.

[HM] N. Habegger and G. Masbaum, *The Kontsevich integral and Milnor's invariants,* Topology **39** (2000) 1253–1289.

[Hab] K. Habiro, *Claspers and finite type invariants of links,* Geom. Topol. **4** (2000) 1–83.

[HKS] K. Habiro, T. Kanenobu, and A. Shima, *Finite Type Invariants of Ribbon 2-Knots,* in *Low Dimensional Topology*, (H. Nencka, ed.) Cont. Math. **233** (1999) 187–196.

[HS] K. Habiro and A. Shima, *Finite Type Invariants of Ribbon 2-Knots, II,* Topology and its Applications **111-3** (2001) 265–287.

[Hav] A. Haviv, *Towards a diagrammatic analogue of the Reshetikhin-Turaev link invariants,* Hebrew University PhD thesis, September 2002, arXiv:math.QA/0211031.

[Jon] V. Jones, *Planar algebras, I,* New Zealand Journal of Mathematics, to appear, arXiv:math.QA/9909027.

[Joy] D. Joyce, *A Classifying Invariant of Knots, the Knot Quandle,* Journal of Pure and Appl. Algebra **23** (1982) 37–65.

[KS] T. Kanenobu, A. Shima, *Two Filtrations of Ribbon 2-konts,* Topology and Appl. **121** (2002) 143–168.

[KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff Formula and Invariant Hyperfunctions,* Invent. Math. **47** (1978) 249–272.

[Ka1] L. H. Kauffman, *On knots,* Princeton Univ. Press, Princeton, 1987.

[Ka2] L. H. Kauffman, *Virtual Knot Theory,* European J. Comb. **20** (1999) 663–690, arXiv:math.GT/9811028.

[KL] L. H. Kauffman and S. Lambropoulou, *Virtual Braids,* Fundamenta Mathematicae **184** (2005) 159–186, arXiv:math.GT/0407349.

[Kn] J. A. Kneissler, *The number of primitive Vassiliev invariants up to degree twelve,* preprint, June 1997, arXiv:q-alg/9706022.

[Ko] T. Kohno, *Vassiliev invariants and de-Rham complex on the space of knots,* Contemp. Math. **179** (1994) 123–138.

[Kr] A. Kricker ...

[Kup] G. Kuperberg, *What is a Virtual Link?,* Algebr. Geom. Topol. **3** (2003) 587–591, arXiv:math.GT/0208039.

[Kur] V. Kurlin, *Compressed Drinfeld associators,* Journal of Algebra **292-1** (2005) 184–242.

[Le] T. Q. T. Le, *An invariant of integral homology 3-spheres which is universal for all finite type invariants,* in *Solitons, geometry and topology: on the crossroad*, (V. Buchstaber and S. Novikov, eds.) AMS Translations Series 2, Providence, arXiv:q-alg/9601002.

[LM1] T. Q. T. Le and J. Murakami, *On Kontsevich's integral for the HOMFLY polynomial and relations of multiple $\zeta$-numbers,* Topology and its Applications **62** (1995) 193–206.

[LM2] T. Q. T. Le and J. Murakami, *The universal Vassiliev-Kontsevich invariant for framed oriented links,* Compositio Math. **102** (1996) 41–64, arXiv:hep-th/9401016.

[LMO] T. Q. T. Le, J. Murakami, and T. Ohtsuki, *On a universal quantum invariant of 3-manifolds,* Topology **37-3** (1998) 539–574, arXiv:q-alg/9512002.

[Lee] P. Lee, *Closed-Form Associators and Braidors in a Partly Commutative Quotient,* University of Toronto preprint, December 2007, `http://individual.utoronto.ca/PetersKnotPage/`.

[Lei] Tom Leinster, *Higher Operads, Higher Categories,* London Mathematical Society Lecture Note Series **298**, Cambridge University Press, ISBN 0-521-53215-9, arXiv:math.CT/0305049.

[Les1] C. Lescop, *Knot Invariants Derived from the Equivariant Linking Pairing,* arXiv:1001.4474.

[Les2] C. Lescop, *On the Cube of the Equivariant Linking Pairing for 3 Manifolds of Rank One,* in preparation.

[Leu] L. Leung, *Combinatorial Formulas for Classical Lie Weight Systems on Arrow Diagrams,* University of Toronto preprint, December 2008, arXiv:0812.2342.

[Lev] J. Levine, *Addendum and Correction to: "Homology Cylinders: an Enlargement of the Mapping Class Group,* Alg. Geom. Top. **2** (2002), 1197–1204, arXiv:math.GT/0207290

[Lie] J. Lieberum, *The Drinfeld Associator of $gl(1|1)$,* arXiv:math.QA/0204346.

[Lin] X-S. Lin, *Power Series Expansions and Invariants of Links,* in *Geometric topology* (proceedings of the Georgia international topology conference), (W. H. Kazez, ed.), 184–202, Amer. Math. Soc. and International Press, Providence, 1997.

[Lod] J-L. Loday, *Une version non commutative des algebres de Lie: des algebres de Leibniz,* Enseign. math. (2) **39** (3-4): 269–293.

[MKS] W. Magnus, A. Karras, and D. Solitar, *Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations,* Wiley, New York, 1966.

[Mc] J. McCool, *On Basis-Conjugating Automorphisms of Free Groups,* Can. J. Math. **38-6** (1986) 1525–1529.

[MM] J. Milnor and J. Moore, *On the structure of Hopf algebras,* Annals of Math. **81** (1965) 211–264.

[Na] G. Naot, *On Chern-Simons Theory with an Inhomogeneous Gauge Group and BF Theory Knot Invariants,* J. Math. Phys. **46** (2005) 122302, arXiv:math.GT/0310366.

[Oh] T. Ohtsuki, *Finite type invariants of integral homology 3-spheres,* Jour. of Knot Theory and its Ramifications **5(1)** (1996) 101–115.

[Po] M. Polyak, *On the Algebra of Arrow Diagrams,* Let. Math. Phys. **51** (2000) 275–291.

[Rol] D. Rolfsen, *Knots and Links,* AMS Chelsea, 2003.

[Rou] F. Roukema, *Goussarov-Polyak-Viro Combinatorial Formulas for Finite Type Invariants,* arXiv:0711.4001.

[Sa] S. Satoh, *Virtual Knot Presentations of Ribbon Torus Knots,* J. of Knot Theory and its Ramifications **9-4** (2000) 531–542.

[Th] D. Thurston, *Integral expressions for the Vassiliev knot invariants,* Harvard University senior thesis, April 1995, arXiv:math.QA/9901110.

[Vai] A. Vaintrob, *Melvin Morton Conjecture and Primitive Feynman Diagrams,* Int. Jour. of Math. **8-4** (1997) 537-553.

[Vas] V. A. Vassiliev, *Cohomology of knot spaces,* in *Theory of Singularities and its Applications (Providence)* (V. I. Arnold, ed.), Amer. Math. Soc., Providence, 1990.

[Win] B. Winter, *On Codimension Two Ribbon Embeddings,* arXiv:0904.0684.

Department of Mathematics, University of Toronto, Toronto Ontario M5S 2E4, Canada
*E-mail address*: `drorbn@math.toronto.edu, zsuzsi@math.toronto.edu`
*URL*: `http://www.math.toronto.edu/~drorbn, http://www.math.toronto.edu/zsuzsi`