Scatter and Glow - Analytic Testing

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Tails Commute and 4T

```
Der[Ar[1, 2] + Ar[1, 3]][Ar[2, 4]]
```

```
Y[1, 2, 4, AH[-1]]
```

```
Expect[0, Der[Ar[1, 3]][Ar[1, 2]]]
```

```
0
```

```
Expect[0,
 Der[Ar[1, 2] + Ar[1, 3]][Ar[2, 3]]
]
```

```
0
```

```
Der[Ar[1, 2]][Ar[1, 3] + Ar[2, 3]]
```

```
Y[1, 2, 3, AH[-1]]
```

```
Expect[0,
 Der[Ar[1, 2]][Ar[3, 1] + Ar[3, 2]]
]
```

```
0
```

```
Expect[{0, 0},
 Der[Ar[1, 1]]@{Ar[1, 2], Ar[2, 1]}
]
```

```
{0, 0}
```

```
Expect[{0, 0},
 Der[Ar[1, 2]]@{Ar[1, 1], Ar[2, 2]}
]
```

```
{0, 0}
```

Antisymmetry of Der

```
Expect[{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 Table[
  ReducePrimitives[
   Der[Y[1, 2, 3, AH[1]]]@Ar[i, j] + Der[Ar[i, j]]@Y[1, 2, 3, AH[1]]
  ], {i, 4}, {j, 4}
 ]
]
```

```
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

```
Expect[{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 Table[
  ReducePrimitives[
   Der[Y[1, 2, 2, AH[1]]]@Ar[i, j] + Der[Ar[i, j]]@Y[1, 2, 2, AH[1]]
  ], {i, 3}, {j, 3}
 ]
]
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
Expect[{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 ToAH[Table[
   ReducePrimitives[
    Der[ToPH[3, Y[1, 2, 3, AH[1]]]]@Ar[i, j] + Der[Ar[i, j]]@Y[1, 2, 3, ToPH[3, PH[1]]]
   ], {i, 4}, {j, 4}
  ]]
]
```

```
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

- **The braid group on two strands is commutative:**

```
Expect[Ar[1, 2],
 Ar[1, 2] // S[sigma[1, 2]]
]
```

```
Ar[1, 2]
```

```
Expect[Ar[1, 2],
 Ar[1, 2] // ToPH[5, S[sigma[1, 2]]]
]
```

```
Ar[1, 2]
```

- **Reidemeister 2**

```
Expect[SnG[S[], 0],
 SnG[sigma[1, 2], sigbar[1, 2]]
]
```

```
SnG[S[], 0]
```

■ Locallity in Scale (global over local)

```
S[sigma[3, 1], sigma[3, 2]]
```

$$S\left[Ar[0,\,1] \to Ar[0,\,1] + Y\left[0,\,3,\,1,\,AH\left[-\frac{-1+e^{-x[3]}}{x[3]}\right]\right],\right.$$

$$Ar[0,\,2] \to Ar[0,\,2] + Y\left[0,\,3,\,2,\,AH\left[-\frac{-1+e^{-x[3]}}{x[3]}\right]\right],$$

$$Ar[0,\,3] \to Ar[0,\,3] + Y\left[0,\,3,\,1,\,AH\left[\frac{-1+e^{-x[3]}}{x[3]}\right]\right] + Y\left[0,\,3,\,2,\,AH\left[\frac{-1+e^{-x[3]}}{x[3]}\right]\right],$$

$$Ar[1,\,0] \to Ar[1,\,0] + Y\left[1,\,3,\,0,\,AH\left[-\frac{-1+e^{x[3]}}{x[3]}\right]\right],$$

$$\left.Ar[2,\,0] \to Ar[2,\,0] + Y\left[2,\,3,\,0,\,AH\left[-\frac{-1+e^{x[3]}}{x[3]}\right]\right]\right]$$

```
Expect[{Ar[1, 2], Ar[2, 1]},
 {Ar[1, 2], Ar[2, 1]} // S[sigma[3, 1], sigma[3, 2]]
]
```

```
{Ar[1, 2], Ar[2, 1]}
```

■ Overcrossings Commute

```
oc1 = SnG[sigma[1, 2], sigma[1, 3]]
```

$$SnG\left[S\left[Ar[0,\,1] \to Ar[0,\,1] + Y\left[0,\,1,\,2,\,AH\left[\frac{-1+e^{-x[1]}}{x[1]}\right]\right] + Y\left[0,\,1,\,3,\,AH\left[\frac{-1+e^{-x[1]}}{x[1]}\right]\right],\right.\right.$$

$$Ar[0,\,2] \to Ar[0,\,2] + Y\left[0,\,1,\,2,\,AH\left[-\frac{-1+e^{-x[1]}}{x[1]}\right]\right],$$

$$Ar[0,\,3] \to Ar[0,\,3] + Y\left[0,\,1,\,3,\,AH\left[-\frac{-1+e^{-x[1]}}{x[1]}\right]\right],$$

$$Ar[2,\,0] \to Ar[2,\,0] + Y\left[1,\,2,\,0,\,AH\left[\frac{-1+e^{x[1]}}{x[1]}\right]\right],$$

$$\left.\left.Ar[3,\,0] \to Ar[3,\,0] + Y\left[1,\,3,\,0,\,AH\left[\frac{-1+e^{x[1]}}{x[1]}\right]\right]\right], Ar[1,\,2] + Ar[1,\,3]\right]$$

```
oc2 = SnG[sigma[1, 3], sigma[1, 2]];
Test[oc1 == oc2]
```

```
True
```

■ Reidemeister 3

```
r31 = SnG[sigma[1, 2], sigma[1, 3], sigma[2, 3]]
```

$$\text{SnG}\Big[\text{S}\Big[\text{Ar}[0,1] \to \text{Ar}[0,1] + \text{Y}\Big[0,1,2,\text{AH}\Big[\frac{-1+e^{-x[1]}}{x[1]}\Big]\Big] + \text{Y}\Big[0,1,3,\text{AH}\Big[-\frac{e^{-x[1]}\left(-1+e^{x[1]}\right)}{x[1]}\Big]\Big],$$

$$\text{Ar}[0,2] \to \text{Ar}[0,2] + \text{Y}\Big[0,1,2,\text{AH}\Big[-\frac{-1+e^{-x[1]}}{x[1]}\Big]\Big] +$$

$$\text{Y}\Big[0,1,3,\text{AH}\Big[\frac{\left(-1+e^{-x[1]}\right)\left(-1+e^{-x[2]}\right)}{x[1]}\Big]\Big] + \text{Y}\Big[0,2,3,\text{AH}\Big[\frac{-1+e^{-x[2]}}{x[2]}\Big]\Big],$$

$$\text{Ar}[0,3] \to \text{Ar}[0,3] + \text{Y}\Big[0,1,3,\text{AH}\Big[\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big]\Big] + \text{Y}\Big[0,2,3,\text{AH}\Big[-\frac{-1+e^{-x[2]}}{x[2]}\Big]\Big],$$

$$\text{Ar}[2,0] \to \text{Ar}[2,0] + \text{Y}\Big[1,2,0,\text{AH}\Big[\frac{-1+e^{x[1]}}{x[1]}\Big]\Big],$$

$$\text{Ar}[3,0] \to \text{Ar}[3,0] + \text{Y}\Big[1,2,0,\text{AH}\Big[-\frac{\left(-1+e^{x[1]}\right)\left(-1+e^{x[2]}\right)x[3]}{x[1]\,x[2]}\Big]\Big] +$$

$$\text{Y}\Big[1,3,0,\text{AH}\Big[\frac{e^{x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big]\Big] + \text{Y}\Big[2,3,0,\text{AH}\Big[\frac{-1+e^{x[2]}}{x[2]}\Big]\Big]\Big], \text{Ar}[1,2] + \text{Ar}[1,3] + \text{Ar}[2,3]\Big]$$

```
r32 = SnG[sigma[2, 3], sigma[1, 3], sigma[1, 2]];
Test[CanonicalForm[r31 == r32]]
```

```
True
```

■ Commutators Commute

```
cc11 = SnG[sigma[2, 1], sigma[3, 1], sigbar[2, 1], sigbar[3, 1]]
```

$$\text{SnG}\Big[\text{S}\Big[\text{Ar}[0,1] \to$$

$$\text{Ar}[0,1] + \text{Y}\Big[0,2,1,\text{AH}\Big[-\frac{\left(-1+e^{x[2]}\right)\left(-1+e^{x[3]}\right)}{x[2]}\Big]\Big] + \text{Y}\Big[0,3,1,\text{AH}\Big[\frac{\left(-1+e^{x[2]}\right)\left(-1+e^{x[3]}\right)}{x[3]}\Big]\Big],$$

$$\text{Ar}[0,2] \to \text{Ar}[0,2] + \text{Y}\Big[0,2,1,\text{AH}\Big[\frac{\left(-1+e^{x[2]}\right)\left(-1+e^{x[3]}\right)}{x[2]}\Big]\Big],$$

$$\text{Ar}[0,3] \to \text{Ar}[0,3] + \text{Y}\Big[0,3,1,\text{AH}\Big[-\frac{\left(-1+e^{x[2]}\right)\left(-1+e^{x[3]}\right)}{x[3]}\Big]\Big], \text{Ar}[1,0] \to$$

$$\text{Ar}[1,0] + \text{Y}\Big[1,2,0,\text{AH}\Big[\frac{\left(-1+e^{x[2]}\right)\left(-1+e^{x[3]}\right)}{x[2]}\Big]\Big] + \text{Y}\Big[1,3,0,\text{AH}\Big[-\frac{\left(-1+e^{x[2]}\right)\left(-1+e^{x[3]}\right)}{x[3]}\Big]\Big]\Big],$$

$$\text{Y}\Big[2,3,1,\text{AH}\Big[\frac{-e^{x[2]}x[2] + e^{x[2]+x[3]}x[2] - e^{x[3]}x[3] + e^{x[2]+x[3]}x[3]}{x[2]\,x[3]}\Big]\Big]\Big]$$

```
cc12 = SnG[sigma[4, 1], sigma[5, 1], sigbar[4, 1], sigbar[5, 1]];
Test[(cc11 ** cc12) == (cc12 ** cc11)]
```

```
True
```

```
cc21 = SnG[sigma[2, 1], sigma[3, 1], sigbar[2, 1], sigbar[3, 1]]
```

$$\text{SnG}\Big[\text{S}\Big[\text{Ar}[0,1] \to \text{Ar}[0,1] + \text{Y}\Big[0,2,1,\text{AH}\Big[-\frac{\left(-1+e^{x[2]}\right)\left(-1+e^{x[3]}\right)}{x[2]}\Big]\Big] + \text{Y}\Big[0,3,1,\text{AH}\Big[\frac{\left(-1+e^{x[2]}\right)\left(-1+e^{x[3]}\right)}{x[3]}\Big]\Big],$$

$$\text{Ar}[0,2] \to \text{Ar}[0,2] + \text{Y}\Big[0,2,1,\text{AH}\Big[\frac{\left(-1+e^{x[2]}\right)\left(-1+e^{x[3]}\right)}{x[2]}\Big]\Big],$$

$$\text{Ar}[0,3] \to \text{Ar}[0,3] + \text{Y}\Big[0,3,1,\text{AH}\Big[-\frac{\left(-1+e^{x[2]}\right)\left(-1+e^{x[3]}\right)}{x[3]}\Big]\Big], \text{Ar}[1,0] \to$$

$$\text{Ar}[1,0] + \text{Y}\Big[1,2,0,\text{AH}\Big[\frac{\left(-1+e^{x[2]}\right)\left(-1+e^{x[3]}\right)}{x[2]}\Big]\Big] + \text{Y}\Big[1,3,0,\text{AH}\Big[-\frac{\left(-1+e^{x[2]}\right)\left(-1+e^{x[3]}\right)}{x[3]}\Big]\Big]\Big],$$

$$\text{Y}\Big[2,3,1,\text{AH}\Big[\frac{-e^{x[2]}\,x[2] + e^{x[2]+x[3]}\,x[2] - e^{x[3]}\,x[3] + e^{x[2]+x[3]}\,x[3]}{x[2]\,x[3]}\Big]\Big]\Big]$$

```
cc22 = SnG[sigma[3, 1], sigma[4, 1], sigbar[3, 1], sigbar[4, 1]];
Test[(cc21 ** cc22) == (cc22 ** cc21)]
```

```
True
```

```
cc31 = SnG[sigma[1, 2], sigma[3, 1], sigbar[1, 2], sigbar[3, 1]]
```

$$\text{SnG}\Big[\text{S}\Big[\text{Ar}[0,1] \to \text{Ar}[0,1] + \text{Y}\Big[0,1,2,\text{AH}\Big[-\frac{e^{-x[3]}\left(-1+e^{x[1]}\right)\left(-1+e^{x[3]}\right)}{x[1]}\Big]\Big], \text{Ar}[0,2] \to \text{Ar}[0,2] +$$

$$\text{Y}\Big[0,1,2,\text{AH}\Big[\frac{e^{-x[3]}\left(-1+e^{x[1]}\right)\left(-1+e^{x[3]}\right)}{x[1]}\Big]\Big] + \text{Y}\Big[0,3,2,\text{AH}\Big[\frac{\left(-1+e^{x[1]}\right)\left(-1+e^{-x[3]}\right)}{x[3]}\Big]\Big],$$

$$\text{Ar}[0,3] \to \text{Ar}[0,3] + \text{Y}\Big[0,3,2,\text{AH}\Big[\frac{e^{-x[3]}\left(-1+e^{x[1]}\right)\left(-1+e^{x[3]}\right)}{x[3]}\Big]\Big],$$

$$\text{Ar}[2,0] \to \text{Ar}[2,0] + \text{Y}\Big[1,3,0,\text{AH}\Big[\frac{e^{-x[3]}\left(-1+e^{x[1]}\right)\left(-1+e^{x[3]}\right)x[2]}{x[1]\,x[3]}\Big]\Big]\Big],$$

$$\text{Y}\Big[1,3,2,\text{AH}\Big[-\frac{e^{-x[3]}\left(-e^{x[1]}\,x[1] + e^{x[1]+x[3]}\,x[1] - x[3] + e^{x[1]}\,x[3]\right)}{x[1]\,x[3]}\Big]\Big]\Big]$$

```
cc32 = SnG[sigma[1, 4], sigma[5, 1], sigbar[1, 4], sigbar[5, 1]];
Test[(cc31 ** cc32) == (cc32 ** cc31)]
```

```
True
```

- **This last one we expect to fail:**

```
cc41 = SnG[sigma[1, 2], sigma[3, 1], sigbar[1, 2], sigbar[3, 1]]
```

$$\text{SnG}\left[\text{S}\left[\text{Ar}[0,1]\to \text{Ar}[0,1]+\text{Y}\left[0,1,2,\text{AH}\left[-\frac{e^{-x[3]}\left(-1+e^{x[1]}\right)\left(-1+e^{x[3]}\right)}{x[1]}\right]\right],\ \text{Ar}[0,2]\to \text{Ar}[0,2]+\text{Y}\left[0,1,2,\text{AH}\left[\frac{e^{-x[3]}\left(-1+e^{x[1]}\right)\left(-1+e^{x[3]}\right)}{x[1]}\right]\right]+\text{Y}\left[0,3,2,\text{AH}\left[\frac{\left(-1+e^{x[1]}\right)\left(-1+e^{-x[3]}\right)}{x[3]}\right]\right],\right.$$
$$\text{Ar}[0,3]\to \text{Ar}[0,3]+\text{Y}\left[0,3,2,\text{AH}\left[\frac{e^{-x[3]}\left(-1+e^{x[1]}\right)\left(-1+e^{x[3]}\right)}{x[3]}\right]\right],$$
$$\text{Ar}[2,0]\to \text{Ar}[2,0]+\text{Y}\left[1,3,0,\text{AH}\left[\frac{e^{-x[3]}\left(-1+e^{x[1]}\right)\left(-1+e^{x[3]}\right)x[2]}{x[1]\,x[3]}\right]\right]\Bigg],$$
$$\left.\text{Y}\left[1,3,2,\text{AH}\left[-\frac{e^{-x[3]}\left(-e^{x[1]}x[1]+e^{x[1]+x[3]}x[1]-x[3]+e^{x[1]}x[3]\right)}{x[1]\,x[3]}\right]\right]\right]$$

```
cc42 = SnG[sigma[4, 1], sigma[5, 1], sigbar[4, 1], sigbar[5, 1]];
Expect[False,
 (cc41 ** cc42) === (cc42 ** cc41)
]
```

```
False
```

- **Commutators Commutators are Central (along strand 1)**

```
(ccc = SnG[
    sigma[1, 2], sigma[3, 1], sigbar[1, 2], sigbar[3, 1],
    sigma[4, 1], sigma[5, 1], sigbar[4, 1], sigbar[5, 1],
    sigma[3, 1], sigma[1, 2], sigbar[3, 1], sigbar[1, 2],
    sigma[5, 1], sigma[4, 1], sigbar[5, 1], sigbar[4, 1]
   ]) // Last
```

$$\text{Y}\left[1,4,2,\right.$$
$$\text{AH}\left[\frac{1}{x[1]\,x[4]}e^{-x[3]}\left(-e^{x[1]}x[1]+e^{x[1]+x[3]}x[1]+e^{x[1]+x[4]}x[1]-e^{x[1]+x[3]+x[4]}x[1]+e^{x[1]+x[5]}x[1]-\right.\right.$$
$$e^{x[1]+x[3]+x[5]}x[1]-e^{x[1]+x[4]+x[5]}x[1]+e^{x[1]+x[3]+x[4]+x[5]}x[1]-x[3]+e^{x[1]}x[3]+$$
$$e^{x[4]}x[3]-e^{x[1]+x[4]}x[3]+e^{x[5]}x[3]-e^{x[1]+x[5]}x[3]-e^{x[4]+x[5]}x[3]+$$
$$e^{x[1]+x[4]+x[5]}x[3]-e^{x[4]}x[4]+e^{x[1]+x[4]}x[4]+e^{x[3]+x[4]}x[4]-e^{x[1]+x[3]+x[4]}x[4]+$$
$$e^{x[4]+x[5]}x[4]-e^{x[1]+x[4]+x[5]}x[4]-e^{x[3]+x[4]+x[5]}x[4]+e^{x[1]+x[3]+x[4]+x[5]}x[4]-$$
$$\left.\left.e^{x[5]}x[5]+e^{x[1]+x[5]}x[5]+e^{x[3]+x[5]}x[5]-e^{x[1]+x[3]+x[5]}x[5]+e^{x[4]+x[5]}x[5]-\right.\right.$$
$$\left.\left.e^{x[1]+x[4]+x[5]}x[5]-e^{x[3]+x[4]+x[5]}x[5]+e^{x[1]+x[3]+x[4]+x[5]}x[5]\right)\right]+$$
$$\text{Y}\left[1,5,2,\text{AH}\left[-\frac{1}{x[1]\,x[5]}e^{-x[3]}\left(-e^{x[1]}x[1]+e^{x[1]+x[3]}x[1]+e^{x[1]+x[4]}x[1]-e^{x[1]+x[3]+x[4]}x[1]+\right.\right.\right.$$
$$e^{x[1]+x[5]}x[1]-e^{x[1]+x[3]+x[5]}x[1]-e^{x[1]+x[4]+x[5]}x[1]+e^{x[1]+x[3]+x[4]+x[5]}x[1]-$$
$$x[3]+e^{x[1]}x[3]+e^{x[4]}x[3]-e^{x[1]+x[4]}x[3]+e^{x[5]}x[3]-e^{x[1]+x[5]}x[3]-$$
$$e^{x[4]+x[5]}x[3]+e^{x[1]+x[4]+x[5]}x[3]-e^{x[4]}x[4]+e^{x[1]+x[4]}x[4]+e^{x[3]+x[4]}x[4]-$$
$$e^{x[1]+x[3]+x[4]}x[4]+e^{x[4]+x[5]}x[4]-e^{x[1]+x[4]+x[5]}x[4]-e^{x[3]+x[4]+x[5]}x[4]+$$
$$e^{x[1]+x[3]+x[4]+x[5]}x[4]-e^{x[5]}x[5]+e^{x[1]+x[5]}x[5]+e^{x[3]+x[5]}x[5]-e^{x[1]+x[3]+x[5]}x[5]+$$
$$\left.\left.\left.e^{x[4]+x[5]}x[5]-e^{x[1]+x[4]+x[5]}x[5]-e^{x[3]+x[4]+x[5]}x[5]+e^{x[1]+x[3]+x[4]+x[5]}x[5]\right)\right]\right]$$

```
Test[ccc ** SnG[sigma[6, 1]] == SnG[sigma[6, 1]] ** ccc]
```

```
True
```

```
Test[ccc ** SnG[sigma[1, 6]] == SnG[sigma[1, 6]] ** ccc]
```

```
True
```

■ Scattering by Exponentials

```
S[Exp[Ar[1, 2]]]
```

$$S\left[Ar[0, 1] \to Ar[0, 1] + Y\left[0, 1, 2, AH\left[-\frac{e^{-x[1]}\left(-1 + e^{x[1]}\right)}{x[1]}\right]\right],\right.$$

$$Ar[0, 2] \to Ar[0, 2] + Y\left[0, 1, 2, AH\left[\frac{e^{-x[1]}\left(-1 + e^{x[1]}\right)}{x[1]}\right]\right],$$

$$\left.Ar[2, 0] \to Ar[2, 0] + Y\left[1, 2, 0, AH\left[\frac{-1 + e^{x[1]}}{x[1]}\right]\right]\right]$$

```
Test[CanonicalForm[
  S[Exp[Ar[1, 2]]] == S[sigma[1, 2]]
 ]]
```

```
True
```

```
Test[CanonicalForm[
  S[Exp[-Ar[1, 2]]] == S[sigbar[1, 2]]
 ]]
```

```
True
```

■ Mixing AH and PH

```
mix1 = ToPH[3, S[sigma[3, 1], sigma[3, 2]]]
```

$$S\left[Ar[0, 1] \to Ar[0, 1] + Y\left[0, 3, 1, PH\left[1 - \frac{1}{2}x[3]\,z + \frac{1}{6}x[3]^2 z^2 + O[z]^3\right]\right],\right.$$

$$Ar[0, 2] \to Ar[0, 2] + Y\left[0, 3, 2, PH\left[1 - \frac{1}{2}x[3]\,z + \frac{1}{6}x[3]^2 z^2 + O[z]^3\right]\right],$$

$$Ar[0, 3] \to Ar[0, 3] + Y\left[0, 3, 1, PH\left[-1 + \frac{1}{2}x[3]\,z - \frac{1}{6}x[3]^2 z^2 + O[z]^3\right]\right] +$$

$$Y\left[0, 3, 2, PH\left[-1 + \frac{1}{2}x[3]\,z - \frac{1}{6}x[3]^2 z^2 + O[z]^3\right]\right],$$

$$Ar[1, 0] \to Ar[1, 0] + Y\left[1, 3, 0, PH\left[-1 - \frac{1}{2}x[3]\,z - \frac{1}{6}x[3]^2 z^2 + O[z]^3\right]\right],$$

$$\left.Ar[2, 0] \to Ar[2, 0] + Y\left[2, 3, 0, PH\left[-1 - \frac{1}{2}x[3]\,z - \frac{1}{6}x[3]^2 z^2 + O[z]^3\right]\right]\right]$$

```
mix2 = ToPH[3, S[sigma[3, 1]] ** S[sigma[3, 2]]
```

$$S\Big[Ar[0,1] \to Ar[0,1] + Y\Big[0,3,1,PH\Big[1 - \frac{1}{2}x[3]\,z + \frac{1}{6}x[3]^2 z^2 + O[z]^3\Big]\Big],$$
$$Ar[0,2] \to Ar[0,2] + Y\Big[0,3,2,AH\Big[-\frac{-1+e^{-x[3]}}{x[3]}\Big]\Big],$$
$$Ar[0,3] \to Ar[0,3] + Y\Big[0,3,1,PH\Big[-1 + \frac{1}{2}x[3]\,z - \frac{1}{6}x[3]^2 z^2 + O[z]^3\Big]\Big] + Y\Big[0,3,2,AH\Big[\frac{-1+e^{-x[3]}}{x[3]}\Big]\Big],$$
$$Ar[1,0] \to Ar[1,0] + Y\Big[1,3,0,PH\Big[-1 - \frac{1}{2}x[3]\,z - \frac{1}{6}x[3]^2 z^2 + O[z]^3\Big]\Big],$$
$$Ar[2,0] \to Ar[2,0] + Y\Big[2,3,0,AH\Big[-\frac{-1+e^{x[3]}}{x[3]}\Big]\Big]\Big]$$

```
Test[ToPH[3, mix1 == mix2]]
```

True

■ The BCH Formula

■ Analytic

```
S[sigma[1, 3], sigma[2, 3]] // Short
```

$$S\Big[Ar[0,1] \to Ar[0,1] + Y\Big[0,1,3,AH\Big[-\frac{e^{\ll 1\gg}\ \ll 1\gg}{x[1]}\Big]\Big],\ \ll 2\gg,\ Ar[3,0] \to \ll 1\gg\Big]$$

```
S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bc]]] // Short
```

$$S\Big[Ar[0,1] \to Ar[0,1] + Y\Big[0,1,3,AH\Big[\frac{\ll 1\gg}{x[1] + \ll 1\gg}\Big]\Big],\ \ll 1\gg,\ \ll 1\gg,\ Ar[3,0] \to \ll 1\gg\Big]$$

```
bch = bc /. First[HSolve[
    Coefficient[
      Ar[0, 1] // S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bc]]],
      Y[0, 1, 3]
     ] == Coefficient[
      Ar[0, 1] // S[sigma[1, 3], sigma[2, 3]],
      Y[0, 1, 3]
     ],
    bc
   ]]
```

$$\frac{-e^{x[1]}\,x[1] + e^{x[1]+x[2]}\,x[1] + x[2] - e^{x[1]}\,x[2]}{\left(-1 + e^{x[1]+x[2]}\right)x[1]\,x[2]}$$

```
Test[CanonicalForm[
  S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, AH[bch]]]] == S[sigma[1, 3], sigma[2, 3]]
 ]]
```

True

- **Compare with Kurlin**

```
Test[Simplify[(bch /. {x[1] → x, x[2] → y}) == 1/y (1 - (e^x - 1)/x (x + y)/(e^(x+y) - 1))]]
```

```
True
```

- **Perturbative**

```
unknowns = DeclareSeries[bc3[x[1], x[2]], 4]
```

```
{bc3[0, 0], bc3[1, 0], bc3[0, 1], bc3[2, 0],
 bc3[1, 1], bc3[0, 2], bc3[3, 0], bc3[2, 1], bc3[1, 2], bc3[0, 3]}
```

```
PH[bc3]
```

$$\text{PH}\left[\text{bc3}[0,0]+(\text{bc3}[1,0]\,x[1]+\text{bc3}[0,1]\,x[2])\,z+\left(\frac{1}{2}\text{bc3}[2,0]\,x[1]^2+\text{bc3}[1,1]\,x[1]\,x[2]+\frac{1}{2}\text{bc3}[0,2]\,x[2]^2\right)z^2+\left(\frac{1}{6}\text{bc3}[3,0]\,x[1]^3+\frac{1}{2}\text{bc3}[2,1]\,x[1]^2\,x[2]+\frac{1}{2}\text{bc3}[1,2]\,x[1]\,x[2]^2+\frac{1}{6}\text{bc3}[0,3]\,x[2]^3\right)z^3+O[z]^4\right]$$

```
S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, PH[bc3]]]] // Short[#, 5] &
```

$$S\left[\text{Ar}[0,1]\to \text{Ar}[0,1]+Y\left[0,1,3,\text{PH}\left[-1+\left(\frac{x[1]}{2}+\frac{x[2]}{2}+\text{bc3}[0,0]\,x[2]\right)z+\left(\ll 10\gg+\text{bc3}[0,1]\ll 1\gg^2\right)z^2+(\ll 1\gg)z^3+\left(-\frac{1}{120}x[1]^4-\frac{1}{30}x[1]^3x[2]-\frac{1}{24}\text{bc3}[0,0]\,x[1]^3x[2]+\ll 30\gg+\frac{1}{6}\text{bc3}[0,1]\,x[2]^4-\frac{1}{4}\text{bc3}[0,2]\,x[2]^4+\frac{1}{6}\text{bc3}[0,3]\,x[2]^4\right)z^4+O[z]^5\right]\right],\ \ll 2\gg,\ \text{Ar}[3,0]\to\ll 1\gg\right]$$

```
(eq = Coefficient[
      Ar[0, 1] // S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, PH[bc3]]]],
      Y[0, 1, 3]
     ] == Coefficient[
      Ar[0, 1] // S[sigma[1, 3], sigma[2, 3]],
      Y[0, 1, 3]
     ]) // Short[#, 5] &
```

$$\text{PH}\left[-1+\left(\frac{x[1]}{2}+\frac{x[2]}{2}+\text{bc3}[0,0]\,x[2]\right)z+\left(\ll 10\gg+\text{bc3}[0,1]\,x[2]^2\right)z^2+\left(\frac{x[1]^3}{24}+\ll 16\gg+\frac{1}{2}\text{bc3}[0,2]\,x[2]^3\right)z^3+\left(-\frac{1}{120}x[1]^4-\frac{1}{30}x[1]^3x[2]-\frac{1}{24}\text{bc3}[0,0]\,x[1]^3x[2]+\ll 30\gg+\frac{1}{6}\text{bc3}[0,1]\,x[2]^4-\frac{1}{4}\text{bc3}[0,2]\,x[2]^4+\frac{1}{6}\text{bc3}[0,3]\,x[2]^4\right)z^4+O[z]^5\right]==\text{AH}\left[-\frac{\ll 1\gg}{\ll 1\gg}\right]$$

```
sol = First[HSolve[eq, unknowns]]
```

$$\left\{bc3[0,0]\to\frac{1}{2},\ bc3[1,0]\to\frac{1}{12},\ bc3[0,1]\to-\frac{1}{12},\ bc3[2,0]\to 0,\ bc3[1,1]\to-\frac{1}{24},\right.$$
$$\left. bc3[0,2]\to 0,\ bc3[3,0]\to-\frac{1}{120},\ bc3[2,1]\to-\frac{1}{90},\ bc3[1,2]\to\frac{1}{90},\ bc3[0,3]\to\frac{1}{120}\right\}$$

```
bch3 = PH[bc3] /. sol
```

$$PH\left[\frac{1}{2}+\left(\frac{x[1]}{12}-\frac{x[2]}{12}\right)z-\frac{1}{24}(x[1]\,x[2])\,z^2+\left(-\frac{1}{720}x[1]^3-\frac{1}{180}x[1]^2x[2]+\frac{1}{180}x[1]\,x[2]^2+\frac{x[2]^3}{720}\right)z^3+O[z]^4\right]$$

```
Test[CanonicalForm[
  S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bch3]]] == ToPH[5, S[sigma[1, 3], sigma[2, 3]]]
 ]]
```

```
True
```

```
Test[Simplify[(bch3 /. {x[1] → x, x[2] → y}) ==
   PH[(((1/y) (1 - (e^x - 1)/x (x + y)/(e^(x+y) - 1))) /. {x → z x, y → z y}) + O[z]^4]]]
```

```
True
```

■ Testing Code

```
SetAttributes[{Test, Expect}, {HoldAll}];
Test[expr_] := If[TrueQ[Check[expr, False]], True,
   If[Head[$FailLog] =!= List, $FailLog = {}];
   AppendTo[$FailLog,
    "On " <> ToString[Date[]] <> " failed in " <> ToString[HoldForm[expr]]];
   Print[Last[$FailLog]]
  ];
Expect[val_, expr_] := If[TrueQ[Test[val == expr]], val];

SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\ScatterAndGlow"]
<< ScatterAndGlow.m
```

```
C:\drorbn\AcademicPensieve\Projects\ScatterAndGlow
```

■ Test Test

```
Test[0 == 1]
```

```
On {2009, 1, 16, 10, 27, 52.1856000} failed in 0 == 1
```

Failed Tests

$FailLog

```
{On {2009, 1, 16, 10, 27, 44.4324000} failed in 0 == 1,
 On {2009, 1, 16, 10, 27, 52.1856000} failed in 0 == 1}
```