FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS: FROM ALEXANDER TO KASHIWARA AND VERGNE

DROR BAR-NATAN

Abstract. W-knots, and more generally, w-knotted objects (w-braids, w-tangles, etc.) make a class of knotted objects which is wider but weaker than their "ordinary" counterparts. To get (say) w-knots from ordinary knots, one has to allow non-planar "virtual" knot diagrams, hence enlarging the the base set of knots. But then one imposes a new relation, the "overcrossings commute" relation, further beyond the ordinary collection of Reidemeister moves, making w-knotted objects a bit weaker once again.

The group of w-braids was studied (under the name "welded braids") by Fenn, Rimanyi and Rourke [FRR] and was shown to be isomorphic to the McCool group [Mc] of "basis-conjugating" automorphisms of a free group F_n — the smallest subgroup of $\mathrm{Aut}(F_n)$ that contains both braids and permutations. Brendle and Hatcher [BH], in work that traces back to Goldsmith [Gol], have shown this group to be a group of movies of flying rings in $\mathbb{R}^3$. Satoh [Sa] studied several classes of w-knotted objects (under the name "weakly-virtual") and has shown them to be closely related to certain classes of knotted surfaces in $\mathbb{R}^4$. So w-knotted objects are topologically and algebraically interesting.

In this article we study finite type invariants of several classes of w-knotted objects. Following Berceanu and Papadima [BP], we construct a homomorphic universal finite type invariant of w-braids, and hence show that the McCool group of automorphisms is "1-formal". We also construct a homomorphic universal finite type invariant of w-tangles. We find that the universal finite type invariant of w-knots is more or less the Alexander polynomial (details inside).

Much as the spaces $\mathcal{A}$ of chord diagrams for ordinary knotted objects are related to metrized Lie algebras, we find that the spaces $\vec{\mathcal{A}}^w$ of "arrow diagrams" for w-knotted objects are related to not-necessarily-metrized Lie algebras. Many questions concerning w-knotted objects turn out to be equivalent to questions about Lie algebras. Most notably we find that a homomorphic universal finite type invariant of w-knotted trivalent graphs is essentially the same as a solution of the Kashiwara-Vergne [KV] conjecture and much of the Alekseev-Torrosian [AT] work on Drinfel'd associators and Kashiwara-Vergne can be re-intepreted as a study of w-knotted trivalent graphs.

Date: Nov. 10, 2008.

1991 *Mathematics Subject Classification.* 57M25.

Key words and phrases. w-braids, w-knots, w-tangles, knotted graphs, finite type invariants, Alexander polynomial, Kashiwara-Vergne, Associators, Free Lie algebras.

This work was partially supported by NSERC grant RGPIN 262178. Electronic version at `http://www.math.toronto.edu/~drorbn/papers/WKO/`.

To Do.

- Move the skeleton outside of $\mathcal{D}$.
- Finish the paper.

Contents

1. Introduction

1.1. **Dreams.** I have a dream[1], at least partially founded on reality, that many of the difficult algebraic equations in mathematics, especially those that are written in graded spaces, more especially those that are related in one way or another to quantum groups [Dr1], and even more especially those related to the work of Etingof and Kazhdan [EK], can be understood, and indeed, would appear more natural, in terms of finite type invariants of various topological objects.

I believe this is the case for Drinfel'd's theory of associators [Dr2], which can be interpreted as a theory of well-behaved universal finite type invariants of paranthesized tangles[2] [LM2, BN3], and even more elegantly, as a theory of universal finite type invariants of knotted trivalent graphs [BN8].

I believe this is the case for Drinfel'd's "Grothendieck-Teichmuller group" [Dr3] which is better understood as a group of automorphisms of a certain algebraic structure, also related to universal finite type invariants of paranthesized tangles [BN5].

And I'm optimistic, indeed I believe, that sooner or later the work of Etingof and Kazhdan [EK] on quantization of Lie bialgebras will be re-interpreted as a construction of a

[1] Understanding an author's history and psychology ought never be necessary to understand his/her papers, but it may be useful. Nothing material in the rest of this paper relies on Section 1.1.

[2] "q-tangles" in [LM2], "non-associative tangles" in [BN3].

well-behaved universal finite type invariant of virtual knots [Ka2] or of some other class of virtually knotted objects. Some preliminary steps in that direction were taken by Haviv [Ha].

I have another dream, to construct a useful "Algebraic Knot Theory". As at least a partial writeup exists [BN7], I'll only state that an important ingredient necessary to fulfil that dream would be a "closed form"[3] formula for an associator, at least in some reduced sense. Formulas for associators or reduced associators were in themselves the goal of several studies undertaken for various other reasons [LM1, Li1, Ku, Lee].

1.2. **Stories.** Thus I was absolutely delighted when in January 2008 Anton Alekseev described to me his joint work [AT] with Charles Torossian — he told me they found a relationship between the Kashiwara-Vergne conjecture [KV], a cousin of the Duflo isomorphism (which I already knew to be knot-theoretic [BLT]), and associators taking values in a space called sder, which I could identify as "tree-level Jacobi diagrams", also a knot-theoretic space related to the Milnor invariants [BN2, HM]. What's more, Anton told me that in certain quotient spaces the Kashiwara-Vergne conjecture can be solved explicitly; this should lead to some explicit associators!

So I spent the following several months trying to understand [AT], and this paper is a summary of my efforts. The main thing I learned is that the Alekseev-Torossian paper, and with it the Kashiwara-Vergne conjecture, fit very nicely with my first dream recalled above, about interpreting algebra in terms of knot theory. Indeed much of [AT] can be reformulated as a construction and a discussion of a well-behaved universal finite type invariant Z of a certain class of knotted objects (which I will call here w-knotted), a certain natural quotient of the space of virtual knots (more precisely, virtual trivalent tangles). And my hopes remain high that later I (or somebody else) will be able to exploit this relationship in directions compatible with my second dream recalled above, on the construction of an "algebraic knot theory".

The story, in fact, is pretier than I was hoping for, for it has the following additional qualities:

- W-knotted objects are quite interesting in themselves: as stated in the abstract, they are related to combinatorial group theory via "basis-conjugating" automorphisms of a free group F_n, to groups of movies of flying rings in $\mathbb{R}^3$, and more generaly, to certain classes of knotted surfaces in $\mathbb{R}^4$. The references include [BH, FRR, Gol, Mc, Sa].
- The "chord diagrams" for w-knotted objects (really, these are "arrow diagrams") describe formulas for invariant tensors in spaces pertaining to not-necessarily-metrized Lie algebras in much of the same way as ordinary chord diagrams for ordinary knotted objects describe formulas for invariant tensors in spaces pertaining to metrized Lie algebras. This observation is bound to have further implications.
- Arrow diagrams also describe the Feynmann diagrams of topological BF theory [CCM, CCFM] and of a certain class of Chern-Simons theories [Na]. Thus it is likely that our story is directly related to quantum field theory[4].
- When composed with the map from knots to w-knots, Z becomes the Alexander polynomial. For links, it becomes an invariant stronger than the multi-variable Alexander

[3]The phrase "closed form" in itself requires an explanation. See Section 4.1.

[4]Some non-perturbative relations between BF theory and w-knots was discussed by Baez, Wise and Crans [BWC].

polynomial whice contains the multi-variable Alexander polynomial as an easily identifiable reduction. On other w-knotted objects Z has easily identifiable reductions that can be considered as "Alexander polynomials" with good behaviour relative to various knot-theoretic operations — cablings, compositions of tangles, etc. There is also a certain specific reduction of Z that can be considered as the "ultimate Alexander polynomial" — in the appropriate sense, it is the minimal extension of the Alexander polynomial to other knotted objects which is well behaved under a whole slew of knot theoretic operations, including the ones named above.

1.3. **Plans.** Our order of proceedings is: w-braids, w-knots, w-tangles, w-tangled graphs, and then some odds and ends. For more detailed information consult the "Section Summary" paragraph at the beginning of each of the sections.

1.4. **Acknowledgement.** I wish to thank Anton Alekseev, Scott Carter, Joel Kamnitzer, Lou Kauffman and Dylan Thurston for comments and suggestions.

2. W-Braids

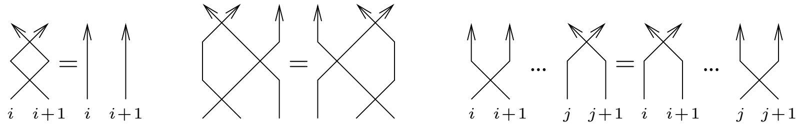
Section Summary. This section is largely a compilation of existing literature, though we also introduce the language of arrow diagrams that we use thoughout the rest of the paper. We define w-braids and survey their relationship with basis-conjugating automorphisms of free groups and with "the group of flying rings in $\mathbb{R}^3$" (really, a group of knotted tubes in $\mathbb{R}^4$). We then play the usual game of introducing finite type invariants, weight systems, chord diagrams (arrow diagrams, for this case), and 4T-like relations. Finally we define and construct a universal finite type invariant for w-braids. It turns out that the only algebraic tool we need to use is the formal exponential function $\exp(a) := \sum a^n/n!$.

2.1. **What are w-braids?** It is simplest to define w-braids in terms of generators and relations, either algebraically or picturially. Algebraically, for a natural number n we set wB_n to be the group generated by symbols σ_i ($1 \leq i \leq n-1$), called "crossings" and graphically represented by an overcrossing ⤫ "between strand i and strand $i+1$" (with inverse ⤬)[5], and s_i, called "virtual crossings" and graphically represented by a non-crossing, ⨯, also "between strand i and strand $i+1$", subject to the following relations:

- The subgroup of wB_n generated by the virtual crossings s_i is the symmetric group S_n, and the s_i's correspond to the transpositions $(i, i+1)$. That is, we have

  $$s_i^2 = 1, \qquad s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \qquad \text{and if } |i-j|>1 \text{ then} \qquad s_is_j = s_js_i.$$

  In pictures, this is



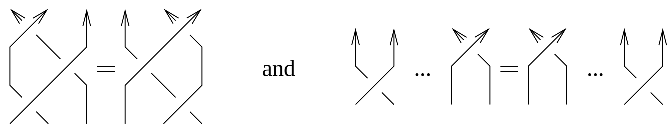
  Note that we read our braids from bottom to top.

[5] We sometimes refer to ⤫ as a "positive crossing" and to ⤬ as a "negative crossing".

- The subgroup of wB_n generated by the crossings σ_i's is the usual braid group B_n, and σ_i corresponds to the braiding of strand i over strand $i+1$. That is, we have

(1) $$\sigma_i\sigma_{i+1}^{\pm 1}\sigma_i = \sigma_{i+1}\sigma_i^{\pm 1}\sigma_{i+1}, \qquad \text{and if } |i-j|>1 \text{ then} \qquad \sigma_i\sigma_j = \sigma_j\sigma_i.$$

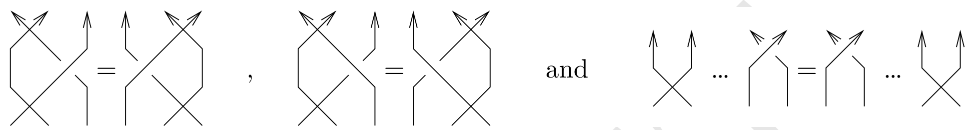
  In pictures, showing only the positive-powers case and dropping the indices, this is



- Some "mixed relations",

(2) $$s_i\sigma_{i+1}^{\pm 1}s_i = s_{i+1}\sigma_i^{\pm 1}s_{i+1}, \qquad \text{and if } |i-j|>1 \text{ then} \qquad s_i\sigma_j = \sigma_j s_i.$$

  In pictures, this is



- Finally, we break the symmetry between over crossings and under crossings by imposing one of the "forbidden moves" virtual knot theory, but not the other:

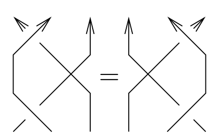
(3) $$\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}, \qquad \text{yet} \qquad \sigma_i^{-1}\sigma_{i+1}^{-1}s_i \neq s_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}.$$

  In pictures, this is

The relation we have just imposed may be called the "unforbidden relation", or, perhaps more appropriately, the "overcrossings commute" relation (OC). Ignoring the non-crossings[6] $\times$, the OC relation says that it is the same if strand i first crosses over strand $i+1$ and then over strand $i+2$, or if it first crosses over strand $i+2$ and then over strand $i+1$. The "undercrossings commute" relation UC, the one we do not impose in (3), would say the same, except with "under" replacing "over".

Exercise 2.1. Show that the OC relation is equivalent to the relation

$$\sigma_i^{-1}s_{i+1}\sigma_i = \sigma_{i+1}s_i\sigma_{i+1}^{-1} \qquad \text{or}$$



Remark 2.2. The group we get without imposing the OC relation (3) is the virtual braid group vB_n (sometimes called "the group of v-braids" below). Thus $wB_n = vB_n/OC$.

Remark 2.3. The "skeleton" of a v-braid or a w-braid B is the set of strands appearing in it, retaining the association between their beginning and ends but ignoring all the crossing information. More precisely, it is the permutation induced by tracing along B, and even more precisely it is the image of B via the "skeleton morphism" $\varsigma : vB_n \to S_n$ (or $\varsigma : wB_n \to S_n$)

[6] Why this is fully appropriate will be explained in Section 3.1.

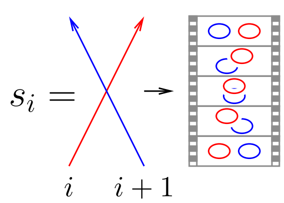
defined by $\varsigma(\sigma_i) = \varsigma(s_i) = s_i$ (or picturially, by $\varsigma(\text{⤧}) = \varsigma(\text{⤧}) = \text{⤫}$). Thus the symmetric group S_n is both a subgroup and a quotient group of vB_n (and/or wB_n).

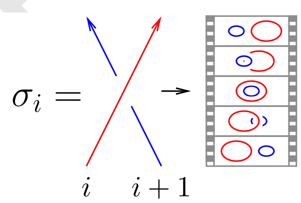
While mostly in this paper the picturial / algebraic definition of w-braids (and other w-knotted objects) will suffice, we ought describe at least briefly 2-3 further interpretations of wB_n:

2.1.1. *The group of flying rings.* Let X_n be the space of all placements of n numbered disjoint geometric circles in $\mathbb{R}^3$, such that all circles are parallel to the xy plane. Such placements will be called horizontal. A horizontal placement is determined by the centers in $\mathbb{R}^3$ of the n circles and by n radii, so $\dim X_n = 3n + n = 4n$. The permutation group S_n acts on X_n by permuting the circles, and one may think of the quotient $\tilde{X}_n := X_n/S_n$ as the space of all horizontal placements of n anonymous circles in $\mathbb{R}^3$. The fundamental group $\pi_1(\tilde{X}_n)$ is a group of paths traced by n disjoint horizontal circles (modulo homotopy), so it is fair to think of it as "the group of flying rings".

Theorem 1. *The group of w-braids wB_n is isomorphic to the group of flying rings $\pi_1(\tilde{X}_n)$.*

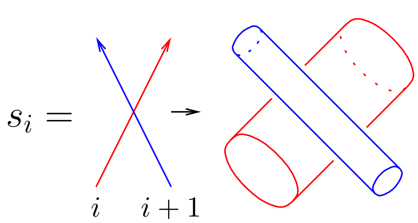
For the proof of this theorem, see [Gol, Sa] and especially [BH]. Here we will contend ourselves with pictures describing the images of the generators of wB_n in $\pi_1(\tilde{X}_n)$ and a few comments:

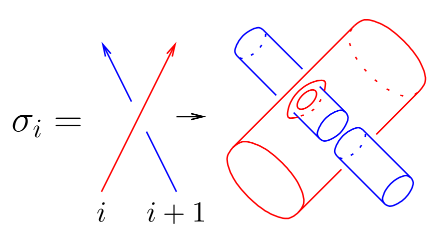




Thus we map the permutation s_i to the movie clip in which ring number i trades its place with ring number $i+1$ by having the two flying around each other. This acrobatic feat is performed in $\mathbb{R}^3$ and it does not matter if ring number i goes "above" or "below" or "left" or "right" of ring number $i+1$ when they trade places, as all of these possibilities are homotopic. More interestingly, we map the braiding σ_i to the movie clip in which ring $i+1$ shrinks a bit and flies through[7] ring i. It is a worthwhile exercise for the reader to verify that the relations in the definition of wB_n become homotopies of movie clips. Of these relations it is most interesting to see why the "overcrossings commute" relation $\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}$ holds, yet the "undercrossings commute" relation $\sigma_i^{-1}\sigma_{i+1}^{-1}s_i = s_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}$ doesn't.

2.1.2. *Certain ribbon tubes in $\mathbb{R}^4$.* With time as the added dimension, a flying ring in $\mathbb{R}^3$ traces a tube (an annulus) in $\mathbb{R}^4$, as shown in the picture below:





[7]To be perfectly precise, we have to specify the fly-through direction. Our convention can be inferred from the pictures. Yet, since it will not be used, we make no effort to make it more explicit.

Note that we adopt here the drawing conventions of Carter and Saito [CS] — we draw surfaces as if they were projected from $\mathbb{R}^4$ to $\mathbb{R}^3$, and we cut them open whenever they are "hidden" by something with a higher t coordinate.

Note also that the tubes we get in $\mathbb{R}^4$ always bound natural 3D "solids" — their "insides", in the pictures above. These solids are disjoint in the case of s_i and have a very specific kind of intersection in the case of σ_i — these are transverse intersections with no triple points, and their inverse images are a meridianal disk on the "thin" solid tube and an interior disk on the "thick" one. By analogy with the case of ribbon knots and ribbon singularities in $\mathbb{R}^3$ (e.g. [Ka1, Chapter V]) and following Satoh [Sa], we call this kind if intersections of solids in $\mathbb{R}^4$ "ribbon singularities" and thus our tubes in $\mathbb{R}^4$ are always "ribbon tubes".

2.1.3. *Basis conjugating automorphisms of F_n.* Let F_n be the free (non-Abelian) group with generators $x_1, \dots, x_n$. Artin's theorem (Theorems 15 and 16 of [Ar]) says that that the (ordinary) braid group B_n (equivalently, the subgroup of wB_n generated by the σ_i's) is isomorphic to the group of automorphisms $B : F_n \to F_n$ of F_n that satisfy the following two conditions:

(1) B maps any generator x_i to a conjugate of a generator (possibly different). That is, there is a permutation $\beta \in S_n$ and elements $a_i \in F_n$ so that for every i,
$$B(x_i) = a_i^{-1} x_{\beta i} a_i.$$
(2) B fixes the ordered product of the generators of F_n,
$$B(x_1 x_2 \cdots x_n) = x_1 x_2 \cdots x_n.$$

McCool's theorem [Mc] says that the same hold true if one replaces the braid group B_n with the bigger group wB_n and drops the second condition above. So wB_n is precisely the group of "basis-conjugating" automorphisms of the free group F_n, the group of those automorphisms which map any "basis element" in $\{x_1 \dots x_n\}$ to a conjugate of a (possibly different) basis element.

We contend ourselves with a quick description of the relevant map $\Psi : wB_n \to \mathrm{Aut}(F_n)$:
$$\Psi(s_i) = \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_i \\ x_j \mapsto x_j & j \neq i, i+1 \end{cases} \qquad \Psi(\sigma_i) = \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1} \\ x_j \mapsto x_j & j \neq i, i+1 \end{cases}$$
It is a worthwhile exercise for the reader to verify that Ψ respects the relations in the definition of wB_n.

2.2. **Finite Type Invariants of v-Braids and w-Braids.** In the standard theory of finite type invariants of knots (also known as Vassiliev or Goussarov-Vassiliev invariants) [Gou, Va, BN1, BN6] one progresses from the definition of finite type via iterated differences to chord diagrams and weight systems, to $4T$ (and other) relations, to the definition of universal finite type invariants, and beyond. The exact same progression (with different objects playing similar roles) is also seen in the theories of finite type invariants of braids [BN4], 3-manifolds [Oh, LMO, Le], virtual knots [GPV, Po] and of several other classes of objects. We thus assume that the reader has familiarity with these basic ideas, and we only indicate briefly how they are implemented in the case of v-braids and w-braids. Some further details are in Section 4.2.

✓ and sometimes, when yet unexplored, with the last steps missing

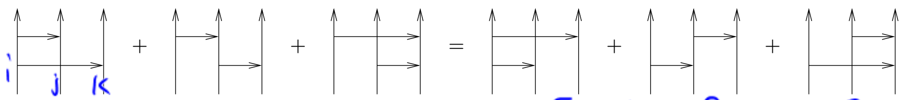


Figure 2. The $6T$ relation. Standard knot theoretic conventions apply — only the relevant parts of each diagram is shown; and reality each diagram may have further vertical strands and horizontal arrows, provided the extras are the same in all 6 diagrams. Also, the vertical strands are in no particular order — other valid $6T$ relations are obtained when those strands are permuted in other ways.

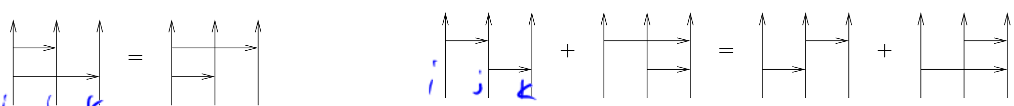


Figure 3. The TC and the $4\vec{T}$ relations.

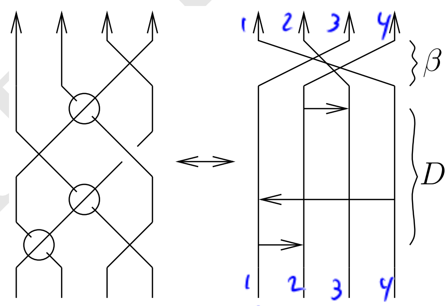


Figure 1. A 3-singular v-braid and its corresponding 3-arrow diagram.

Much like the formula $\times = \times - \times$ of the Vassiliev-Goussarov fame, given a v-braid invariant $V : vB_n \to A$ valued in some Abelian group A, we extend it to "singular" v-braids, braids that contain "semi-virtual crossings" like $\times$ and $\times$ using the formulas $V(\times) := V(\times) - V(\times)$ and $V(\times) := V(\times) - V(\times)$ (see [GPV, Po]). We say that "V is of type m" if its exention vanishes on singular v-braids having more than m semi-virtual crossings. Up to invariants of lower type, an invariant of type m is determined by its "weight system", which is a functional $W = W_m(V)$ defined on "m-singular v-braids modulo $\times = \times = \times$". Let us denote the set of all such equivalence classes by $\mathcal{G}_m\mathcal{D}^v_n$. Much as m-singular knots modulo $\times = \times$ can be identified with chord diagrams, the elements of $\mathcal{G}_m\mathcal{D}^v_n$ can be identified with pairs (D, β), where D is a horizontal arrow diagram and β is a "skeleton permutation". See the figure on the right.

We assmble the spaces $\mathcal{G}_m\mathcal{D}^v_n$ together to form a single graded space, $\mathcal{D}^v_m := \oplus_{m=0}^{\infty}\mathcal{G}_m\mathcal{D}^v_n$. Note that throughout this paper, whenever we write an infinite direct sum, we automatically complete it. Thus in $\mathcal{D}^v_n$ we allow infinite sums with one term in each homogeneous piece $\mathcal{G}_m\mathcal{D}^v_n$.

In the standard finite-type theory for knots, weight systems always satisfy the $4T$ relation, and are therefore functionals on $\mathcal{A} := \mathcal{D}/4T$. Likewise, in the case of v-braids, weight systems satisfy the "$6T$ relation" of [GPV, Po], shown in Figure 2, and are therefore functionals on $\mathcal{A}^v_n := \mathcal{D}^v_n/6T$. In the case of w-braids, the "overcrossings commute" relation (3) implies the "tails commute" (TC) relation on the level of arrow diagrams, and in the presence of the TC relation, two of the terms in the $6T$ relation drop out, and what remains is the "$4T$" relation. These relations are shown in Figure 3. Thus weight systems of finite type invariants of w-braids are linear functionals on $\mathcal{A}^w_n := \mathcal{D}^v_n/TC, 4\vec{T}$.

The next question that arises is whether we have already found *all* the relations that weight systems always satisfy. More precisely, given a degree m linear functional on $\mathcal{A}_n^v = \mathcal{D}_n^v/6T$ (or on $\mathcal{A}_n^w = \mathcal{D}_n^v/TC, 4\vec{T}$), is it always the weight system of some type m invariant V of v-braids (or w-braids)? As in every other theory of finite type invariants, the answer to this question is affirmative if and only if there exists a "Universal Finite Type Invariant" (UFTI) of v-braids (w-braids):

Definition 2.4. A universal finite type invariant of v-braids (w-braids) is an invariant $Z : vB_n \to \mathcal{A}_n^v$ (or $Z : wB_n \to \mathcal{A}_n^w$) satisfying the following "universality condition":

- If B is an m-singular v-braid (w-braid) and $D \in \mathcal{G}_m\mathcal{D}_n^v$ is its underlying arrow diagram as in Figure 1, then
$$Z(B) = D + (\text{terms of degree } > m).$$

Indeed if Z is a UFTI and $W \in \mathcal{G}_m\mathcal{A}^\star$,[8] the universality condition implies that $W \circ Z$ is a finite type invariant whose weight system is W. To go the other way, if (D_i) is a basis of $\mathcal{A}$ consisting of homogeneous elements, if (W_i) is the dual basis of $\mathcal{A}^\star$ and (V_i) are finite type invariants whose weight systems are the W_i's, then $Z(B) := \sum_i D_i V_i(B)$ defines a universal finite type invariant.

In general, constructing a universal finite type invariant is a hard task. For knots, one uses either the Kontsevich integral or perturbative Chern-Simons theory (also known as "configuration space integrals" [BoTa] or "tinkertoy towers" [Th]) or the rather fancy algebraic theory of "Drinfel'd associators" (a summary of all those approaches is at [BS]. For homology spheres, this is the "LMO invariant" [LMO, Le] (also the "Århus integral" [BGRT]). For v-braids, we still don't know if a UFTI exists. As we shall see below, the construction of a UFTI for w-braids is quite easy.

2.3. A Universal Finite Type Invariant of w-Braids. MORE.

3. w-Tangles

MORE, but first more on Section 4.2.

3.1. Circuit Algebras. MORE.

4. Odds and Ends

4.1. What means "closed form"?
As stated earlier, one of my hopes for this paper is that it will lead to closed-form formulas for tree-level associators. The notion "closed-form" in itself requires an explanation (see footnote 3). Is e^x a closed form expression for $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, or is it just an artificial name given for a trancendental expression we cannot otherwise reduce? Likewise, why not call some tree-level associator Φ^{tree} and now it is "in closed form"?

For us, "closed-form" should mean "useful for computations". More precisely, it means that the quantity in question is an element of some space $\mathcal{A}^{cf}$ of "useful closed-form thingies" whose elements have finite descriptions (hopefully, finite and short) and on which some operations are defined by algorithms which terminate in finite time (hopefully, finite and short). Furthermore, there should be a finite-time algorithm to decide whether two descriptions of elements of $\mathcal{A}^{cf}$ describe the same element. (In our context, if it is hard to decide within the

[8] $\mathcal{A}$ here denotes either $\mathcal{A}_n^v$ or $\mathcal{A}_n^w$, or in fact, any of many similar spaces that we will discuss later on.

target space of an invariant whether two elements are equal or not, the invariant is not too useful in deciding whether two knotted objects are equal or not).

Thus for example, polynomials in a variable x are always of closed form, for they are simply described by finite sequences of integers (which in themselves are finite sequences of digits), the standard operations on polynomials ($+$, $\times$, and, say, $\frac{d}{dx}$) are algorithmically computable, and it is easy to write the "polynomial equality" computer program. Likewise for rational functions and even for rational functions of x and e^x.

On the other hand, general elements Φ of the space $\mathcal{A}^{\text{tree}}(\uparrow_3)$ of potential tree-level associators are not closed-form, for they are determined by infinitely many coefficients. Thus iterative constructions of asosciators, such as the one in [BN3] are computationally useful only within bounded-degree quotients of $\mathcal{A}^{\text{tree}}(\uparrow_3)$ and not as all-degree closed-form formulas. Likewise, "explicit" formulas for an associator Φ in terms of multiple ζ-values (e.g. [LM1]) are not useful for computations as it is not clear how to apply tangle-theoretic operations to Φ (such as $\Phi \mapsto \Phi^{1342}$ or $\Phi \mapsto (1 \otimes \Delta \otimes 1)\Phi$) while staying within some space of "objects with finite description in terms of multiple ζ-values". And even if a reasonable space of such objects could be defined, it remains an open problem to decide whether a given rational linear combination of multiple ζ-values is equal to 0.

4.2. **Finite Type Invariants of v-Braids and w-Braids, in some Detail.** As mentioned in Remark 2.2, w-braids are v-braids modulo an additional relation. So we start with a discussion of finite type invariants of v-braids. For simplicity we take our base ring to be $\mathbb{Q}$; everywhere we could replace it by an arbitrary field of characteristic 0[9], and many definitions make sense also over $\mathbb{Z}$ or even with $\mathbb{Q}$ replaced by an arbitrary Abelian group.

4.2.1. *Basic Definitions.* Let $\mathbb{Q}vB_n$ denote group ring of vB_n, the algebra of formal linear combinations of elements of vB_n, and let $\mathbb{Q}S_n$ be the group ring of S_n. The skeleton homomorphism of Remark 2.3 extends to a homomorphism $\varsigma : \mathbb{Q}vB_n \to \mathbb{Q}S_n$. Let $\mathcal{I}$ (or $\mathcal{I}_n$ when we need to be more precise) denote the kernel of the skeleton homomorphism; it is the ideal in $\mathbb{Q}vB_n$ generated by formal differences of v-braids having the same skeleton. One may easily check that $\mathcal{I}$ is generated by differences of the form $\overcrossing - \virtualcrossing$ and $\virtualcrossing - \undercrossing$. Following [GPV] we call such differences "semi-virtual crossings" and denote them by $\semivirtualover$ and $\semivirtualunder$, respectively[10]. In a similar manner, for any natural number m the mth power $\mathcal{I}^m$ of $\mathcal{I}$ is generated by "m-fold iterated differences" of v-braids, or equally well, by "m-singular v-braids", which are v-braids that are also have exactly m semi-virtual crossings (subject to relations which we don't need to specify).

Let $V : vB_n \to A$ be an invariant of v-braids with values in some vector space A. We say that V is "of type m" (for some $m \in \mathbb{Z}_{\geq 0}$) if its linear extention to $\mathbb{Q}vB_n$ vanishes on $\mathcal{I}^{m+1}$ (alternatively, on all $m+1$-singular v-braids, in clear analogy with the standard definition of finite type invariants). If V is of type m for some unspecified m, we say that V is "of finite type". Given a type m invariant V, we can restrict it to $\mathcal{I}^m$ and as it vanishes on $\mathcal{I}^{m+1}$, this restriction can be regarded as an element of $(\mathcal{I}^m/\mathcal{I}^{m+1})^\star$. If two type m invariants define the same element of $(\mathcal{I}^m/\mathcal{I}^{m+1})^\star$ then their difference vanishes on $\mathcal{I}^m$, and so it is an invariant

[9] Or using the variation of constants method, we can simply declare that $\mathbb{Q}$ is an arbitrary field of characteristic 0.

[10] The signs in $\semivirtualover \leftrightarrow \overcrossing - \virtualcrossing$ and $\semivirtualunder \leftrightarrow \virtualcrossing - \undercrossing$ are "crossings come with their sign and their virtual counterparts come with the opposite sign".

of type $m-1$. Thus it is clear that an understanding of $\mathcal{I}^m/\mathcal{I}^{m+1}$ will be instrumental to an inductive understanding of finite type invariants. Hence the following definition.

Definition 4.1. The projectivization[11] $\operatorname{proj} vB_n$ is the direct sum

$$\operatorname{proj} vB_n := \bigoplus_{m\geq 0} \mathcal{I}^m/\mathcal{I}^{m+1}.$$

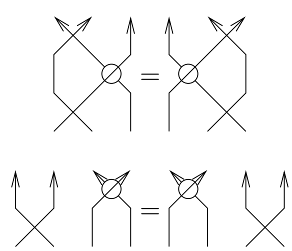
Note that throughout this paper, whenever we write an infinite direct sum, we automatically complete it. Therefore an element in $\operatorname{proj} vB_n$ is an infinite sequence $D=(D_0, D_1, \dots)$, where $D_m \in \mathcal{I}^m/\mathcal{I}^{m+1}$. The projectivization $\operatorname{proj} vB_n$ is a graded space, with the degree m piece being $\mathcal{I}^m/\mathcal{I}^{m+1}$.

We proceed with the study of $\operatorname{proj} vB_n$ (and thus of finite type invariants of v-braids) in three steps. In Section 4.2.2 we introduce a space $\mathcal{D}_n^v$ and a surjection $\rho_0 : \mathcal{D}_n^v \to \operatorname{proj} vB_n$. In Section 4.2.3 we find some relations in $\ker \rho_0$, most notably the $6T$ relation, and introduce the quotient $\mathcal{A}_n^v := \mathcal{D}_n^v/6T$. And then in Section 4.2.4 we introduce the notion of a "universal finite type invariant" and explain how the existance of such a gadget proves that $\operatorname{proj} vB_n$ is isomorphic to $\mathcal{A}_n^v$ (in a more traditional language this is the statement that every weight system integrates to an invariant).

Unfortunately, we do not know if there is a universal finite type invarint of v-braids. Thus in Section 4.3 we return to the subject of w-braids and prove the weaker statement that there exists a universal finite type invariant of w-braids.

4.2.2. *Arrow Diagrams.* We are looking for a space that will surject on $\mathcal{I}^m/\mathcal{I}^{m+1}$. In other words, we are looking for a set of generators for $\mathcal{I}^m$, and we are willing to call two such generators the same if their difference is in $\mathcal{I}^{m+1}$. But that's easy. Left and right multiples of the formal differences $\semivirtualover = \overcrossing - \virtualcrossing$ and $\semivirtualunder = \undercrossing - \virtualcrossing$ generate I, so products of the schematic form

$$B_0(\semivirtualover|\semivirtualunder)B_1(\semivirtualover|\semivirtualunder)B_2\cdots B_{m-1}(\semivirtualover|\semivirtualunder)B_m \tag{4}$$



generate $\mathcal{I}^m$ (here $(\semivirtualover|\semivirtualunder)$ means "either a $\semivirtualover$ or a $\semivirtualunder$", and there are exactly m of those in any product). Furtheremore, inside such a product any B_k can be replaced by any other v-braid B'_k having the same skeleton (e.g., with $\varsigma(B_k)$), for then $B_k - B'_k \in \mathcal{I}$ and the whole product changes by something in $\mathcal{I}^{m+1}$. Also, the relations in (1) and in (2) imply the relations shown on the right for $\semivirtualover$, and similar relations for $\semivirtualunder$. With this freedom, a product as in (4) is determined by the strand-placements of the $\semivirtualover$'s and the $\semivirtualunder$'s. That is, for each semi-virtual crossing in such a product, we only need to know which strand number is the "over" strand, which strand number is the "under" strand, and a sign that determines whether it is the positive semi-virtual $\semivirtualover$ or the negative semi-virtual $\semivirtualunder$. This motivates the following definition.

Definition 4.2. A "horizontal m-arrow diagrams" (analogues to the "chord diagrams" of, say, [BN1]) is an ordered pair (D,β) in which D is a word of length m in the alphabet $\{a_{ij}^+, a_{ij}^- : i,j \in \{1,\dots,n\},\ i\neq j\}$ and β is a permutation in S_n. Let $\mathcal{D}_m^{vh}$ be the space of formal linear combinations of horizontal m-arrow diagrams. We usually use a picturial notation for horizontal arrow diagram, as demonstrated in Figure 4.

[11] Why "projectivization"? See Section 4.4.

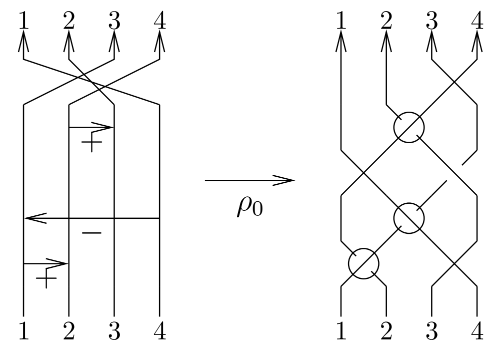


Figure 4. The horizontal 3-arrow diagram $(D, \beta) = (a_{12}^+ a_{41}^- a_{23}^+, 3421)$ and its image via ρ_0: The first arrow, a_{12}^+ starts at strand 1, ends at strand 2 and carries a + sign, so it is mapped to a positive semi-virtual crossing of strand 1 over strand 2. Likewise the second arrow a_{41}^- maps to a negative semi-virtual crossing of strand 4 over strand 1, and a_{23}^+ to a positive semi-virtual crossing of strand 2 over strand 3. We also show one possible choice for a representative of the image of $\rho_0(D, \beta)$ in $\mathcal{I}^m/\mathcal{I}^{m+1}$: it is a v-braid with semi-virtual crossings as specified by D and whose overall skeleton is 3421.

There is a surjection $\rho_0 : \mathcal{D}_m^{vh} \to \mathcal{I}^m/\mathcal{I}^{m+1}$. The definition of ρ_0 is suggested by the first paragraph of this section and an example is shown in Figure 4; we will skip the formal definition here. We also skip the formal proof of the surjectivity of ρ_0.

Finally, consider the product $\boxtimes \cdot \boxtimes$ and use the second Reidemeister move for both virtual and non-virtual crossings:

$$\boxtimes\boxtimes = (\times - \times)(\times - \times) = \times\times + \times\times - \times\times - \times\times = (\times\times - 1) + (\times\times) = \boxtimes\times - \times\boxtimes.$$

If a total of $m-1$ further semi-virtual crossings are multiplied into this equality on the left and on the right, along with arbitrary further crossings and virtual crossings, the left hand side of the equality becomes a member of $\mathcal{I}^{m+1}$, and therefore, as a member of $\mathcal{I}^m/\mathcal{I}^{m+1}$, it is 0. Thus with "..." standing for extras added on the left and on the right, we have that in $\mathcal{I}^m/\mathcal{I}^{m+1}$,

$$0 = \ldots(\boxtimes\times - \times\boxtimes)\ldots = \rho_0(\ldots ?? \ldots)$$

MORE.

4.2.3. *The 6T Relations.* MORE.

4.2.4. *The Notion of a Universal Finite Type Invariant.* MORE.

4.3. **Finite type invariants of w-braids.** MORE.

4.4. **Why "Projectivization"?** The operation we call "projectivization" is exceedingly general and makes sense for an arbitrary "algebraic structures" $\mathcal{O}$. That is, $\mathcal{O}$ is some collection $(\mathcal{O}_\alpha)$ of sets of objects of different kinds, where the subscript α denotes the "kind" of the objects in $\mathcal{O}_\alpha$, along with some collection of "operations" ψ_β, where each ψ_β is an arbitrary map with domain some product $\mathcal{O}_{\alpha_1} \times \cdots \times \mathcal{O}_{\alpha_k}$ of sets of objects, and range a single set $\mathcal{O}_{\alpha_0}$ (so operations may be unary or binary or multinary, but they always return a value of some fixed kind). We also allow some named "constants" within some $calO_\alpha$'s (or equivalently, allow some 0-nary operations).* The operations may or may not be subject to

foot: * one may alternatively define "algebraic structures" using the theory of "multi categories" [ref]. Using this language, an algebraic structure is simply a functor from some "structure" multi-category $\mathcal{C}$ into sets (or into Vect, if all $\mathcal{O}_i$ are assumed to be V.S. and all operations are linear). Using this language, a "morphism" . . .

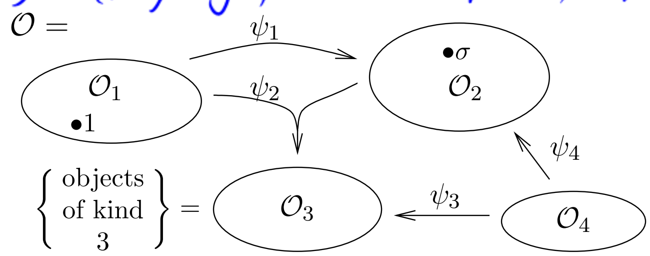


Figure 5. An algebraic structure $\mathcal{O}$ with 4 kinds of objects and one binary, 3 unary and two 0-nary operations (the constants 1 and σ).

axioms — an "axiom" is an identity asserting that some composition of operations is equal to some other composition of operations.

Figure 5 illustrates the general notion of an algebraic structure. Here are a few specific examples:

- Groups: one kind of objects, one binary "multiplication", one unary "inverse", one constant "the identity", and some axioms.
- Group homomorphisms: Two kinds of objects, one for each group. 7 operations — 3 for each of the two groups and the homomorphism itself, going between the two groups. Many axioms.
- A group acting on a set, a group extension, a split group extension and many other examples from group theory.
- A quandle. It is worthwhile to quote the abstract of the paper that introduced the definition (Joyce, [Joy]):

  The two operations of conjugation in a group, $x \triangleright y = y^{-1}xy$ and $x \triangleright^{-1} y = yxy^{-1}$ satisfy certain identities. A set with two operations satisfying these identities is called a quandle. The Wirtinger presentation of the knot group involves only relations of the form $y^{-1}xy = z$ and so may be construed as presenting a quandle rather than a group. This quandle, called the knot quandle, is not only an invariant of the knot, but in fact a classifying invariant of the knot.
- Planar algebras as in [Jon] and circuit algebras as in Section 3.1.
- The algebra of knotted trivalent graphs as in [BN7, Da].
- Let $\varsigma : B \to S$ be an arbitrary homomorphism of groups (though our notation suggests what we have in mind — B may well be braids, and S may well be permutations). We can consider an algebraic structure $\mathcal{O}$ whose kinds are the elements of S, for which the objects of kind $s \in S$ are the elements of $\mathcal{O}_s := \varsigma^{-1}(s)$, and with the product in B definning operations $\mathcal{O}_{s_1} \times \mathcal{O}_{s_2} \to \mathcal{O}_{s_1 s_2}$.
- Clearly, many more examples appear throughout mathematics.

Any algebraic structure $\mathcal{O}$ has a projectivization. First extend $\mathcal{O}$ to allow formal linear combinations of objects of the same kind (extending the operations in a linear or multi-linear manner), then let $\mathcal{I}$ be the sub-structure made out of all such combinations in which the sum of coefficients is 0, then let $\mathcal{I}^m$ be the set of all outputs of algebraic expressions (that is, arbitrary compositions of the operations in $\mathcal{O}$) that have at least m inputs in $\mathcal{I}$, and finally, set

(5) $$\operatorname{proj} \mathcal{O} := \bigoplus_{m \geq 0} \mathcal{I}^m / \mathcal{I}^{m+1}.$$

Clearly, $\mathrm{proj}\,\mathcal{O}$ is again algebraic structure, with the same multi-graph of spaces and operations as $\mathcal{O}$, but with new objects and with new operations that may or may not satisfy the axioms satisfied by the operations of $\mathcal{O}$. The main new feature in $\mathrm{proj}\,\mathcal{O}$ is that it is quite clearly a "graded" structure.

I believe that many of the most interesting graded structures that appear in mathematics are the result of this construction, and that many of the interesting graded equations that appear in mathematics arise when one tries to find "expansions", or "universal finite type invariants", which are also morphisms[12] $Z : \mathcal{O} \to \mathrm{proj}\,\mathcal{O}$ or when one studies "automorphisms" of such expansions[13] Indeed, the paper you are reading now is really the study of the projectivizations of various algebraic structures associated with w-knotted objects. I would like to believe that much of the theory of quantum groups (at "generic" $\hbar$) will eventually be shown to be a study of the projectivizations of various algebraic structures associated with v-knotted objects.

Thus I believe that the operation described in Equation (5) is truly fundamental and therefore worthy of a catchy name. So why "projectivization"? Well, it reminds me of graded spaces, but really, that's all. I simply found no better name. I'm open to suggestions.

References

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators,* arXiv:0802.4300.

[Ar] E. Artin, *Theory of Braids,* Ann. of Math. **48-1** (1947) 101–126.

[BWC] J. C. Baez, D. K. Wise and A. S. Crans, *Exotic Statistics for Strings in 4d BF Theory,* Adv. Theor. Math. Phys. **11** (2007) 707–749, arXiv:gr-qc/0603085.

[BN1] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology **34** (1995) 423–472.

[BN2] D. Bar-Natan, *Vassiliev homotopy string link invariants,* Jour. of Knot Theory and its Ramifications **4** (1995) 13–32.

[BN3] D. Bar-Natan, *Non-associative tangles,* in *Geometric topology* (proceedings of the Georgia international topology conference), (W. H. Kazez, ed.), 139–183, Amer. Math. Soc. and International Press, Providence, 1997.

[BN4] D. Bar-Natan, *Vassiliev and quantum invariants of braids,* in Proc. of Symp. in Appl. Math. **51** (1996) 129–144, *The interface of knots and physics,* (L. H. Kauffman, ed.), Amer. Math. Soc., Providence.

[BN5] D. Bar-Natan, *On Associators and the Grothendieck-Teichmuller Group I,* Selecta Mathematica, New Series **4** (1998) 183–212.

[BN6] D. Bar-Natan, *Finite Type Invariants,* in *Encyclopedia of Mathematical Physics,* (J.-P. Francoise, G. L. Naber and Tsou S. T., eds.) Elsevier, Oxford, 2006 (vol. 2 p. 340).

[BN7] D. Bar-Natan, *Algebraic Knot Theory — A Call for Action,* web document, 2006, `http://www.math.toronto.edu/~drorbn/papers/AKT-CFA.html`.

[BN8] D. Bar-Natan, *in preparation.*

[BGRT] D. Bar-Natan, S. Garoufalidis, L. Rozansky and D. P. Thurston, *The Århus integral of rational homology 3-spheres I–III,* Selecta Mathematica, New Series **8** (2002) 315–339, arXiv:q-alg/9706004, **8** (2002) 341–371, arXiv:math.QA/9801049, **10** (2004) 305–324, arXiv:math.QA/9808013.

[BLT] D. Bar-Natan, T. Q. T. Le and D. P. Thurston, *Two applications of elementary knot theory to Lie algebras and Vassiliev invariants,* Geometry and Topology **7-1** (2003) 1–31, arXiv:math.QA/0204311.

[12] Indeed, if $\mathcal{O}$ is finitely presented then finding such a morphism $Z : \mathcal{O} \to \mathrm{proj}\,\mathcal{O}$ amounts to finding its values on the generators of $\mathcal{O}$, subject to the relations of $\mathcal{O}$. Thus it is equivalent to solving a system of equations written in some graded spaces.

[13] The Drinfel'd graded Grothendieck-Teichmuller group GRT is an example of such an automorphism group. See [Dr3, BN5].

[BS] D. Bar-Natan and A. Stoimenow, *The fundamental theorem of Vassiliev invariants,* in Proc. of the Århus Conf. *Geometry and physics,* (J. E. Andersen, J. Dupont, H. Pedersen, and A. Swann, eds.), lecture notes in pure and applied mathematics **184** (1997) 101–134, Marcel Dekker, New-York. Also arXiv:q-alg/9702009.

[BP] B. Berceanu and S. Papadima, *Universal Representations of Braid and Braid-Permutation Groups,* arXiv:0708.0634.

[BoTa] R. Bott and C. Taubes, *On the self-linking of knots,* Jour. Math. Phys. **35** (1994).

[BH] T. Brendle and A. Hatcher, *Configuration Spaces of Rings and Wickets,* arXiv:0805.4354.

[CS] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams,* Mathematical Surveys and Monographs **55**, American Mathematical Society, Providence 1998.

[CCM] A. S. Cattaneo, P. Cotta-Ramusino and M. Martellini, *Three-dimensional BF Theories and the Alexander-Conway Invariant of Knots,* Nucl. Phys. **B436** (1995) 355–384, arXiv:hep-th/9407070.

[CCFM] A. S. Cattaneo, P. Cotta-Ramusino, J. Froehlich and M. Martellini, *Topological BF Theories in 3 and 4 Dimensions,* J. Math. Phys. **36** (1995) 6137–6160, arXiv:hep-th/9505027.

[Da] Z. Dancso, *On a Kontsevich Integral for Knotted Trivalent Graphs,* in preparation.

[Dr1] V. G. Drinfel'd, *Quantum Groups,* in *Proceedings of the International Congress of Mathematicians,* 798–820, Berkeley, 1986.

[Dr2] V. G. Drinfel'd, *Quasi-Hopf Algebras,* Leningrad Math. J. **1** (1990) 1419–1457.

[Dr3] V. G. Drinfel'd, *On Quasitriangular Quasi-Hopf Algebras and a Group Closely Connected with* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$*,* Leningrad Math. J. **2** (1991) 829–860.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I,* Selecta Mathematica, New Series **2** (1996) 1–41, arXiv:q-alg/9506005.

[FRR] R. Fenn, R. Rimanyi and C. Rourke, *The Braid-Permutation Group,* Topology **36** (1997) 123–135.

[Gol] D. L. Goldsmith, *The Theory of Motion Groups,* Mich. Math. J. **28-1** (1981) 3–17.

[Gou] M. Goussarov, *On n-equivalence of knots and invariants of finite degree,* Zapiski nauch. sem. POMI **208** (1993) 152–173 (English translation in *Topology of manifolds and varieties* (O. Viro, editor), Amer. Math. Soc., Providence 1994, 173–192).

[GPV] M. Goussarov, M. Polyak and O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

[HM] N. Habegger and G. Masbaum, *The Kontsevich integral and Milnor's invariants,* Topology **39** (2000) 1253–1289.

[Ha] A. Haviv, *Towards a diagrammatic analogue of the Reshetikhin-Turaev link invariants,* Hebrew University PhD thesis, September 2002, arXiv:math.QA/0211031.

[Jon] V. Jones, *Planar algebras, I,* New Zealand Journal of Mathematics, to appear, arXiv:math.QA/9909027.

[Joy] D. Joyce, *A Classifying Invariant of Knots, the Knot Quandle,* Journal of Pure and Appl. Algebra **23** (1982) 37–65.

[KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff Formula and Invariant Hyperfunctions,* Invent. Math. **47** (1978) 249–272.

[Ka1] L. H. Kauffman, *On knots,* Princeton Univ. Press, Princeton, 1987.

[Ka2] L. H. Kauffman, *Virtual Knot Theory,* European J. Comb. **20** (1999) 663–690, arXiv:math.GT/9811028.

[Ku] V. Kurlin, *Compressed Drinfeld associators,* Journal of Algebra **292-1** (2005) 184–242.

[Le] T. Q. T. Le, *An invariant of integral homology 3-spheres which is universal for all finite type invariants,* in *Solitons, geometry and topology: on the crossroad*, (V. Buchstaber and S. Novikov, eds.) AMS Translations Series 2, Providence, arXiv:q-alg/9601002.

[LM1] T. Q. T. Le and J. Murakami, *On Kontsevich's integral for the HOMFLY polynomial and relations of multiple ζ-numbers,* Topology and its Applications **62** (1995) 193–206.

[LM2] T. Q. T. Le and J. Murakami, *The universal Vassiliev-Kontsevich invariant for framed oriented links,* Compositio Math. **102** (1996) 41–64, arXiv:hep-th/9401016.

[LMO] T. Q. T. Le, J. Murakami and T. Ohtsuki, *On a universal quantum invariant of 3-manifolds,* Topology **37-3** (1998) 539–574, arXiv:q-alg/9512002.

[Lee] P. Lee, *Closed-Form Associators and Braidors in a Partly Commutative Quotient,* University of Toronto preprint, December 2007, `http://individual.utoronto.ca/PetersKnotPage/`.
[Li1] J. Lieberum, *The Drinfeld associator of* $gl(1|1)$*,* arXiv:math.QA/0204346.
[Mc] J. McCool, *On Basis-Conjugating Automorphisms of Free Groups,* Can. J. Math. **38-6** (1986) 1525–1529.
[Na] G. Naot, *On Chern-Simons Theory with an Inhomogeneous Gauge Group and BF Theory Knot Invariants,* J. Math. Phys. **46** (2005) 122302, arXiv:math.GT/0310366.
[Oh] T. Ohtsuki, *Finite type invariants of integral homology 3-spheres,* Jour. of Knot Theory and its Ramifications **5(1)** (1996) 101–115.
[Po] M. Polyak, *On the Algebra of Arrow Diagrams,* Let. Math. Phys. **51** (2000) 275–291.
[Sa] S. Satoh, *Virtual Knot Presentations of Ribbon Torus Knots,* J. of Knot Theory and its Ramifications **9-4** (2000) 531–542.
[Th] D. Thurston, *Integral expressions for the Vassiliev knot invariants,* Harvard University senior thesis, April 1995, arXiv:math.QA/9901110.
[Va] V. A. Vassiliev, *Cohomology of knot spaces,* in *Theory of Singularities and its Applications (Providence)* (V. I. Arnold, ed.), Amer. Math. Soc., Providence, 1990.

Department of Mathematics, University of Toronto, Toronto Ontario M5S 2E4, Canada
E-mail address: `drorbn@math.toronto.edu`
URL: `http://www.math.toronto.edu/~drorbn`