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In[*]:= A3[Knot[10, 121]]
```

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Out[*]=
```

$$\frac{2 - 11 \omega + 27 \omega^2 - 35 \omega^3 + 27 \omega^4 - 11 \omega^5 + 2 \omega^6}{\omega^3}$$

```
In[*]:= M // MatrixForm
```

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Out[*]//MatrixForm=
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(-1 + \sqrt{\omega})(1 + \sqrt{\omega})}{\sqrt{\omega}} & -\sqrt{\omega} & \sqrt{\omega} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{\omega}} \\ 0 & 0 & \frac{1}{\sqrt{\omega}} & -\frac{(-1 + \sqrt{\omega})(1 + \sqrt{\omega})}{\sqrt{\omega}} & -\frac{1}{\sqrt{\omega}} & 0 & 0 & \sqrt{\omega} & 0 & 0 & -\frac{1}{\sqrt{\omega}} \\ 0 & 0 & -\frac{1}{\sqrt{\omega}} & \sqrt{\omega} & -\frac{(-1 + \sqrt{\omega})(1 + \sqrt{\omega})}{\sqrt{\omega}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{(-1 + \sqrt{\omega})(1 + \sqrt{\omega})}{\sqrt{\omega}} & 0 & -\frac{1}{\sqrt{\omega}} & 0 & 0 & \frac{1}{\sqrt{\omega}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{\omega}} & 0 & \sqrt{\omega} & 0 & -\frac{2(-1 + \sqrt{\omega})(1 + \sqrt{\omega})}{\sqrt{\omega}} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\omega} & \sqrt{\omega} & 0 & -\sqrt{\omega} & 0 & 0 & 0 & 0 & \frac{(-1 + \sqrt{\omega})}{\sqrt{\omega}} \\ 0 & 0 & \frac{1}{\sqrt{\omega}} & 0 & -\sqrt{\omega} & \sqrt{\omega} & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{\omega}} \\ 0 & 0 & 0 & 0 & \sqrt{\omega} & -\frac{1}{\sqrt{\omega}} & 0 & \frac{(-1 + \sqrt{\omega})(1 + \sqrt{\omega})}{\sqrt{\omega}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[*]:= NullSpace[M] // MatrixForm
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Out[*]//MatrixForm=
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$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[*]:= FirstPosition[1] /@ NullSpace[M]
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Out[*]=
```

$$\{\{12\}, \{4\}, \{3\}, \{7\}, \{2\}, \{1\}\}$$

```
In[*]:= keeps = Complement[Range@Length@M, Union@@FirstPosition[1] /@ NullSpace[M]]
```

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Out[*]=
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$$\{5, 6, 8, 9, 10, 11\}$$

```
In[*]:= Factor@Det[M[[keeps, keeps]]]
```

```
Out[*]=
```

$$\frac{2 - 11 \omega + 27 \omega^2 - 35 \omega^3 + 27 \omega^4 - 11 \omega^5 + 2 \omega^6}{\omega^3}$$

In[\*]:= **A4[Knot[10, 121]]**

Out[\*]=

$$\frac{(2 + 17 \omega + 66 \omega^2 + 17 \omega^3 + 2 \omega^4) (2 - 11 \omega + 27 \omega^2 - 35 \omega^3 + 27 \omega^4 - 11 \omega^5 + 2 \omega^6)^2}{\omega^8}$$

In[\*]:= **M // MatrixForm**

Out[\*]//MatrixForm=

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & -1 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & 0 & & \\ 0 & 3 & \sqrt{\frac{(1+\omega)^2}{\omega}} & 1 & 1 & 1 & 0 & 0 & \sqrt{\frac{(1+\omega)^2}{\omega}} & \sqrt{\frac{(1+\omega)^2}{\omega}} & \\ 0 & \sqrt{\frac{(1+\omega)^2}{\omega}} & \frac{1-\omega+\omega^2}{\omega} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 1 & 0 & -\frac{1-\omega+\omega^2}{\omega} & -1 & 0 & 0 & -1 & 0 & & \\ 0 & 1 & 0 & -1 & -\frac{1-\omega+\omega^2}{\omega} & 0 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & 0 & -\frac{1-\omega+\omega^2}{\omega} & -\sqrt{\frac{(1+\omega)^2}{\omega}} & -1 & 0 & & \\ -1 & 0 & 0 & 0 & 0 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & -3 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & -1 & & \\ -\sqrt{\frac{(1+\omega)^2}{\omega}} & 0 & 0 & -1 & 0 & -1 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & -\frac{2(1+\omega^2)}{\omega} & -\sqrt{\frac{(1+\omega)^2}{\omega}} & & \\ 0 & \sqrt{\frac{(1+\omega)^2}{\omega}} & 1 & 0 & 0 & 0 & -1 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & \frac{(-1+\omega)^2}{\omega} & & \\ 0 & \sqrt{\frac{(1+\omega)^2}{\omega}} & 1 & 0 & 0 & 0 & -1 & 0 & 1 & (-) & \\ -\sqrt{\frac{(1+\omega)^2}{\omega}} & 0 & 0 & 0 & -1 & -1 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & -2 & 0 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & \\ -1 & 0 & -1 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & -\sqrt{\frac{(1+\omega)^2}{\omega}} & 0 & 0 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & -1 & & \end{pmatrix}$$

In[\*]:= **NullSpace[M] // MatrixForm**

Out[\*]//MatrixForm=

$$\begin{pmatrix} -1 & \sqrt{\frac{(1+\omega)^2}{\omega}} & -1 & 0 & 0 & 0 & 1 & 0 & -1 & & \\ -\sqrt{\frac{(1+\omega)^2}{\omega}} & -\frac{-1-\omega-\omega^2}{\omega} & -\sqrt{\frac{(1+\omega)^2}{\omega}} & 1 & 1 & 1 & 0 & 1 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & -\frac{20\omega\sqrt{\frac{(1+\omega)^2}{\omega}}-156\omega^2\sqrt{\frac{(1+\omega)^2}{\omega}}+536\omega^3\sqrt{\frac{(1+\omega)^2}{\omega}}-1134\omega^4\sqrt{\frac{(1+\omega)^2}{\omega}}}{\omega} & \end{pmatrix}$$

In[\*]:= **RowReduce@NullSpace[M] // MatrixForm**

Out[\*]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 1 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & -\sqrt{\frac{(1+\omega)^2}{\omega}} & -\sqrt{\frac{(1+\omega)^2}{\omega}} & \frac{1+\omega+\omega^2}{\omega} & -\sqrt{\frac{(1+\omega)^2}{\omega}} & 1 & 1 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & \frac{1+\omega+\omega^2}{\omega} \\ 0 & 1 & 0 & -1 & -1 & -1 & \sqrt{\frac{(1+\omega)^2}{\omega}} & -1 & 0 & 0 & -1 & \sqrt{\frac{(1+\omega)^2}{\omega}} \end{pmatrix}$$

In[\*]:= **pivots = Union @@ FirstPosition[1] /@ RowReduce@NullSpace[M]**

Out[\*]=

{1, 2}

In[ ]:= **keeps = Complement [Range@Length@M, Union @@ FirstPosition [1] /@ RowReduce@NullSpace [M] ]**

Out[ ]:=  
 $\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

In[ ]:= **Factor@Det [M[[keeps, keeps]]]**

Out[ ]:=  

$$\frac{(1 - \omega + \omega^2) (4 - 16 \omega + 35 \omega^2 - 40 \omega^3 + 42 \omega^4 - 31 \omega^5 + 42 \omega^6 - 40 \omega^7 + 35 \omega^8 - 16 \omega^9 + 4 \omega^{10})}{\omega^6}$$

In[ ]:= **A5 [Knot [4, 1] ]**

Out[ ]:=  

$$\frac{(1 - 3 \omega + \omega^2)^2 (1 + \omega + \omega^2)^2}{\omega^4}$$

In[ ]:= **M // MatrixForm**

Out[ ]//MatrixForm=  

$$\begin{pmatrix} 2 & 0 & \sqrt{\frac{(1+\omega)^2}{\omega}} & 1 & \sqrt{\frac{(1+\omega)^2}{\omega}} & 1 \\ 0 & -2 & -1 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & -1 & -\sqrt{\frac{(1+\omega)^2}{\omega}} \\ \sqrt{\frac{(1+\omega)^2}{\omega}} & -1 & \frac{1-\omega+\omega^2}{\omega} & 0 & 2 & 0 \\ 1 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & 0 & -\frac{1-\omega+\omega^2}{\omega} & 0 & -2 \\ \sqrt{\frac{(1+\omega)^2}{\omega}} & -1 & 2 & 0 & \frac{1-\omega+\omega^2}{\omega} & 0 \\ 1 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & 0 & -2 & 0 & -\frac{1-\omega+\omega^2}{\omega} \end{pmatrix}$$

In[ ]:= **NullSpace [M] // MatrixForm**

Out[ ]//MatrixForm=  

$$\begin{pmatrix} -1 & -\sqrt{\frac{(1+\omega)^2}{\omega}} & 0 & 1 & 0 & 1 \\ -\sqrt{\frac{(1+\omega)^2}{\omega}} & -1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

In[ ]:= **PowerExpand@RowReduce@NullSpace [M] // MatrixForm**

Out[ ]//MatrixForm=  

$$\begin{pmatrix} 1 & 0 & -\frac{\sqrt{\omega} (1+\omega)}{1+\omega+\omega^2} & \frac{\omega}{1+\omega+\omega^2} & -\frac{\sqrt{\omega} (1+\omega)}{1+\omega+\omega^2} & \frac{\omega}{1+\omega+\omega^2} \\ 0 & 1 & \frac{\omega}{1+\omega+\omega^2} & -\frac{\sqrt{\omega} (1+\omega)}{1+\omega+\omega^2} & \frac{\omega}{1+\omega+\omega^2} & -\frac{\sqrt{\omega} (1+\omega)}{1+\omega+\omega^2} \end{pmatrix}$$

In[ ]:= **Det [NullSpace [M] [{{1, 2}}, {1, 2}]]**

Out[ ]:=  

$$-1 - \frac{1}{\omega} - \omega$$

In[\*]:= **rad = RowReduce@NullSpace[M]**

Out[\*]=

$$\left\{ \left\{ 1, 1, 0, -\frac{\omega \sqrt{\frac{(1+\omega)^2}{\omega}}}{1+\omega+\omega^2}, \frac{\omega}{1+\omega+\omega^2}, \frac{\omega}{1+\omega+\omega^2}, -\frac{\omega \sqrt{\frac{(1+\omega)^2}{\omega}}}{1+\omega+\omega^2}, \frac{\omega}{1+\omega+\omega^2}, -\frac{\omega \sqrt{\frac{(1+\omega)^2}{\omega}}}{1+\omega+\omega^2} \right\}, \right. \\ \left. \left\{ 0, 0, 1, \frac{\omega}{1+\omega+\omega^2}, -\frac{\omega \sqrt{\frac{(1+\omega)^2}{\omega}}}{1+\omega+\omega^2}, -\frac{\omega \sqrt{\frac{(1+\omega)^2}{\omega}}}{1+\omega+\omega^2}, \frac{\omega}{1+\omega+\omega^2}, -\frac{\omega \sqrt{\frac{(1+\omega)^2}{\omega}}}{1+\omega+\omega^2}, \frac{\omega}{1+\omega+\omega^2} \right\} \right\}$$

In[\*]:= **Subsets[Range[5], {2}]**

Out[\*]=

$$\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$$