REVIEW OF “AN UNEXPECTED CYCLIC SYMMETRY OF $I\mathfrak{u}_n$”

1. Recommendation

Motivated from symmetries of knot invariants coming from Lie algebras, Theorem 2 of this paper gives an explicit cyclic automorphism of $\mathfrak{gl}^{\epsilon}_{n+}$ (essentially two copies of the Borel glued along the Cartan, motivated by connections to knot theory in the paragraph before Theorem 2). The construction is simple, and the proof is a quick check. While the cyclic symmetry is not as unexpected as possible[1], and while the construction is not as general as possible[2], I feel that this paper could be accepted to Algebras and Representation Theory because

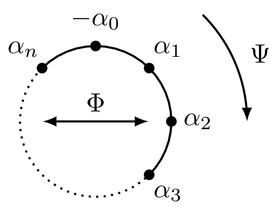
(1) it originated and fits into a larger research program of the authors (*over then under tangles*), so that
(2) it may introduce an unfamiliar new tool to the knot-theory community.

Additionally,

(3) it has already sparked additional research ([BR]), and
(4) it has the virtue of being very short (5 pages, with references; it may even be possible to shorten it by an additional page by proving only Theorem 2, and then specializing to Theorem 1).

2. Overview

As might be expected, the cyclic automorphism Ψ is explained by the cyclic symmetry of the affine Dynkin diagram of type $\widetilde{A}_n$



where $\alpha_1, \ldots, \alpha_n$ are a choice of simple roots of $\mathfrak{sl}_n$ and $\alpha_0 = \sum_{i=1}^{n} \alpha_i$ is the corresponding highest root. In this notation, the map Ψ acts as the cyclic diagram automorphism

$$\alpha_i \mapsto \alpha_{i+1} \text{ for } 1 \leq i < n$$
$$\alpha_n \mapsto -\alpha_0,$$

and extends linearly to all roots.

Other appearances of (a subgroup of) the symmetry group of the affine Dynkin diagram (at the generality of finite-dimensional simple Lie algebras $\mathfrak{g}$):

- the symmetries of the fundamental alcove for the affine Weyl group $\widetilde{W}$ (essentially the dual of the affine Dynkin diagram),

[1]Notes 5 and 6 indicate that explanations comming from the symmetry of affine Dynkin diagrams occurred immediately and independently to several different researchers; perhaps the title could be changed to reflect this?

[2]Bulois and Ressayre have subsequently generalized the result to simple Lie algebras of finite type in [BR], and Section 4 of their paper further completely describes the structure of $\mathrm{Aut}(I\mathfrak{b})$.

- the quotient of the weight lattice by the coroot lattice (the order of this group is the *index of connection* f, which appears in Weyl's formula for the order of the Weyl group $|W| = n!f\prod_{i=1}^n a_i$, where $\alpha_0 = \sum_{i=1}^n a_i\alpha_i$),
- the center of the corresponding simply-connected Lie group G,
- ways to linearly map the affine vertex $-\alpha_0$ to a root that appears with multiplicity one in the expansion of $\alpha_0 = \sum_{i=1}^n a_i\alpha_i$ (the so-called *minuscule* nodes, whose corresponding weights have particularly simple highest-weight representations),
- etc.

The additional automorphism on page denoted Φ comes from the symmetry of the ordinary type A_n Dynkin diagram (and together, Φ and Ψ genenerate the dihedral group of order $2n$; see also [BR]). These automorphisms Φ, Ψ specialize to automorphisms of $I\mathfrak{u}_n := \mathfrak{u}_n\mathfrak{u}_n^*$ when $\epsilon = 0$ (here, again, $\mathfrak{u}_n$ is a Borel subalgebra of $\mathfrak{gl}_n$ or $\mathfrak{sl}_n$).

3. Specific Comments

The paper is well-written and the exposition is clear.

(1) It may be worth revisiting the title.
(2) I wonder if it was a conscious choice to write gl_n instead of $\mathfrak{gl}_n$?
(3) Perhaps the clause regarding the indicator of truth (in Proof of Theorem 1) might be better served in a separate sentence.
(4) It might be nice to draw the picture of the affine Dynkin diagram (as I did, perhaps as an addition to Figure 1?), and address where this symmetry group also arises.
(5) It seems possible to shorten the paper further by only giving the proof of Theorem 2 and then specializing to give Theorem 1.