Pensieve header: Signatures using the Meyer cocycle, following A. Conway’s arXiv://1903.04477.

From A. Conway’s
paper:

Let B_n denote the n-stranded braid group. Given $\omega \in S^1$, Gambaudo and Ghys study the map $B_n \to \mathbb{Z}, \beta \mapsto \sigma_{\widehat{\beta}}(\omega)$ obtained by sending a braid to the Levine-Tristram signature of its closure. While this map is not a homomorphism, these authors express the homomorphism defect $\sigma_{\widehat{\alpha\beta}}(\omega) - \sigma_{\widehat{\alpha}}(\omega) - \sigma_{\widehat{\beta}}(\omega)$ in terms of the reduced Burau representation

$$\overline{\mathscr{B}}_t \colon B_n \to GL_{n-1}(\mathbb{Z}[t^{\pm 1}])\,.$$

We briefly recall the definition of $\overline{\mathscr{B}}_t$. Any braid $\beta \in B_n$ can be represented by (an isotopy class of) a homeomorphism $h_\beta \colon D_n \to D_n$ of the punctured disk D_n. This punctured disk has a canonical infinite cyclic cover D_n^∞ (corresponding to the kernel of the map $\pi_1(D_n) \to \mathbb{Z}$ sending the obvious generators of $\pi_1(D_n)$ to 1) and, after fixing basepoints, the homeomorphism h_β lifts to a homeomorphism $\widetilde{h}_\beta \colon D_n^\infty \to D_n^\infty$. It turns out that $H_1(D_n^\infty;\mathbb{Z})$ is a free $\mathbb{Z}[t^{\pm1}]$-module of rank $n-1$ and the *reduced Burau representation* is the $\mathbb{Z}[t^{\pm1}]$-linear automorphism of $H_1(D_n^\infty;\mathbb{Z})$ induced by $\widetilde{h}_\beta$. This representation is unitary with respect to the equivariant skew-Hermitian form on $H_1(D_n^\infty;\mathbb{Z})$ which is defined by mapping $x, y \in H_1(D_n^\infty;\mathbb{Z})$ to

$$\xi(x,y) = \sum_{n\in\mathbb{Z}} \langle x, t^n y\rangle t^{-n}.$$

In particular, evaluating any matrix for $\overline{\mathscr{B}}_t(\beta)$ at $t=\omega$, the matrix $\overline{\mathscr{B}}_\omega(\alpha)$ preserves the skew-Hermitian form obtained by evaluating a matrix for ξ at $t=\omega$. Therefore, given two braids $\alpha, \beta \in B_n$ and $\omega \in S^1$, one can consider the Meyer cocycle of the two unitary matrices $\overline{\mathscr{B}}_\omega(\alpha)$ and $\overline{\mathscr{B}}_\omega(\beta)$. Here, given a skew-Hermitian form ξ on a complex vector space $\mathbb{C}$ and two unitary automorphisms γ_1, γ_2 of (V,ξ), the *Meyer cocycle* $\operatorname{Meyer}(\gamma_1,\gamma_2)$ is computed by considering the space $E_{\gamma_1,\gamma_2} = \operatorname{im}(\gamma_1^{-1} - \operatorname{id}) \cap \operatorname{im}(\operatorname{id} - \gamma_2)$ and taking the signature of the Hermitian form obtained by setting $b(e,e') = \xi(x_1 + x_2, e')$ for $e = \gamma_1^{-1}(x_1) - x_1 = x_2 - \gamma_2(x_2) \in E_{\gamma_1,\gamma_2}$ [71, 72].

The following result is due to Gambaudo and Ghys [31, Theorem A].

Theorem 5.2. *For all $\alpha, \beta \in B_n$ and $\omega \in S^1$ of order coprime to n, the following equation holds:*

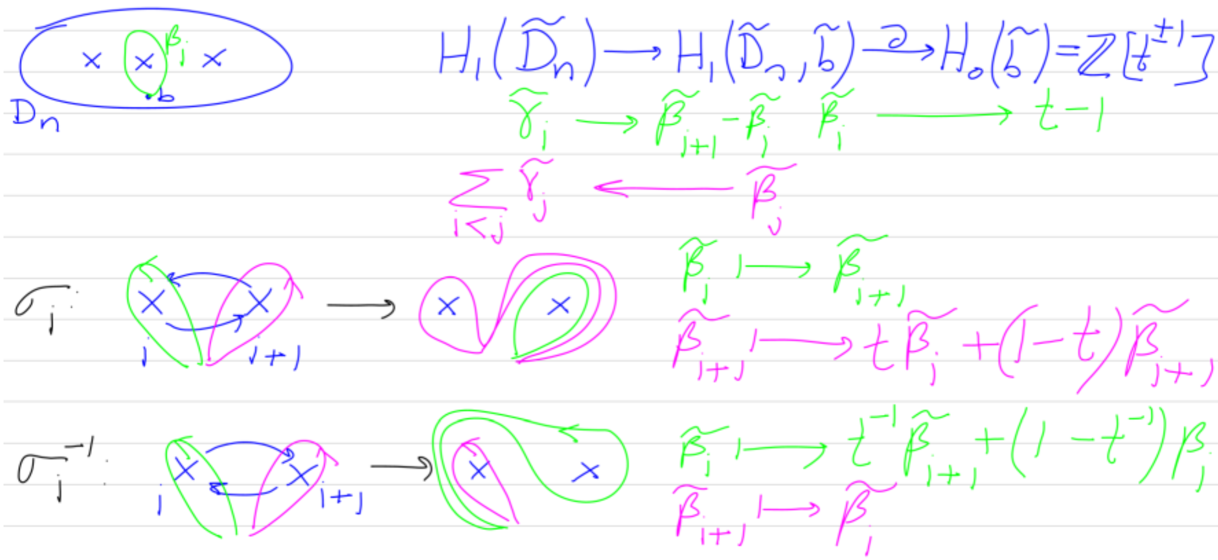
$$\sigma_{\widehat{\alpha\beta}}(\omega) - \sigma_{\widehat{\alpha}}(\omega) - \sigma_{\widehat{\beta}}(\omega) = -\operatorname{Meyer}(\overline{\mathscr{B}}_\omega(\alpha), \overline{\mathscr{B}}_\omega(\beta)). \tag{3}$$

In fact, since both sides of (3) define locally constant functions on S^1, Theorem 5.2 holds on a dense subset of S^1. The proof of Theorem 5.2 is 4-dimensional; can it also be understood using the constructions of Section 4? The answer ought to follow from [8], where a result analogous to Theorem 5.2 is established for Blanchfield pairings; see also [33].

We conclude this survey by applying Theorem 5.2 recursively in order to provide a formula for the Levine-Tristram signature purely in terms of braids. Indeed, using $\sigma_1, \ldots, \sigma_{n-1}$ to denote the generators of the braid group B_n (and recalling that the signature vanishes on trivial links), the next result follows from Theorem 5.2:

Corollary 5.3. *If an oriented link L is the closure of a braid $\sigma_{i_1} \cdots \sigma_{i_l}$, then the following equality holds on a dense subset of S^1:*

$$\sigma_L(\omega) = -\sum_{j=1}^{l-1} \operatorname{Meyer}(\overline{\mathscr{B}}_\omega(\sigma_{i_1}\cdots\sigma_{i_j}), \overline{\mathscr{B}}_\omega(\sigma_{i_{j+1}})).$$



In[∘]:=
```
γ2β = {γi_ ⧴ βi+1 - βi}; β2γ = {βj_ ⧴ Sum[γi, {i, j - 1}]};
σi_[x_] /; i > 0 := Expand[x /. γ2β /. {βi → βi+1, βi+1 → t βi + (1 - t) βi+1} /. β2γ];
σi_[x_] /; i < 0 := Expand[x /. γ2β /. {β-i → t^-1 β-i+1 + (1 - t^-1) β-i, β-i+1 → β-i} /. β2γ];
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In[∘]:=
```
Table[γi, {i, 7}] // σ3 // σ5
Table[γi, {i, 7}] // σ5 // σ3
Table[γi, {i, 6}] // σ3 // σ4 // σ3
Table[γi, {i, 6}] // σ4 // σ3 // σ4
Table[γi, {i, 5}] // σ3 // σ-3
```

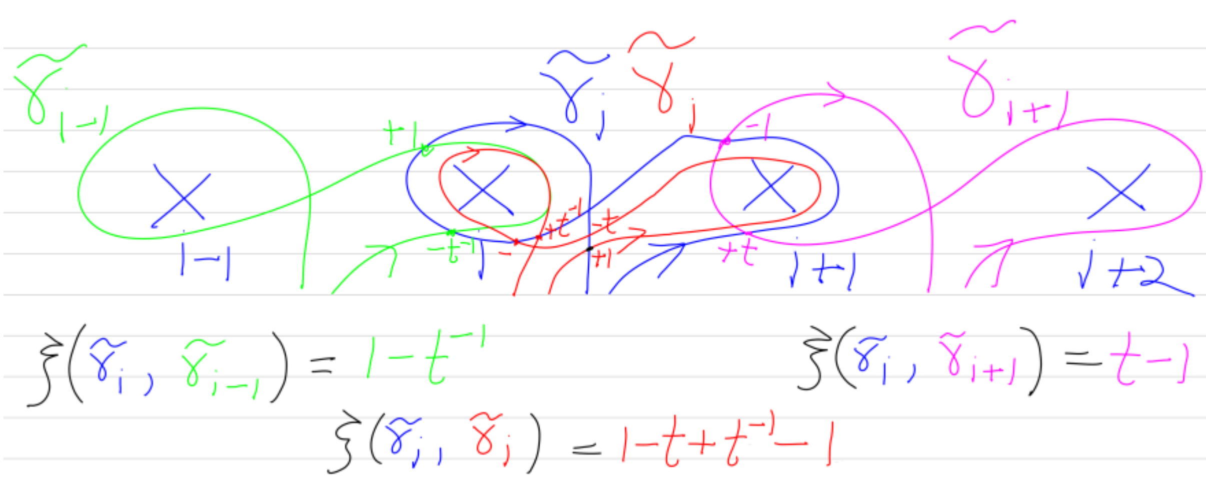
Out[∘]= $\{\gamma_1, \gamma_2 + \gamma_3, -t\,\gamma_3, t\,\gamma_3 + \gamma_4 + \gamma_5, -t\,\gamma_5, t\,\gamma_5 + \gamma_6, \gamma_7\}$

Out[∘]= $\{\gamma_1, \gamma_2 + \gamma_3, -t\,\gamma_3, t\,\gamma_3 + \gamma_4 + \gamma_5, -t\,\gamma_5, t\,\gamma_5 + \gamma_6, \gamma_7\}$

Out[∘]= $\{\gamma_1, \gamma_2 + \gamma_3 + \gamma_4, -t\,\gamma_4, -t^2\,\gamma_3, t^2\,\gamma_3 + t\,\gamma_4 + \gamma_5, \gamma_6\}$

Out[∘]= $\{\gamma_1, \gamma_2 + \gamma_3 + \gamma_4, -t\,\gamma_4, -t^2\,\gamma_3, t^2\,\gamma_3 + t\,\gamma_4 + \gamma_5, \gamma_6\}$

Out[∘]= $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$



In[∘]:=

$$\xi[x\_,\ y\_] :=$$

$$\text{Expand}\left[\text{Expand}\left[\left(x\ /.\ \{t \to t^{-1}\}\right)\ \left(y\ /.\ \{\gamma_{i\_} :\!\to \overline{\gamma}_i\}\right)\right]\ /.\ \gamma_{i\_}\ \overline{\gamma}_{j\_} :\!\to \begin{cases} 1 - t^{-1} & j == i - 1 \\ t^{-1} - t & j == i \\ t - 1 & j == i + 1 \\ 0 & \text{True} \end{cases}\right]$$

In[∘]:= $\text{MatrixForm}\ /@\ \{\text{Table}[\xi[\gamma_i, \gamma_j], \{i, 9\}, \{j, 9\}],\ \text{Table}[\xi[\gamma_i\ //\ \sigma_3, \gamma_j\ //\ \sigma_3], \{i, 9\}, \{j, 9\}]\}$

Out[∘]=

$$\left\{\begin{pmatrix}
\frac{1}{t}-t & -1+t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\frac{1}{t} & \frac{1}{t}-t & -1+t & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-\frac{1}{t} & \frac{1}{t}-t & -1+t & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-\frac{1}{t} & \frac{1}{t}-t & -1+t & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1-\frac{1}{t} & \frac{1}{t}-t & -1+t & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1-\frac{1}{t} & \frac{1}{t}-t & -1+t & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1-\frac{1}{t} & \frac{1}{t}-t & -1+t & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1-\frac{1}{t} & \frac{1}{t}-t & -1+t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\frac{1}{t} & \frac{1}{t}-t
\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}
\frac{1}{t}-t & -1+t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\frac{1}{t} & \frac{1}{t}-t & -1+t & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-\frac{1}{t} & \frac{1}{t}-t & -1+t & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-\frac{1}{t} & \frac{1}{t}-t & -1+t & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1-\frac{1}{t} & \frac{1}{t}-t & -1+t & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1-\frac{1}{t} & \frac{1}{t}-t & -1+t & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1-\frac{1}{t} & \frac{1}{t}-t & -1+t & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1-\frac{1}{t} & \frac{1}{t}-t & -1+t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\frac{1}{t} & \frac{1}{t}-t
\end{pmatrix}\right\}$$

In[∘]:= $\text{Table}[\xi[\beta_i\ /.\ \beta2\gamma,\ \beta_j\ /.\ \beta2\gamma], \{i, 7\}, \{j, 7\}]\ //\ \text{MatrixForm}$

Out[∘]//MatrixForm=

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{t}-t & -1+\frac{1}{t} & -1+\frac{1}{t} & -1+\frac{1}{t} & -1+\frac{1}{t} & -1+\frac{1}{t} \\
0 & 1-t & \frac{1}{t}-t & -1+\frac{1}{t} & -1+\frac{1}{t} & -1+\frac{1}{t} & -1+\frac{1}{t} \\
0 & 1-t & 1-t & \frac{1}{t}-t & -1+\frac{1}{t} & -1+\frac{1}{t} & -1+\frac{1}{t} \\
0 & 1-t & 1-t & 1-t & \frac{1}{t}-t & -1+\frac{1}{t} & -1+\frac{1}{t} \\
0 & 1-t & 1-t & 1-t & 1-t & \frac{1}{t}-t & -1+\frac{1}{t} \\
0 & 1-t & 1-t & 1-t & 1-t & 1-t & \frac{1}{t}-t
\end{pmatrix}$$