FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS: FROM ALEXANDER TO KASHIWARA AND VERGNE

DROR BAR-NATAN

Abstract. W-knots, and more generally, w-knotted objects (w-braids, w-tangles, etc.) make a class of knotted objects which is wider but weaker than their "ordinary" counterparts. To get (say) w-knots from ordinary knots, one has to allow non-planar "virtual" knot diagrams, hence enlarging the the base set of knots. But then one imposes a new relation, the "overcrossings commute" relation, further beyond the ordinary collection of Reidemeister moves, making w-knotted objects a bit weaker once again.

The group of w-braids was studied (under the name "welded braids") by Fenn, Rimanyi and Rourke [FRR] and was shown to be isomorphic to the McCool group [Mc] of "basis-conjugating" automorphisms of a free group F_n — the smallest subgroup of $\mathrm{Aut}(F_n)$ that contains both braids and permutations. Brendle and Hatcher [BH], in work that traces back to Goldsmith [Gol], have shown this group to be a group of movies of flying rings in $\mathbb{R}^3$. Satoh [Sa] studied several classes of w-knotted objects (under the name "weakly-virtual") and has shown them to be closely related to certain classes of knotted surfaces in $\mathbb{R}^4$. So w-knotted objects are topologically and algebraically interesting.

In this article we study finite type invariants of several classes of w-knotted objects. Following Berceanu and Papadima [BP], we construct a homomorphic universal finite type invariant of w-braids, and hence show that the McCool group of automorphisms is "1-formal". We also construct a homomorphic universal finite type invariant of w-tangles. We find that the universal finite type invariant of w-knots is more or less the Alexander polynomial (details inside).

Much as the spaces $\mathcal{A}$ of chord diagrams for ordinary knotted objects are related to metrized Lie algebras, we find that the spaces $\vec{\mathcal{A}}^w$ of "arrow diagrams" for w-knotted objects are related to not-necessarily-metrized Lie algebras. Many questions concerning w-knotted objects turn out to be equivalent to questions about Lie algebras. Most notably we find that a homomorphic universal finite type invariant of w-knotted trivalent graphs is essentially the same as a solution of the Kashiwara-Vergne [KV] conjecture and much of the Alekseev-Torrosian [AT] work on Drinfel'd associators and Kashiwara-Vergne can be re-intepreted as a study of w-knotted trivalent graphs.

Date: Oct. 14, 2008.

1991 *Mathematics Subject Classification.* 57M25.

Key words and phrases. w-braids, w-knots, w-tangles, knotted graphs, finite type invariants, Alexander polynomial, Kashiwara-Vergne, Associators, Free Lie algebras.

This work was partially supported by NSERC grant RGPIN 262178. Electronic version at `http://www.math.toronto.edu/~drorbn/papers/WKO/`.

To Do.

- Finish the paper.

Contents

1. Introduction

1.1. **Dreams.** I have a dream[1], at least partially founded on reality, that many of the difficult algebraic equations in mathematics, especially those that are written in graded spaces, more especially those that are related in one way or another to quantum groups [Dr1], and even more especially those related to the work of Etingof and Kazhdan [EK], can be understood, and indeed, would appear more natural, in terms of finite type invariants of various topological objects.

I believe this is the case for Drinfel'd's theory of associators [Dr2], which can be interpreted as a theory of well-behaved universal finite type invariants of paranthesized tangles[2] [LM2, BN3], and even more elegantly, as a theory of universal finite type invariants of knotted trivalent graphs [BN8].

I believe this is the case for Drinfel'd's "Grothendieck-Teichmuller group" [Dr3] which is better understood as a group of automorphisms of a certain algebraic structure, also related to universal finite type invariants of paranthesized tangles [BN5].

And I'm optimistic, indeed I believe, that sooner or later the work of Etingof and Kazhdan [EK] on quantization of Lie bialgebras will be re-interpreted as a construction of a well-behaved universal finite type invariant of virtual knots [Ka2] or of some other class of virtually knotted objects. Some preliminary steps in that direction were taken by Haviv [Ha].

I have another dream, to construct a useful "Algebraic Knot Theory". As at least a partial writeup exists [BN7], I'll only state that an important ingredient necessary to fulfil that dream would be a "closed form"[3] formula for an associator, at least in some reduced

[1] Understanding an author's history and psychology ought never be necessary to understand his/her papers, but it may be useful. Nothing material in the rest of this paper relies on Section 1.1.

[2] "q-tangles" in [LM2], "non-associative tangles" in [BN3].

[3] The phrase "closed form" in itself requires an explanation. See Section 4.1.

sense. Formulas for associators or reduced associators were in themselves the goal of several studies undertaken for various other reasons [LM1, Li1, Ku, Lee].

1.2. **Stories.** Thus I was absolutely delighted when in January 2008 Anton Alekseev described to me his joint work [AT] with Charles Torossian — he told me they found a relationship between the Kashiwara-Vergne conjecture [KV], a cousin of the Duflo isomorphism (which I already knew to be knot-theoretic [BLT]), and associators taking values in a space called sder, which I could identify as "tree-level Jacobi diagrams", also a knot-theoretic space related to the Milnor invariants [BN2, HM]. What's more, Anton told me that in certain quotient spaces the Kashiwara-Vergne conjecture can be solved explicitly; this should lead to some explicit associators!

So I spent the following several months trying to understand [AT], and this paper is a summary of my efforts. The main thing I learned is that the Alekseev-Torossian paper, and with it the Kashiwara-Vergne conjecture, fit very nicely with my first dream recalled above, about interpreting algebra in terms of knot theory. Indeed much of [AT] can be reformulated as a construction and a discussion of a well-behaved universal finite type invariant Z of a certain class of knotted objects (which I will call here w-knotted), a certain natural quotient of the space of virtual knots (more precisely, virtual trivalent tangles). And my hopes remain high that later I (or somebody else) will be able to exploit this relationship in directions compatible with my second dream recalled above, on the construction of an "algebraic knot theory".

The story, in fact, is pretier than I was hoping for, for it has the following additional qualities:

- W-knotted objects are quite interesting in themselves: as stated in the abstract, they are related to combinatorial group theory via "basis-conjugating" automorphisms of a free group F_n, to groups of movies of flying rings in $\mathbb{R}^3$, and more generaly, to certain classes of knotted surfaces in $\mathbb{R}^4$. The references include [BH, FRR, Gol, Mc, Sa].
- The "chord diagrams" for w-knotted objects (really, these are "arrow diagrams") describe formulas for invariant tensors in spaces pertaining to not-necessarily-metrized Lie algebras in much of the same way as ordinary chord diagrams for ordinary knotted objects describe formulas for invariant tensors in spaces pertaining to metrized Lie algebras. This observation is bound to have further implications.
- Arrow diagrams also describe the Feynmann diagrams of topological BF theory [CCM, CCFM] and of a certain class of Chern-Simons theories [Na]. Thus it is likely that our story is directly related to quantum field theory[4].
- When composed with the map from knots to w-knots, Z becomes the Alexander polynomial. For links, it becomes an invariant stronger than the multi-variable Alexander polynomial whice contains the multi-variable Alexander polynomial as an easily identifiable reduction. On other w-knotted objects Z has easily identifiable reductions that can be considered as "Alexander polynomials" with good behaviour relative to various knot-theoretic operations — cablings, compositions of tangles, etc. There is also a certain specific reduction of Z that can be considered as the "ultimate Alexander polynomial" — in the appropriate sense, it is the minimal extension of the

[4]Some non-perturbative relations between BF theory and w-knots was discussed by Baez, Wise and Crans [BWC].

Alexander polynomial to other knotted objects which is well behaved under a whole slew of knot theoretic operations, including the ones named above.

1.3. **Plans.** Our order of proceedings is: w-braids, w-knots, w-tangles, w-tangled graphs, and then some odds and ends. For more detailed information consult the "Section Summary" paragraph at the beginning of each of the sections.

1.4. **Acknowledgement.** I wish to thank Anton Alekseev, Scott Carter, Lou Kauffman and Dylan Thurston for comments and suggestions.

2. W-Braids

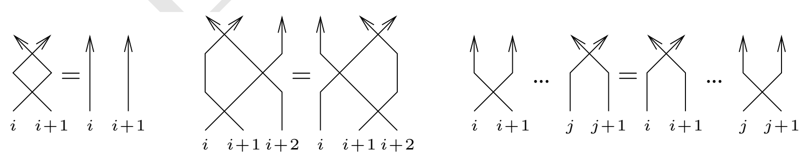
Section Summary. This section is largely a compilation of existing literature, though we also introduce the language of arrow diagrams that we use thoughout the rest of the paper. We define w-braids and survey their relationship with basis-conjugating automorphisms of free groups and with "the group of flying rings in $\mathbb{R}^3$" (really, a group of knotted tubes in $\mathbb{R}^4$). We then play the usual game of introducing finite type invariants, weight systems, chord diagrams (arrow diagrams, for this case), and 4T-like relations. Finally we define and construct a universal finite type invariant for w-braids. It turns out that the only algebraic tool we need to use is the formal exponential function $\exp(a) := \sum a^n/n!$.

2.1. **What are w-braids?** It is simplest to define w-braids in terms of generators and relations, either algebraically or picturially. Algebraically, for a natural number n we set wB_n to be the group generated by symbols σ_i ($1 \leq i \leq n-1$), called "crossing" and graphically represented by an overcrossing "between strand i and strand $i+1$" (with inverse), and s_i, called "virtual crossings" and graphically represented by a non-crossing, , also "between strand i and strand $i+1$", subject to the following relations:

- The subgroup of wB_n generated by the virtual crossings s_i is the symmetric group S_n, and the s_i's correspond to the transpositions $(i, i+1)$. That is, we have

  $$s_i^2 = 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad \text{and if } |i-j|>1 \text{ then} \qquad s_i s_j = s_j s_i.$$

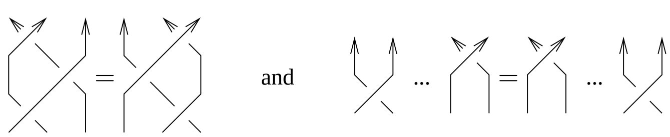
  In pictures, this is

  Note that we read our braids from bottom to top.
- The subgroup of wB_n generated by the crossings σ_i's is the usual braid group B_n, and σ_i corresponds to the braiding of strand i over strand $i+1$. That is, we have

  $$\sigma_i \sigma_{i+1}^{\pm 1} \sigma_i = \sigma_{i+1} \sigma_i^{\pm 1} \sigma_{i+1}, \qquad \text{and if } |i-j|>1 \text{ then} \qquad \sigma_i \sigma_j = \sigma_j \sigma_i.$$

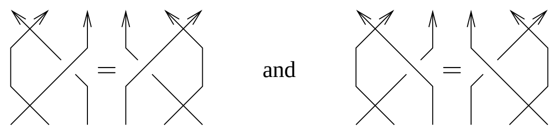
  In pictures, showing only the positive-powers case and dropping the indices, this is

- Some "mixed relations",

$$s_i\sigma_{i+1}^{\pm1}s_i = s_{i+1}\sigma_i^{\pm1}s_{i+1}.$$

In pictures, this is



- Finally, we break the symmetry between over crossings and under crossings by imposing one of the "forbidden moves" virtual knot theory, but not the other:

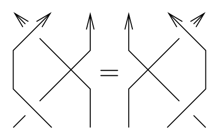
$$\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}, \qquad \text{yet} \qquad \sigma_i^{-1}\sigma_{i+1}^{-1}s_i \neq s_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}. \tag{1}$$

In pictures, this is

The relation we have just imposed may be called the "unforbidden relation", or, perhaps more appropriately, the "overcrossings commute" relation (OC). Ignoring the non-crossings[5] $\times$, the OC relation says that it is the same if strand i first crosses over strand $i+1$ and then over strand $i+2$, or if it first crosses over strand $i+2$ and then over strand $i+1$. The "undercrossings commute" relation UC, the one we do not impose in (1), would say the same, except with "under" replacing "over".

Exercise 2.1. Show that the OC relation is equivalent to the relation

$$\sigma_i^{-1}s_{i+1}\sigma_i = \sigma_{i+1}s_i\sigma_{i+1}^{-1} \qquad \text{or}$$



Remark 2.2. The group we get without imposing the OC relation (1) is the virtual braid group vB_n (sometimes called "the group of v-braids" below). Thus $wB_n = vB_n/OC$.

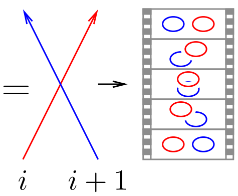
While mostly in this paper the picturial / algebraic definition of w-braids (and other w-knotted objects) will suffice, we ought describe at least briefly 2-3 further interpretations of wB_n:

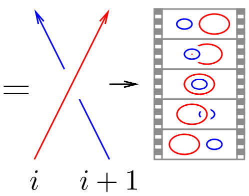
2.1.1. *The group of flying rings.* Let X_n be the space of all placements of n numbered disjoint geometric circles in $\mathbb{R}^3$, such that all circles are parallel to the xy plane. Such placements will be called horizontal. A horizontal placement is determined by the centers in $\mathbb{R}^3$ of the n circles and by n radii, so $\dim X_n = 3n + n = 4n$. The permutation group S_n acts on X_n by permuting the circles, and one may think of the quotient $\tilde{X}_n := X_n/S_n$ as the space of all horizontal placements of n anonymous circles in $\mathbb{R}^3$. The fundamental group $\pi_1(\tilde{X}_n)$ is a group of paths traced by n disjoint horizontal circles (modulo homotopy), so it is fair to think of it as "the group of flying rings".

Theorem 1. *The group of w-braids wB_n is isomorphic to the group of flying rings $\pi_1(\tilde{X}_n)$.*

[5]Why this is fully appropriate will be explained in Section 3.1.

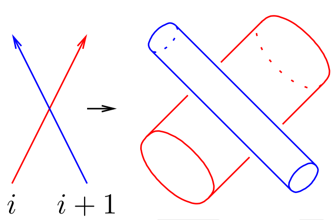
For the proof of this theorem, see [Gol, Sa] and especially [BH]. Here we will contend ourselves with pictures describing the images of the generators of wB_n in $\pi_1(\tilde{X}_n)$ and a few comments:

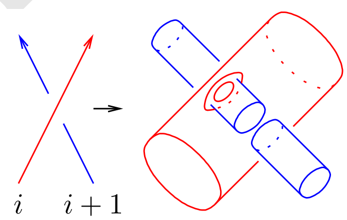
$s_i =$ 

$\sigma_i =$ 

Thus we map the permutation s_i to the movie clip in which ring number i trades its place with ring number $i+1$ by having the two flying around each other. This acrobatic feat is performed in $\mathbb{R}^3$ and it does not matter if ring number i goes "above" or "below" or "left" or "right" of ring number $i+1$ when they trade places, as all of these possibilities are homotopic. More interestingly, we map the braiding σ_i to the movie clip in which ring $i+1$ shrinks a bit and flies through[6] ring i. It is a worthwhile exercise for the reader to verify that the relations in the definition of wB_n become homotopies of movie clips. Of these relations it is most interesting to see why the "overcrossings commute" relation $\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}$ holds, yet the "undercrossings commute" relation $\sigma_i^{-1}\sigma_{i+1}^{-1}s_i = s_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}$ doesn't.

2.1.2. *Certain ribbon tubes in* $\mathbb{R}^4$. With time as the added dimension, a flying ring in $\mathbb{R}^3$ traces a tube (an annulus) in $\mathbb{R}^4$, as shown in the picture below:

$s_i =$ 

$\sigma_i =$ 

Note that we adopt here the drawing conventions of Carter and Saito [CS] — we draw surfaces as if they were projected from $\mathbb{R}^4$ to $\mathbb{R}^3$, and we cut them open whenever they are "hidden" by something with a higher t coordinate.

Note also that the tubes we get in $\mathbb{R}^4$ always bound natural 3D "solids" — their "insides", in the pictures above. These solids are disjoint in the case of s_i and have a very specific kind of intersection in the case of σ_i — these are transverse intersections with no triple points, and their inverse images are a meridianal disk on the "thin" solid tube and an interior disk on the "thick" one. By analogy with the case of ribbon knots and ribbon singularities in $\mathbb{R}^3$ (e.g. [Ka1, Chapter V]) and following Satoh [Sa], we call this kind if intersections of solids in $\mathbb{R}^4$ "ribbon singularities" and thus our tubes in $\mathbb{R}^4$ are always "ribbon tubes".

2.1.3. *Basis conjugating automorphisms of* F_n. Let F_n be the free (non-Abelian) group with generators $x_1, \dots, x_n$. Artin's theorem (Theorems 15 and 16 of [Ar]) says that that the (ordinary) braid group B_n (equivalently, the subgroup of wB_n generated by the σ_i's) is isomorphic to the group of automorphisms $B : F_n \to F_n$ of F_n that satisfy the following two conditions:

[6] To be perfectly precise, we have to specify the fly-through direction. Our convention can be inferred from the pictures. Yet, since it will not be used, we make no effort to make it more explicit.

(1) B maps any generator x_i to a conjugate of a generator (possibly different). That is, there is a permutation $\beta \in S_n$ and elements $a_i \in F_n$ so that for every i,

$$B(x_i) = a_i^{-1} x_{\beta i} a_i.$$

(2) B fixes the ordered product of the generators of F_n,

$$B(x_1 x_2 \cdots x_n) = x_1 x_2 \cdots x_n.$$

McCool's theorem [Mc] says that the same hold true if one replaces the braid group B_n with the bigger group wB_n and drops the second condition above. So wB_n is precisely the group of "basis-conjugating" automorphisms of the free group F_n, the group of those automorphisms which map any "basis element" in $\{x_1 \dots x_n\}$ to a conjugate of a (possibly different) basis element.

We contend ourselves with a quick description of the relevant map $\Psi : wB_n \to \mathrm{Aut}(F_n)$:

$$\Psi(s_i) = \begin{cases} x_i \mapsto x_{i+1} & \\ x_{i+1} \mapsto x_i & \\ x_j \mapsto x_j & j \neq i, i+1 \end{cases} \qquad \Psi(\sigma_i) = \begin{cases} x_i \mapsto x_{i+1} & \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1} & \\ x_j \mapsto x_j & j \neq i, i+1 \end{cases}$$

It is a worthwhile exercise for the reader to verify that Ψ respects the relations in the definition of wB_n.

2.2. **Finite type invariants of w-braids.** In the standard theory of finite type invariants of knots (also known as Vassiliev or Goussarov-Vassiliev invariants) [Gou, Va, BN1, BN6] one progresses from the definition of finite type via iterated differences to chord diagrams and weight systems, to $4T$ (and other) relations, to the definition of a universal finite type invariants, and beyond. The exact same progression (with different objects playing similar roles) is also seen in the theories of finite type invariants of braids [BN4], 3-manifolds [Oh, LMO, Le], virtual knots [GPV, Po] and of several other classes of objects. We thus assume that the reader has familiarity with these basic ideas, and we only indicate briefly how they are implemented in the case of w-braids.

As mentioned in Remark 2.2, w-braids are v-braids modulo an additional relation. So we start with a discussion of finite type invariants of v-braids.

vB

Let $V : wB \to A$ be an invariant of v-braids with values in some Abelian group A. As always and especially as in [GPV] we extend V to "singular v-braids" — v-braids that are also allowed to have "semi-virtual crossings" $\overcrossing_{sv}$ and $\undercrossing_{sv}$* — using the formulas[7]

(2) $$V(\overcrossing_{sv}) := V(\overcrossing) - V(\virtualcrossing) \qquad \text{and} \qquad V(\undercrossing_{sv}) := V(\virtualcrossing) - V(\undercrossing).$$

We then declare that "V is of type m" (for some $m \in \mathbb{Z}_{\geq 0}$) if it vanishes on any singular v-braid that has more than m semi-virtual crossings. If V is of type m for some unspecified m, we say that V is "of finite type". Just as in all other theories of finite type invariants, and it is useful to restrict invariant of type m ~~can be restricted~~ to m-singular v-braids (w-braids with precisely m semi-virtual crossings). This restriction is called "the weight system" $W = W_m(V)$ of V. It can be interpreted as a linear functional on the space $\mathcal{D}_m^{vh}$ of formal $\mathbb{Z}$-linear combinations of "horizontal m-arrow diagrams" — these are the analogues of the "chord diagrams" of, say,

[7]The signs in (2) are "crossings come with their sign and their virtual counterparts come with the opposite sign".

* One may bother to give a formal definition for singular v-braids, stating precisely the relations that the new crossings satisfy along with the old ones. But for our purposes, this is unnecessary and we loose nothing by regarding singular v-braids as short hand notations for

differences or iterated differences with, $x \leftrightarrow X - x$ &

$\cancel{x} \Leftrightarrow X - X$

New par A slightly more mature perspective . . . . .

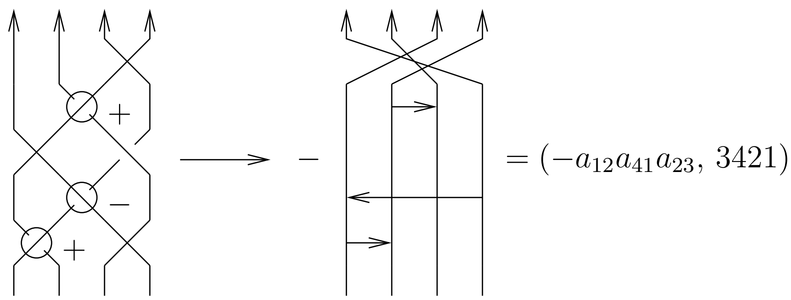


Figure 1. A 3-singular v-braid and its corresponding horizontal arrow diagram. The arrow diagram ignores the crossings and the virtual crossings but retains the underlying permutation (3421, in this case) and some combinatorial information about the semi-virtual crossings: In the first (lowest) the first strands crosses over the second, hences the first arrow starts on the first strand and ends on the second, ~~which may also be denoted as a_{12}~~. In the second strand 4 goes over strand 1, so it is an a_{41}, and the third is an a_{23}. There is an overall minus sign because an odd number of the semi-virtual crossings is negative.

Symbolically, this arrow is denoted a_{12}

[BN1], and a sample is shown in Figure 1. More specifically, a horizontal m-arrow diagrams is formal linear combination of ordered pairs (D, β) in which D is a word of length m in the alphabet $(a_{ij})_{i,j\in\{1,\dots,n\},\, i\neq j}$ and β is a permutation in S_n.

new par above

MORE.

3. W-Tangles

MORE.

3.1. **Circuit Algebras.** MORE.

4. Odds and Ends

4.1. What means "closed form"? As stated earlier, one of my hopes for this paper is that it will lead to closed-form formulas for tree-level associators. The notion "closed-form" in itself requires an explanation (see footnote 3). Is e^x a closed form expression for $\sum_{n=0}^{\infty}\frac{x^n}{n!}$, or is it just an artificial name given for a trancendental expression we cannot otherwise reduce? Likewise, why not call some tree-level associator Φ^{tree} and now it is "in closed form"?

For us, "closed-form" should mean "useful for computations". More precisely, it means that the quantity in question is an element of some space $\mathcal{A}^{cf}$ of "useful closed-form thingies" whose elements have finite descriptions (hopefully, finite and short) and on which some operations are defined by algorithms which terminate in finite time (hopefully, finite and short). Furthermore, there should be a finite-time algorithm to decide whether two descriptions of elements of $\mathcal{A}^{cf}$ describe the same element. (In our context, if it is hard to decide within the target space of an invariant whether two elements are equal or not, the invariant is not too useful in deciding whether two knotted objects are equal or not).

Thus for example, polynomials in a variable x are always of closed form, for they are simply described by finite sequences of integers (which in themselves are finite sequences of digits), the standard operations on polynomials ($+$, $\times$, and, say, $\frac{d}{dx}$) are algorithmically computable, and it is easy to write the "polynomial equality" computer program. Likewise for rational functions and even for rational functions of x and e^x.

On the other hand, general elements Φ of the space $\mathcal{A}^{\text{tree}}(\uparrow_3)$ of potential tree-level associators are not closed-form, for they are determined by infinitely many coefficients. Thus iterative constructions of asosciators, such as the one in [BN3] are computationally useful only within bounded-degree quotients of $\mathcal{A}^{\text{tree}}(\uparrow_3)$ and not as all-degree closed-form formulas. Likewise, "explicit" formulas for an associator Φ in terms of multiple ζ-values (e.g. [LM1]) are not useful for computations as it is not clear how to apply tangle-theoretic operations to Φ (such as $\Phi \mapsto \Phi^{1342}$ or $\Phi \mapsto (1 \otimes \Delta \otimes 1)\Phi$) while staying within some space of "objects with finite description in terms of multiple ζ-values". And even if a reasonable space of such objects could be defined, it remains an open problem to decide whether a given rational linear combination of multiple ζ-values is equal to 0.

References

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators,* arXiv:0802.4300.

[Ar] E. Artin, *Theory of Braids,* Ann. of Math. **48-1** (1947) 101–126.

[BWC] J. C. Baez, D. K. Wise and A. S. Crans, *Exotic Statistics for Strings in 4d BF Theory,* Adv. Theor. Math. Phys. **11** (2007) 707–749, arXiv:gr-qc/0603085.

[BN1] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology **34** (1995) 423–472.

[BN2] D. Bar-Natan, *Vassiliev homotopy string link invariants,* Jour. of Knot Theory and its Ramifications **4** (1995) 13–32.

[BN3] D. Bar-Natan, *Non-associative tangles,* in *Geometric topology* (proceedings of the Georgia international topology conference), (W. H. Kazez, ed.), 139–183, Amer. Math. Soc. and International Press, Providence, 1997.

[BN4] D. Bar-Natan, *Vassiliev and quantum invariants of braids,* in Proc. of Symp. in Appl. Math. **51** (1996) 129–144, *The interface of knots and physics,* (L. H. Kauffman, ed.), Amer. Math. Soc., Providence.

[BN5] D. Bar-Natan, *On Associators and the Grothendieck-Teichmuller Group I,* Selecta Mathematica, New Series **4** (1998) 183–212.

[BN6] D. Bar-Natan, *Finite Type Invariants,* in *Encyclopedia of Mathematical Physics,* (J.-P. Francoise, G. L. Naber and Tsou S. T., eds.) Elsevier, Oxford, 2006 (vol. 2 p. 340).

[BN7] D. Bar-Natan, *Algebraic Knot Theory — A Call for Action,* web document, 2006, `http://www.math.toronto.edu/~drorbn/papers/AKT-CFA.html`.

[BN8] D. Bar-Natan, *in preparation.*

[BLT] D. Bar-Natan, T. Q. T. Le and D. P. Thurston, *Two applications of elementary knot theory to Lie algebras and Vassiliev invariants,* Geometry and Topology **7-1** (2003) 1–31, arXiv:math.QA/0204311.

[BP] B. Berceanu and S. Papadima, *Universal Representations of Braid and Braid-Permutation Groups,* arXiv:0708.0634.

[BH] T. Brendle and A. Hatcher, *Configuration Spaces of Rings and Wickets,* arXiv:0805.4354.

[CS] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams,* Mathematical Surveys and Monographs **55**, American Mathematical Society, Providence 1998.

[CCM] A. S. Cattaneo, P. Cotta-Ramusino and M. Martellini, *Three-dimensional BF Theories and the Alexander-Conway Invariant of Knots,* Nucl. Phys. **B436** (1995) 355–384, arXiv:hep-th/9407070.

[CCFM] A. S. Cattaneo, P. Cotta-Ramusino, J. Froehlich and M. Martellini, *Topological BF Theories in 3 and 4 Dimensions,* J. Math. Phys. **36** (1995) 6137–6160, arXiv:hep-th/9505027.

[Dr1] V. G. Drinfel'd, *Quantum Groups,* in *Proceedings of the International Congress of Mathematicians,* 798–820, Berkeley, 1986.

[Dr2] V. G. Drinfel'd, *Quasi-Hopf Algebras,* Leningrad Math. J. **1** (1990) 1419–1457.

[Dr3] V. G. Drinfel'd, *On Quasitriangular Quasi-Hopf Algebras and a Group Closely Connected with* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. **2** (1991) 829–860.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I,* Selecta Mathematica, New Series **2** (1996) 1–41, arXiv:q-alg/9506005.

[FRR] R. Fenn, R. Rimanyi and C. Rourke, *The Braid-Permutation Group,* Topology **36** (1997) 123–135.

[Gol] D. L. Goldsmith, *The Theory of Motion Groups,* Mich. Math. J. **28-1** (1981) 3–17.

[Gou] M. Goussarov, *On n-equivalence of knots and invariants of finite degree,* Zapiski nauch. sem. POMI **208** (1993) 152–173 (English translation in *Topology of manifolds and varieties* (O. Viro, editor), Amer. Math. Soc., Providence 1994, 173–192).

[GPV] M. Goussarov, M. Polyak and O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

[HM] N. Habegger and G. Masbaum, *The Kontsevich integral and Milnor's invariants,* Topology **39** (2000) 1253–1289.

[Ha] A. Haviv, *Towards a diagrammatic analogue of the Reshetikhin-Turaev link invariants,* Hebrew University PhD thesis, September 2002, arXiv:math.QA/0211031.

[KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff Formula and Invariant Hyperfunctions,* Invent. Math. **47** (1978) 249–272.

[Ka1] L. H. Kauffman, *On knots,* Princeton Univ. Press, Princeton, 1987.

[Ka2] L. H. Kauffman, *Virtual Knot Theory,* European J. Comb. **20** (1999) 663–690, arXiv:math.GT/9811028.

[Ku] V. Kurlin, *Compressed Drinfeld associators,* Journal of Algebra **292-1** (2005) 184–242.

[Le] T. Q. T. Le, *An invariant of integral homology 3-spheres which is universal for all finite type invariants,* in *Solitons, geometry and topology: on the crossroad*, (V. Buchstaber and S. Novikov, eds.) AMS Translations Series 2, Providence, arXiv:q-alg/9601002.

[LM1] T. Q. T. Le and J. Murakami, *On Kontsevich's integral for the HOMFLY polynomial and relations of multiple ζ-numbers,* Topology and its Applications **62** (1995) 193–206.

[LM2] T. Q. T. Le and J. Murakami, *The universal Vassiliev-Kontsevich invariant for framed oriented links,* Compositio Math. **102** (1996) 41–64, arXiv:hep-th/9401016.

[LMO] T. Q. T. Le, J. Murakami and T. Ohtsuki, *On a universal quantum invariant of 3-manifolds,* Topology **37-3** (1998) 539–574, arXiv:q-alg/9512002.

[Lee] P. Lee, *Closed-Form Associators and Braidors in a Partly Commutative Quotient,* University of Toronto preprint, December 2007, `http://individual.utoronto.ca/PetersKnotPage/`.

[Li1] J. Lieberum, *The Drinfeld associator of* $gl(1|1)$, arXiv:math.QA/0204346.

[Mc] J. McCool, *On Basis-Conjugating Automorphisms of Free Groups,* Can. J. Math. **38-6** (1986) 1525–1529.

[Na] G. Naot, *On Chern-Simons Theory with an Inhomogeneous Gauge Group and BF Theory Knot Invariants,* J. Math. Phys. **46** (2005) 122302, arXiv:math.GT/0310366.

[Oh] T. Ohtsuki, *Finite type invariants of integral homology 3-spheres,* Jour. of Knot Theory and its Ramifications **5(1)** (1996) 101–115.

[Po] M. Polyak, *On the Algebra of Arrow Diagrams,* Let. Math. Phys. **51** (2000) 275–291.

[Sa] S. Satoh, *Virtual Knot Presentations of Ribbon Torus Knots,* J. of Knot Theory and its Ramifications **9-4** (2000) 531–542.

[Va] V. A. Vassiliev, *Cohomology of knot spaces,* in *Theory of Singularities and its Applications (Providence)* (V. I. Arnold, ed.), Amer. Math. Soc., Providence, 1990.

Department of Mathematics, University of Toronto, Toronto Ontario M5S 2E4, Canada
E-mail address: `drorbn@math.toronto.edu`
URL: `http://www.math.toronto.edu/~drorbn`