Pensieve header: An analysis of K15a55264 and K15n90489 and more following their discovery on November 5, 2022 and on by SideBySide.nb. Continued in PossibleCounterexamples.nb at pensieve://Talks/ICERM-2305/.

In[∘]:=
```
Once[<< KnotTheory`]
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[∘]:=
```
MatrixSignature[A_] := Total[Sign[Select[Eigenvalues[A], Abs[#] > 10^-12 &]]];
Writhe[K_] := Sum[If[PositiveQ[x], 1, -1], {x, List @@ PD@K}];
```

In[∘]:=
```
MS[{}] = 0;
MS[A_?MatrixQ] /; A == Aᵀ := Module[{k, a, A1, j, l},
  {k} = FirstPosition[A⟦1⟧, x_ /; x ≠ 0, {None}];
  Switch[k,
   None, MS[A⟦2 ;;, 2 ;;⟧],
   1, (
    a = A⟦1, 1⟧;
    Sign[a] + MS[
      A⟦2 ;;, 2 ;;⟧ - Outer[Times, A⟦2 ;;, 1⟧, A⟦1, 2 ;;⟧] / a
     ]),
   _, (
    A1 = A; A1⟦k⟧ /= A⟦k, 1⟧; A1⟦All, k⟧ /= A⟦1, k⟧;
    a = A1⟦k, k⟧; A1⟦k⟧ -= a A1⟦1⟧ / 2; A1⟦All, k⟧ -= a A1⟦All, 1⟧ / 2;
    For[j = 2, j ≤ Length@A, ++j, If[j ≠ k,
      A1⟦j⟧ -= A1⟦j, 1⟧ A1⟦k⟧ + A1⟦j, k⟧ A1⟦1⟧;
      A1⟦ ;; , j⟧ -= A1⟦1, j⟧ A1⟦ ;; , k⟧ + A1⟦k, j⟧ A1⟦ ;; , 1⟧
     ]];
    l = Complement[Range@Length@A, {1, k}];
    MS[A1⟦l, l⟧]
   )
  ]
 ]
```

pdf

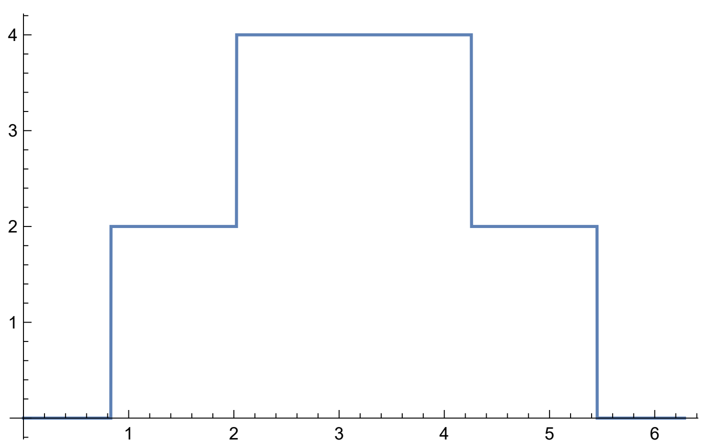
The Bedlewo program

pdf

In[ ]:=
```
Bed[K_, ω_] := Module[{t, r, XingsByArmpits, bends, faces, p, A, is},
  t = 1 - ω; r = t + t*;
  XingsByArmpits = List @@ PD[K] /.
    x : X[i_, j_, k_, l_] :> If[PositiveQ[x], X+[-i, j, k, -l], X-[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] :> p[a,-d] p[b,-a] p[c,-b] p[d,-c];
  faces = bends //. p[x__,y_] p[y_,z__] :> p[x,y,z];
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[[is, is]] += If[Head[x] === X+,
     {{-r, -t, 2 t, t*}, {-t*, 0, t*, 0}, {2 t*, t, -r, -t*}, {t, 0, -t, 0}},
     {{r, -t, -2 t*, t*}, {-t*, 0, t*, 0}, {-2 t, t, r, -t*}, {t, 0, -t, 0}}],
   {x, XingsByArmpits}];
  MatrixSignature[A]
];
```

In[ ]:=
```
Plot[Bed[K = Knot[8, 2], e^(i t)], {t, 0, 2 π}]
PositiveQ /@ (List @@ PD[K])
```

Out[ ]=



Out[ ]=

```
{False, False, True, True, False, False, False, False}
```

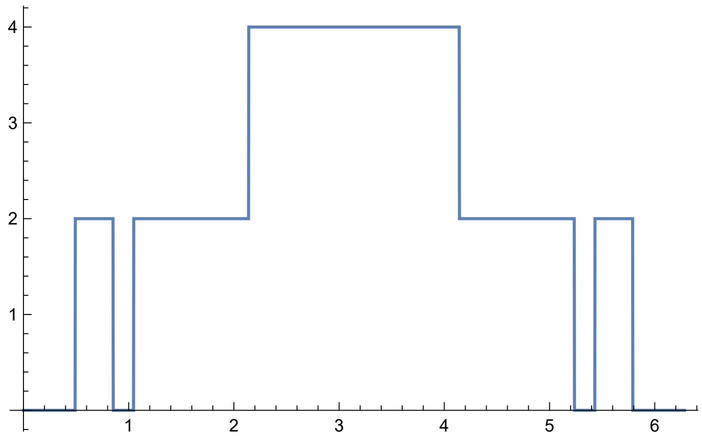
In[ ]:=
```
Draw[Knot[8, 2]]
```

Out[ ]=

```
Draw[Knot[8, 2]]
```

In[∘]:=
```
Plot[Bed[Knot@"K12a422", ⅇ^(ⅈ t)], {t, 0, 2 π}]
```

Out[∘]=



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The Kashaev Program

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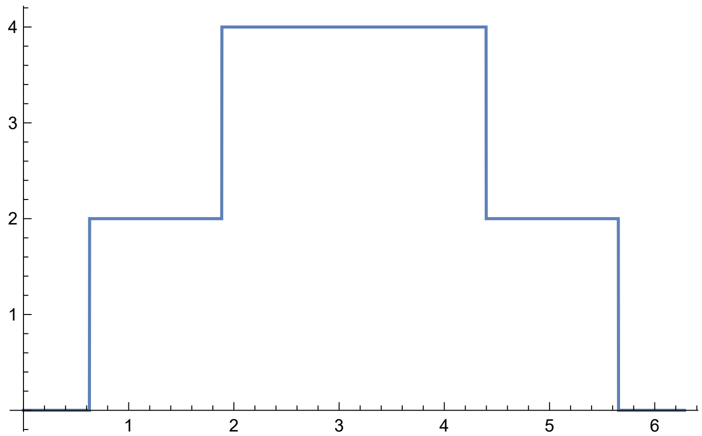
In[∘]:=
```
Kas[K_, ω_] := Module[{u, v, XingsByArmpits, bends, faces, p, A, is},
  u = Re[ω^(1/2)]; v = Re[ω]; (* so v=2u^2-1 *);
  XingsByArmpits = List @@ PD[K] /.
    x : X[i_, j_, k_, l_] :→ If[PositiveQ[x], X₊[-i, j, k, -l], X₋[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] :→ p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
  faces = bends //. p_{x__,y_} p_{y_,z__} :→ p_{x,y,z};
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #]〚1, 1〛 & /@ List @@ x;
   A〚is, is〛 += If[Head[x] === X₊,
     ( v u 1 u
       u 1 u 1
       1 u v u
       u 1 u 1 ), - ( v u 1 u
                      u 1 u 1
                      1 u v u
                      u 1 u 1 )],
   {x, XingsByArmpits}];
  (MatrixSignature[A] - Writhe[K]) / 2];
```

In[∘]:=
```
Plot[Kas[Knot[5, 1], e^(i t)], {t, 0, 2 π}]
```

Out[∘]=



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Comparisons

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In[∘]:=
```
Sum[ω = e^(i RandomReal[{0,2π}]); Bed[K, ω] == Kas[K, ω], {10}, {K, AllKnots[{3, 10}]}]
```

Out[∘]=
pdf

2490 True

In[∘]:=
```
Table[Bed[K, i], {K, AllKnots[{3, 5}]}]
```

Out[∘]=

{2, 0, 2, 2}

In[∘]:=
```
Table[KnotSignature[K], {K, AllKnots[{3, 5}]}]
```

Out[∘]=

{-2, 0, -4, -2}

In[∘]:=
```
Total@Table[Bed[K, -1] == -KnotSignature[K], {K, AllKnots[{3, 10}]}]
```

Out[∘]=

249 True

10_{98}

In[∘]:=
```
K = Knot[10, 98]
```

Out[∘]=

Knot[10, 98]

In[∘]:=
```
Alexander[K][T] // Factor
```

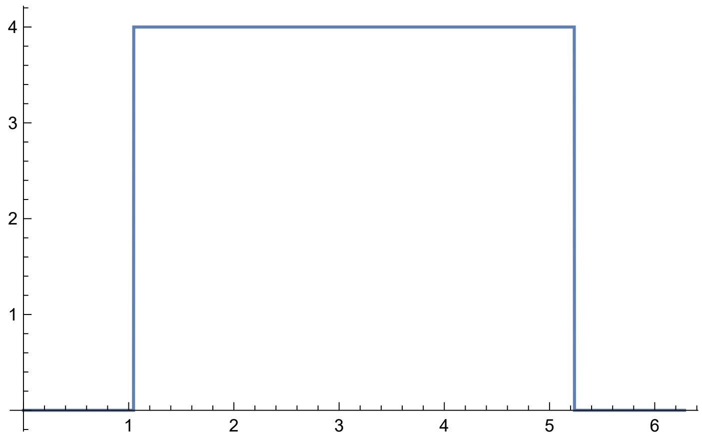
KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

$$-\frac{(-2+T)\ (-1+2T)\ \left(1-T+T^2\right)^2}{T^3}$$

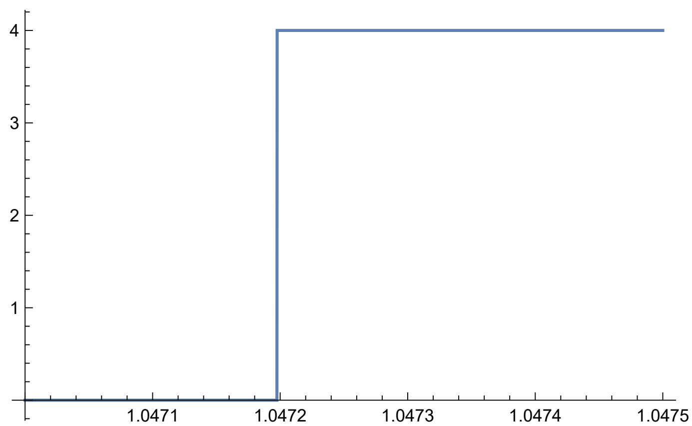
In[∘]:= **Plot[Kas[K, ⅇ^(ⅈ t)], {t, 0, 2 π}]**

Out[∘]=



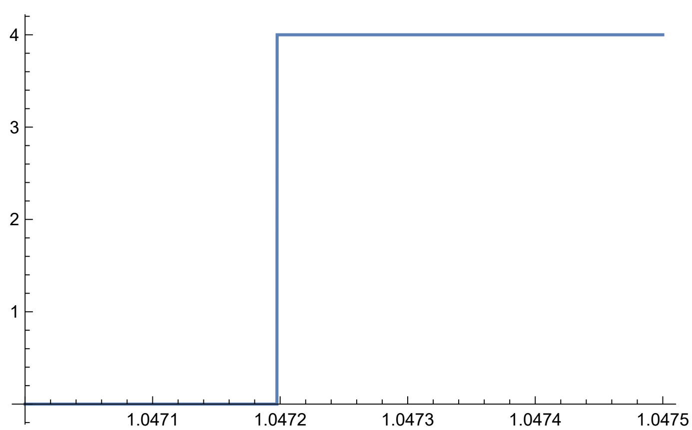
In[∘]:= **Plot[Bed[K, ⅇ^(ⅈ t)], {t, 1.047, 1.0475}]**

Out[∘]=



In[∘]:= **Plot[Kas[K, ⅇ^(ⅈ t)], {t, 1.047, 1.0475}]**

Out[∘]=



K11n72

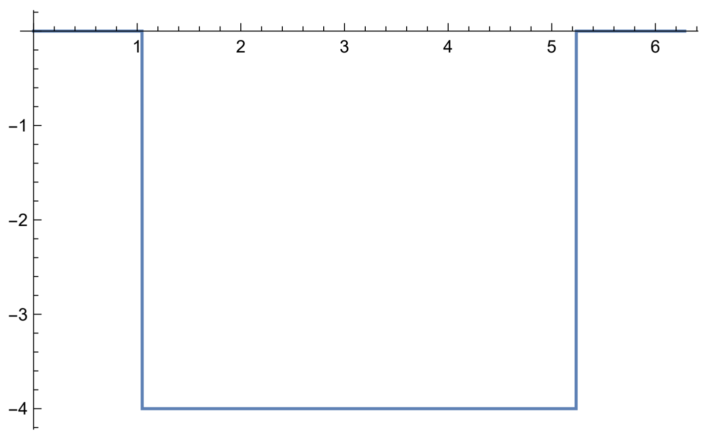
In[∘]:=
```
K = Knot[11, NonAlternating, 72];
Alexander[K][T] // Factor
```

Out[∘]=

$$-\frac{(-2+T)\ (-1+2T)\ \left(1-T+T^2\right)^2}{T^3}$$

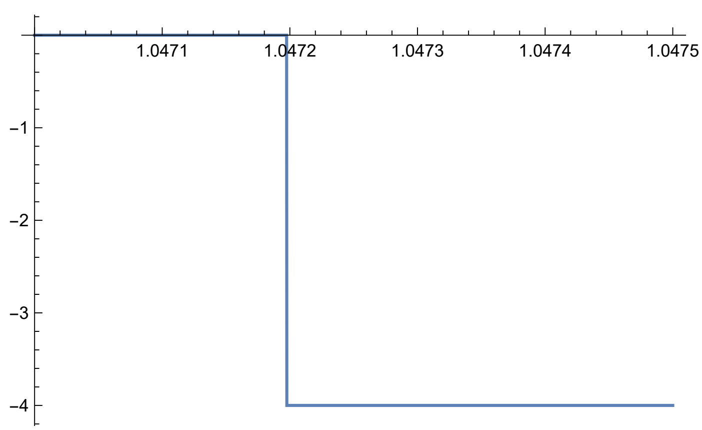
In[∘]:=
```
Plot[Kas[K, ⅇ^(ⅈ t)], {t, 0, 2 π}]
```

Out[∘]=



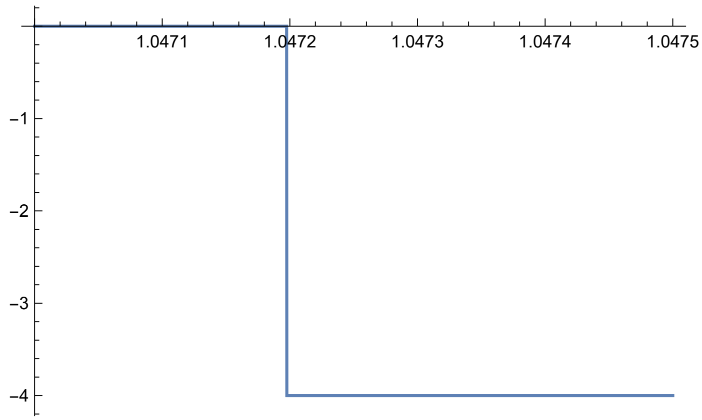
In[∘]:=
```
Plot[Bed[K, ⅇ^(ⅈ t)], {t, 1.047, 1.0475}]
```

Out[∘]=



In[ ]:=
```
Plot[Kas[K, e^(i t)], {t, 1.047, 1.0475}]
```

Out[ ]=



K15a55264 and K15n90489 and more

In[ ]:=
```
{K1, ω1} = {Knot[15, Alternating, 55 264], 0.5023868946758996` - 0.8646429367420442` i};
{K2, ω2} =
  {Knot[15, NonAlternating, 90 489], 0.4973889914966496` - 0.8675276313397434` i};
{K3, ω3} = {Knot[16, Alternating, 144 399], 0.5000006342255959` + 0.8660250376138104` i};
{K4, ω4} =
  {Knot[16, NonAlternating, 225 282], 0.5005898353703419` - 0.8656845942512169` i};
{K5, ω5} =
  {Knot[16, NonAlternating, 761 158], 0.4999164509765885` + 0.8660736354623504` i};
```

In[ ]:=
```
Factor[Alexander[#][T]] & /@ {K1, K2, K3, K4, K5}
```

KnotTheory: Loading precomputed data in KnotTheory/15A.dts.

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

KnotTheory: Loading precomputed data in KnotTheory/15N.dts.

KnotTheory: Loading precomputed data in KnotTheory/16A.dts.

General: Further output of KnotTheory::loading will be suppressed during this calculation.

Out[ ]=

$$\left\{\frac{(1-T+T^2)^6}{T^6}, \frac{(1-T+T^2)^5}{T^5}, \frac{(1-T+T^2)^2\,(2-10T+20T^2-23T^3+20T^4-10T^5+2T^6)}{T^5}, -\frac{(1-3T+T^2)\,(1-T+T^2)^4}{T^5}, \frac{(1-T+T^2)^3\,(1-4T+7T^2-4T^3+T^4)}{T^5}\right\}$$

In[ ]:=
```
{Bed[K1, ω1], Kas[K1, ω1]}
```

Out[ ]=

{0, -1}

In[ ]:= θ1 = Abs@Arg[ω1]

Out[ ]=

1.04444

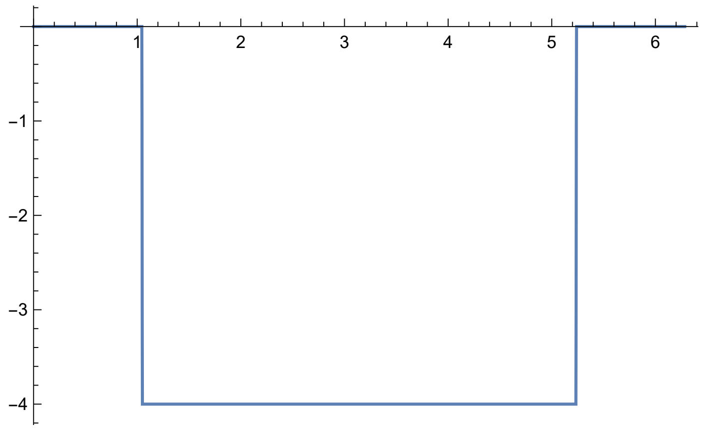
In[ ]:= {Bed[K1, $e^{i \theta 1}$], Kas[K1, $e^{i \theta 1}$]}

Out[ ]=

{0, -1}

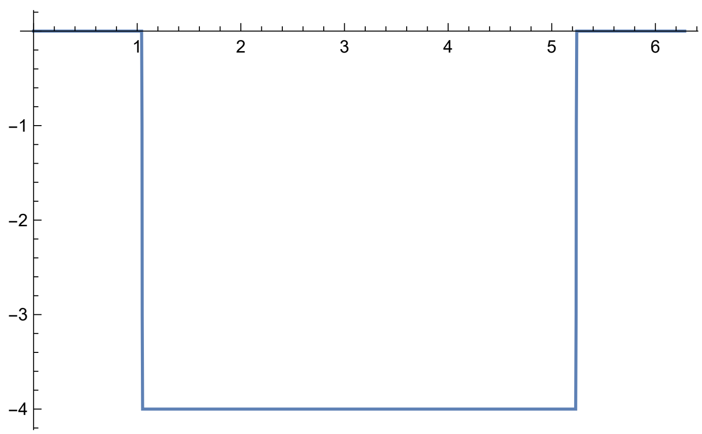
In[ ]:= Plot[Bed[K1, $e^{i t}$], {t, 0, 2 π}]

Out[ ]=



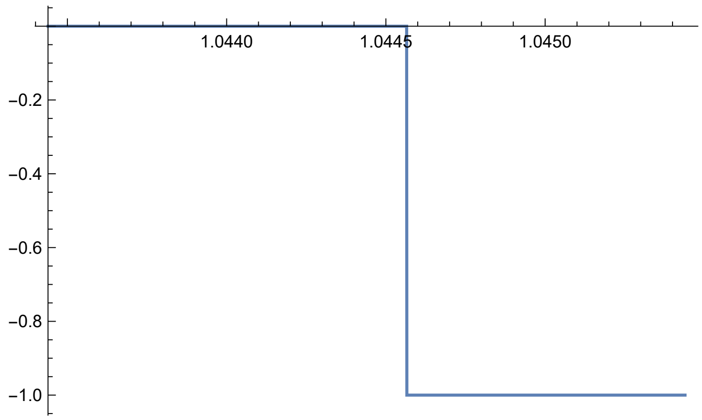
In[ ]:= Plot[Kas[K1, $e^{i t}$], {t, 0, 2 π}]

Out[ ]=



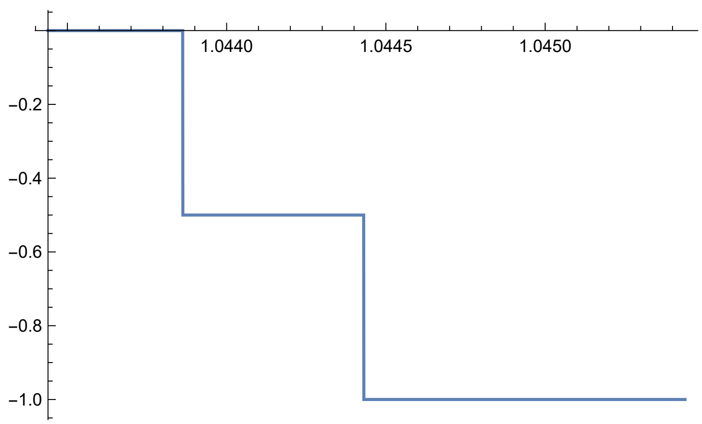
In[∘]:= **Plot[Bed[K1, $e^{i t}$], {t, θ1 - 0.001, θ1 + 0.001}]**

Out[∘]=



In[∘]:= **Plot[Kas[K1, $e^{i t}$], {t, θ1 - 0.001, θ1 + 0.001}]**

Out[∘]=



In[∘]:= **K = K1; sqrtω = N[$e^{\frac{i}{2} 10442/10000}$, 64]**

Out[∘]=

0.8667738186145244140436223354152894733945125092827166380196722824 +
0.4987014611612798657579447606726259380164654403012664107223872815 i

In[∘]:= **t0 = $\frac{\text{Re[sqrtω]}}{1 - \text{Im[sqrtω]}}$**

Out[∘]=

1.729057141523778743182000509933836814079077130503251239315497674

In[∘]:= **t0 = -1 729 057 / 1 000 000;**

x0 = $\frac{2\,t0}{1 + t0^2}$; y0 = $\frac{1 - t0^2}{1 + t0^2}$;

{sqrtω = x0 + i y0, N[sqrtω], Abs[sqrtω]}

Out[∘]=

$\left\{-\frac{3\,458\,114\,000\,000}{3\,989\,638\,109\,249} - \frac{1\,989\,638\,109\,249\, i}{3\,989\,638\,109\,249}, -0.866774 - 0.498701\, i, 1\right\}$

In[ ]:=
```
Kas[K, N[sqrtω^2]]
```

Part: The expression K cannot be used as a part specification.

Set: The expression K cannot be used as a part specification.

Eigenvalues: Argument {} at position 1 is not a non-empty square matrix.

Eigenvalues: Eigenvalues called with 0 arguments; 1 or 2 arguments are expected.

Out[ ]=

$$\frac{1}{2}\left(-\text{If}[\text{PositiveQ}[K], 1, -1] + \text{Total}[\text{Sign}[\text{Eigenvalues}[\,]]]\right)$$

In[ ]:=
```
sqrtω^2
```

Out[ ]=

$$\frac{7\,999\,892\,631\,220\,064\,340\,655\,999}{15\,917\,212\,242\,771\,935\,659\,344\,001} + \frac{13\,760\,790\,801\,054\,992\,772\,000\,000\,i}{15\,917\,212\,242\,771\,935\,659\,344\,001}$$

In[ ]:=
```
Kas[K, sqrtω^2]
```

Out[ ]=

$$-\frac{1}{2}$$

In[ ]:=
```
u = Re[sqrtω]; v = Re[sqrtω^2];
```

In[◦]:=
```
XingsByArmpits = List @@ PD[K] /.
    x : X[i_, j_, k_, l_] :> If[PositiveQ[x], X₊[-i, j, k, -l], X₋[-j, k, l, -i]];
bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] :> p[a,-d] p[b,-a] p[c,-b] p[d,-c];
faces = bends //. p[x__,y_] p[y_,z__] :> p[x,y,z];
A = Table[0, Length@faces, Length@faces];
Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
  A[[is, is]] += If[Head[x] === X₊,
    ( v u 1 u )      ( v u 1 u )
    ( u 1 u 1 )      ( u 1 u 1 )
    ( 1 u v u ) , -  ( 1 u v u ) ],
    ( u 1 u 1 )      ( u 1 u 1 )
  {x, XingsByArmpits}];
A // MatrixForm
```

Out[◦]//MatrixForm=

$\frac{15\,999\,785\,262\,440\,128\,681\,311\,998}{15\,917\,212\,242\,771\,935\,659\,344\,001}$	0	0	0
0	$-\frac{15\,999\,785\,262\,440\,128\,681\,311\,998}{15\,917\,212\,242\,771\,935\,659\,344\,001}$	0	0
0	0	$-\frac{15\,999\,785\,262\,440\,128\,681\,311\,998}{15\,917\,212\,242\,771\,935\,659\,344\,001}$	0
0	0	0	$\frac{15\,999\,785\,262\,440\,128\,681\,311\,998}{15\,917\,212\,242\,771\,935\,659\,344\,001}$
0	0	0	0
0	0	0	0
0	0	0	0
1	0	0	1
$-\frac{6\,916\,228\,000\,000}{3\,989\,638\,109\,249}$	0	-1	0
0	0	0	1
0	$\frac{6\,916\,228\,000\,000}{3\,989\,638\,109\,249}$	0	0
1	0	$\frac{6\,916\,228\,000\,000}{3\,989\,638\,109\,249}$	0
0	$\frac{6\,916\,228\,000\,000}{3\,989\,638\,109\,249}$	$\frac{6\,916\,228\,000\,000}{3\,989\,638\,109\,249}$	0
0	0	0	0
$-\frac{6\,916\,228\,000\,000}{3\,989\,638\,109\,249}$	0	0	$-\frac{6\,916\,228\,000\,000}{3\,989\,638\,109\,249}$
0	-1	-1	0
0	-1	0	$-\frac{6\,916\,228\,000\,000}{3\,989\,638\,109\,249}$

In[◦]:=
```
Eigenvalues[A] // Sort
```

Out[◦]=

$\{0, 0, -9.17\ldots, -3.33\ldots, -1.62\ldots, -0.627\ldots, -0.503\ldots, 5.86\ldots\times 10^{-13}, 1.49\ldots\times 10^{-12}, 1.25\ldots\times 10^{-3}, 1.57\ldots\times 10^{-3}, 0.916\ldots, 1.60\ldots, 2.64\ldots, 5.02\ldots, 8.17\ldots, 11.9\ldots\}$

In[∘]:= **MatrixSignature[A]**

Out[∘]=

4

In[∘]:= **MS[A]**

Out[∘]=

5

In[∘]:= **(MatrixSignature[A] - Writhe[K]) / 2**

Out[∘]=

$-\frac{1}{2}$

In[∘]:= **(MS[A] - Writhe[K]) / 2**

Out[∘]=

0

In[∘]:= **Alexander[K][T] // Factor**

Out[∘]=

$$\frac{\left(1-T+T^2\right)^6}{T^6}$$

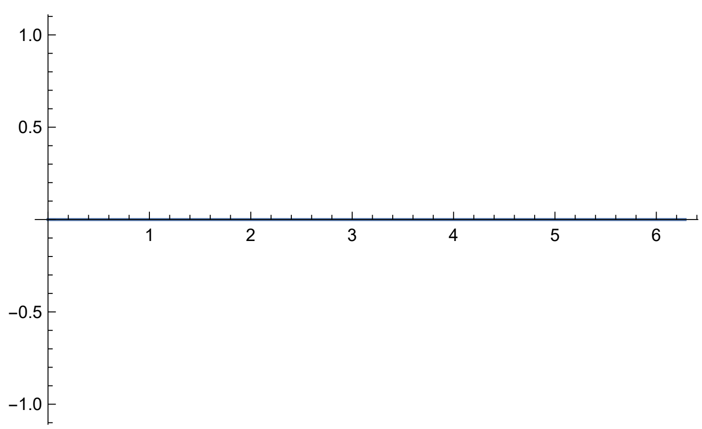
In[∘]:= **Solve[Alexander[K][T] == 0, T]**

Out[∘]=

$\{\{T \to (-1)^{1/3}\}, \{T \to (-1)^{1/3}\}, \{T \to (-1)^{1/3}\}, \{T \to (-1)^{1/3}\},$
$\{T \to (-1)^{1/3}\}, \{T \to (-1)^{1/3}\}, \{T \to -(-1)^{2/3}\}, \{T \to -(-1)^{2/3}\},$
$\{T \to -(-1)^{2/3}\}, \{T \to -(-1)^{2/3}\}, \{T \to -(-1)^{2/3}\}, \{T \to -(-1)^{2/3}\}\}$

In[∘]:= **Plot[Bed[K4, $e^{i t}$], {t, 0, 2 π}]**

Out[∘]=



In[∘]:=

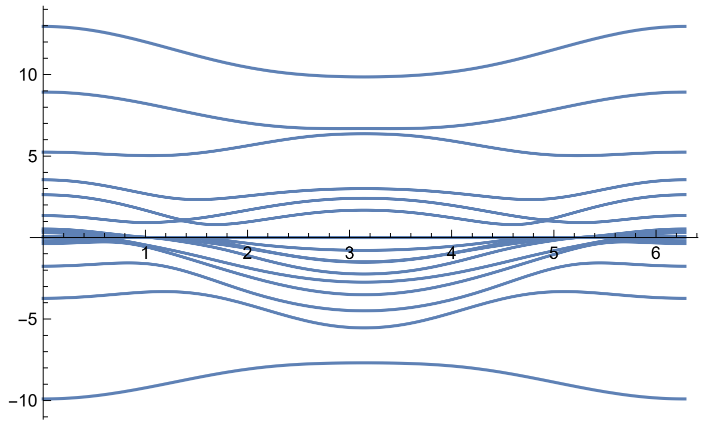
```
Clear[u, v];
XingsByArmpits = List @@ PD[K] /.
    x : X[i_, j_, k_, l_] :→ If[PositiveQ[x], Subscript[X, +][-i, j, k, -l], Subscript[X, -][-j, k, l, -i]];
bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] :→ Subscript[p, a,-d] Subscript[p, b,-a] Subscript[p, c,-b] Subscript[p, d,-c];
faces = bends //. Subscript[p, x__,y_] Subscript[p, y_,z__] :→ Subscript[p, x,y,z];
A = Table[0, Length@faces, Length@faces];
Do[is = Position[faces, #]⟦1, 1⟧ & /@ List @@ x;
  A⟦is, is⟧ += If[Head[x] === Subscript[X, +],
     {{v, u, 1, u}, {u, 1, u, 1}, {1, u, v, u}, {u, 1, u, 1}}, -{{v, u, 1, u}, {u, 1, u, 1}, {1, u, v, u}, {u, 1, u, 1}}],
  {x, XingsByArmpits}];
A // MatrixForm
```

Out[∘]//MatrixForm=

2 v	0	0	0	0	0	0	1	2 u	0	0	1	0	0	2 u	0	0
0	−2 v	0	0	0	0	0	0	0	0	−2 u	0	−2 u	0	0	−1	−1
0	0	−2 v	0	0	0	0	0	−1	0	0	−2 u	−2 u	0	0	−1	0
0	0	0	2 v	0	0	0	1	0	1	0	0	0	0	2 u	0	2 u
0	0	0	0	2 v	0	0	0	0	1	1	0	0	2 u	0	0	2 u
0	0	0	0	0	2 v	1	0	0	0	1	0	0	2 u	0	2 u	0
0	0	0	0	0	1	2 v	0	0	1	0	0	0	2 u	0	2 u	0
1	0	0	1	0	0	0	−1 + 2 v	0	0	0	0	−1	0	2 u	0	0
2 u	0	−1	0	0	0	0	0	2 − 2 v	0	0	0	−2 u	0	2	0	−1
0	0	0	1	1	0	1	0	0	4 v	0	1	0	2 u	2 u	2 u	2 u
0	−2 u	0	0	1	1	0	0	0	0	−2 + 2 v	0	−2	2 u	0	0	0
1	0	−2 u	0	0	0	0	0	0	1	0	−2 + 2 v	−2	0	2 u	0	0
0	−2 u	−2 u	0	0	0	0	−1	−2 u	0	−2	−2	−5	0	0	−2 u	−2 u
0	0	0	0	2 u	2 u	2 u	0	0	2 u	2 u	0	0	5	0	3	2
2 u	0	0	2 u	0	0	0	2 u	2	2 u	0	2 u	0	0	5	1	2
0	−1	−1	0	0	2 u	2 u	0	0	2 u	0	0	−2 u	3	1	4 − 2 v	0
0	−1	0	2 u	2 u	0	0	0	−1	2 u	0	0	−2 u	2	2	0	4 − 2 v

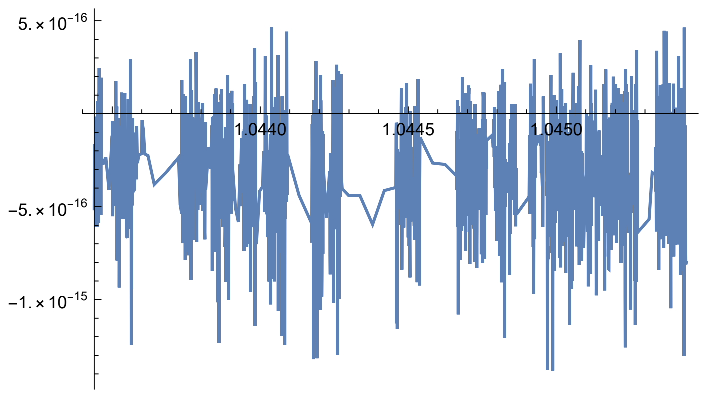
In[∘]:=
```
Plot[ω = ⅇ^(ⅈ t); Sort[Eigenvalues[A /. {u → Re[ω^(1/2)], v → Re[ω]}]], {t, 0, 2 π}]
```

Out[∘]=



In[∘]:=
```
Plot[ω = N[ⅇ^(ⅈ t), 64];
 Sort[Eigenvalues[A /. {u → Re[ω^(1/2)], v → Re[ω]}]]〚6〛, {t, θ1 - 0.001, θ1 + 0.001}]
```

Out[∘]=



In[∘]:=
```
Plot[ω = N[ⅇ^(ⅈ t), 64];
 Sort[Eigenvalues[A /. {u → Re[ω^(1/2)], v → Re[ω]}]]〚7〛, {t, θ1 - 0.001, θ1 + 0.001}]
```

Out[∘]=