Scatter and Glow - Perturbative Testing

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Mixing AH and PH

```
mix1 = ToPH[3, S[sigma[3, 1], sigma[3, 2]]]
```

$$S\Big[Ar[0,1]\to Ar[0,1]+Y\Big[0,3,1,PH\Big[1-\frac{1}{2}x[3]\,z+\frac{1}{6}x[3]^2 z^2+O[z]^3\Big]\Big],$$
$$Ar[0,2]\to Ar[0,2]+Y\Big[0,3,2,PH\Big[1-\frac{1}{2}x[3]\,z+\frac{1}{6}x[3]^2 z^2+O[z]^3\Big]\Big],$$
$$Ar[0,3]\to Ar[0,3]+Y\Big[0,3,1,PH\Big[-1+\frac{1}{2}x[3]\,z-\frac{1}{6}x[3]^2 z^2+O[z]^3\Big]\Big]+$$
$$Y\Big[0,3,2,PH\Big[-1+\frac{1}{2}x[3]\,z-\frac{1}{6}x[3]^2 z^2+O[z]^3\Big]\Big],$$
$$Ar[1,0]\to Ar[1,0]+Y\Big[1,3,0,PH\Big[-1-\frac{1}{2}x[3]\,z-\frac{1}{6}x[3]^2 z^2+O[z]^3\Big]\Big],$$
$$Ar[2,0]\to Ar[2,0]+Y\Big[2,3,0,PH\Big[-1-\frac{1}{2}x[3]\,z-\frac{1}{6}x[3]^2 z^2+O[z]^3\Big]\Big]\Big]$$

```
mix2 = ToPH[3, S[sigma[3, 1]]] ** S[sigma[3, 2]]
```

$$S\Big[Ar[0,1]\to Ar[0,1]+Y\Big[0,3,1,PH\Big[1-\frac{1}{2}x[3]\,z+\frac{1}{6}x[3]^2 z^2+O[z]^3\Big]\Big],$$
$$Ar[0,2]\to Ar[0,2]+Y\Big[0,3,2,AH\Big[-\frac{-1+e^{-x[3]}}{x[3]}\Big]\Big],$$
$$Ar[0,3]\to Ar[0,3]+Y\Big[0,3,1,PH\Big[-1+\frac{1}{2}x[3]\,z-\frac{1}{6}x[3]^2 z^2+O[z]^3\Big]\Big]+Y\Big[0,3,2,AH\Big[\frac{-1+e^{-x[3]}}{x[3]}\Big]\Big],$$
$$Ar[1,0]\to Ar[1,0]+Y\Big[1,3,0,PH\Big[-1-\frac{1}{2}x[3]\,z-\frac{1}{6}x[3]^2 z^2+O[z]^3\Big]\Big],$$
$$Ar[2,0]\to Ar[2,0]+Y\Big[2,3,0,AH\Big[-\frac{-1+e^{x[3]}}{x[3]}\Big]\Big]\Big]$$

```
Test[ToPH[3, mix1 == mix2]]
```

True

The braid group on two strands is commutative:

```
Expect[Ar[1, 2],
 Ar[1, 2] // ToPH[5, S[sigma[1, 2]]]
]
```

Ar[1, 2]

Reidemeister 2

```
Expect[SnG[S[], 0],
 ToPH[5, SnG[sigma[1, 2], sigbar[1, 2]]]
]
```

```
SnG[S[], 0]
```

Locallity in Scale (global over local)

```
ToPH[3, S[sigma[3, 1], sigma[3, 2]]]
```

$$\begin{aligned}
&\mathrm{S}\Big[\mathrm{Ar}[0,1]\to\mathrm{Ar}[0,1]+\mathrm{Y}\Big[0,3,1,\mathrm{PH}\Big[1-\frac{1}{2}\,\mathrm{x}[3]\,\mathrm{z}+\frac{1}{6}\,\mathrm{x}[3]^2\,\mathrm{z}^2+\mathrm{O}[\mathrm{z}]^3\Big]\Big],\\
&\quad\mathrm{Ar}[0,2]\to\mathrm{Ar}[0,2]+\mathrm{Y}\Big[0,3,2,\mathrm{PH}\Big[1-\frac{1}{2}\,\mathrm{x}[3]\,\mathrm{z}+\frac{1}{6}\,\mathrm{x}[3]^2\,\mathrm{z}^2+\mathrm{O}[\mathrm{z}]^3\Big]\Big],\\
&\quad\mathrm{Ar}[0,3]\to\mathrm{Ar}[0,3]+\mathrm{Y}\Big[0,3,1,\mathrm{PH}\Big[-1+\frac{1}{2}\,\mathrm{x}[3]\,\mathrm{z}-\frac{1}{6}\,\mathrm{x}[3]^2\,\mathrm{z}^2+\mathrm{O}[\mathrm{z}]^3\Big]\Big]+\\
&\qquad\mathrm{Y}\Big[0,3,2,\mathrm{PH}\Big[-1+\frac{1}{2}\,\mathrm{x}[3]\,\mathrm{z}-\frac{1}{6}\,\mathrm{x}[3]^2\,\mathrm{z}^2+\mathrm{O}[\mathrm{z}]^3\Big]\Big],\\
&\quad\mathrm{Ar}[1,0]\to\mathrm{Ar}[1,0]+\mathrm{Y}\Big[1,3,0,\mathrm{PH}\Big[-1-\frac{1}{2}\,\mathrm{x}[3]\,\mathrm{z}-\frac{1}{6}\,\mathrm{x}[3]^2\,\mathrm{z}^2+\mathrm{O}[\mathrm{z}]^3\Big]\Big],\\
&\quad\mathrm{Ar}[2,0]\to\mathrm{Ar}[2,0]+\mathrm{Y}\Big[2,3,0,\mathrm{PH}\Big[-1-\frac{1}{2}\,\mathrm{x}[3]\,\mathrm{z}-\frac{1}{6}\,\mathrm{x}[3]^2\,\mathrm{z}^2+\mathrm{O}[\mathrm{z}]^3\Big]\Big]\Big]
\end{aligned}$$

```
Expect[{Ar[1, 2], Ar[2, 1]},
 ToAH[
  {Ar[1, 2], Ar[2, 1]} // ToPH[9, S[sigma[3, 1], sigma[3, 2]]]
 ]
]
```

```
{Ar[1, 2], Ar[2, 1]}
```

Overcrossings Commute

```
oc1 = ToPH[3, SnG[sigma[1, 2], sigma[1, 3]]]
```

$$\begin{aligned}
&\mathrm{SnG}\Big[\mathrm{S}\Big[\mathrm{Ar}[0,1]\to\mathrm{Ar}[0,1]+\mathrm{Y}\Big[0,1,2,\mathrm{PH}\Big[-1+\frac{1}{2}\,\mathrm{x}[1]\,\mathrm{z}-\frac{1}{6}\,\mathrm{x}[1]^2\,\mathrm{z}^2+\mathrm{O}[\mathrm{z}]^3\Big]\Big]+\\
&\qquad\mathrm{Y}\Big[0,1,3,\mathrm{PH}\Big[-1+\frac{1}{2}\,\mathrm{x}[1]\,\mathrm{z}-\frac{1}{6}\,\mathrm{x}[1]^2\,\mathrm{z}^2+\mathrm{O}[\mathrm{z}]^3\Big]\Big],\\
&\quad\mathrm{Ar}[0,2]\to\mathrm{Ar}[0,2]+\mathrm{Y}\Big[0,1,2,\mathrm{PH}\Big[1-\frac{1}{2}\,\mathrm{x}[1]\,\mathrm{z}+\frac{1}{6}\,\mathrm{x}[1]^2\,\mathrm{z}^2+\mathrm{O}[\mathrm{z}]^3\Big]\Big],\\
&\quad\mathrm{Ar}[0,3]\to\mathrm{Ar}[0,3]+\mathrm{Y}\Big[0,1,3,\mathrm{PH}\Big[1-\frac{1}{2}\,\mathrm{x}[1]\,\mathrm{z}+\frac{1}{6}\,\mathrm{x}[1]^2\,\mathrm{z}^2+\mathrm{O}[\mathrm{z}]^3\Big]\Big],\\
&\quad\mathrm{Ar}[2,0]\to\mathrm{Ar}[2,0]+\mathrm{Y}\Big[1,2,0,\mathrm{PH}\Big[1+\frac{1}{2}\,\mathrm{x}[1]\,\mathrm{z}+\frac{1}{6}\,\mathrm{x}[1]^2\,\mathrm{z}^2+\mathrm{O}[\mathrm{z}]^3\Big]\Big],\\
&\quad\mathrm{Ar}[3,0]\to\mathrm{Ar}[3,0]+\mathrm{Y}\Big[1,3,0,\mathrm{PH}\Big[1+\frac{1}{2}\,\mathrm{x}[1]\,\mathrm{z}+\frac{1}{6}\,\mathrm{x}[1]^2\,\mathrm{z}^2+\mathrm{O}[\mathrm{z}]^3\Big]\Big]\Big],\ \mathrm{Ar}[1,2]+\mathrm{Ar}[1,3]\Big]
\end{aligned}$$

```
oc2 = ToPH[3, SnG[sigma[1, 3], sigma[1, 2]]];
Test[oc1 == oc2]
```

```
True
```

- **Reidemeister 3**

```
r31 = ToPH[4, CanonicalForm[SnG[sigma[1, 2], sigma[1, 3], sigma[2, 3]]]]
```

$$
\begin{aligned}
&\mathrm{SnG}\Big[\mathrm{S}\Big[\mathrm{Ar}[0,1]\to\mathrm{Ar}[0,1]+\mathrm{Y}\Big[0,1,2,\mathrm{PH}\Big[-1+\frac{1}{2}x[1]\,z-\frac{1}{6}x[1]^2z^2+\frac{1}{24}x[1]^3z^3+O[z]^4\Big]\Big]+\\
&\quad\mathrm{Y}\Big[0,1,3,\mathrm{PH}\Big[-1+\frac{1}{2}x[1]\,z-\frac{1}{6}x[1]^2z^2+\frac{1}{24}x[1]^3z^3+O[z]^4\Big]\Big],\\
&\ \mathrm{Ar}[0,2]\to\mathrm{Ar}[0,2]+\mathrm{Y}\Big[0,1,2,\mathrm{PH}\Big[1-\frac{1}{2}x[1]\,z+\frac{1}{6}x[1]^2z^2-\frac{1}{24}x[1]^3z^3+O[z]^4\Big]\Big]+\mathrm{Y}\Big[0,1,3,\\
&\qquad\mathrm{PH}\Big[x[2]\,z+\left(-\frac{1}{2}x[1]\,x[2]-\frac{x[2]^2}{2}\right)z^2+\frac{1}{12}\left(2x[1]^2x[2]+3x[1]\,x[2]^2+2x[2]^3\right)z^3+O[z]^4\Big]\Big]+\\
&\quad\mathrm{Y}\Big[0,2,3,\mathrm{PH}\Big[-1+\frac{1}{2}x[2]\,z-\frac{1}{6}x[2]^2z^2+\frac{1}{24}x[2]^3z^3+O[z]^4\Big]\Big],\\
&\ \mathrm{Ar}[0,3]\to\mathrm{Ar}[0,3]+\mathrm{Y}\Big[0,1,3,\mathrm{PH}\Big[1+\frac{1}{2}\left(-x[1]-2x[2]\right)z+\frac{1}{6}\left(x[1]^2+3x[1]\,x[2]+3x[2]^2\right)z^2+\\
&\qquad\frac{1}{24}\left(-x[1]^3-4x[1]^2x[2]-6x[1]\,x[2]^2-4x[2]^3\right)z^3+O[z]^4\Big]\Big]+\\
&\quad\mathrm{Y}\Big[0,2,3,\mathrm{PH}\Big[1-\frac{1}{2}x[2]\,z+\frac{1}{6}x[2]^2z^2-\frac{1}{24}x[2]^3z^3+O[z]^4\Big]\Big],\\
&\ \mathrm{Ar}[2,0]\to\mathrm{Ar}[2,0]+\mathrm{Y}\Big[1,2,0,\mathrm{PH}\Big[1+\frac{1}{2}x[1]\,z+\frac{1}{6}x[1]^2z^2+\frac{1}{24}x[1]^3z^3+O[z]^4\Big]\Big],\\
&\ \mathrm{Ar}[3,0]\to\mathrm{Ar}[3,0]+\mathrm{Y}\Big[1,2,0,\mathrm{PH}\Big[-x[3]\,z+\left(-\frac{1}{2}x[1]\,x[3]-\frac{1}{2}x[2]\,x[3]\right)z^2+\\
&\qquad\frac{1}{12}\left(-2x[1]^2x[3]-3x[1]\,x[2]\,x[3]-2x[2]^2x[3]\right)z^3+O[z]^4\Big]\Big]+\\
&\quad\mathrm{Y}\Big[1,3,0,\mathrm{PH}\Big[1+\left(\frac{x[1]}{2}+x[2]\right)z+\frac{1}{6}\left(x[1]^2+3x[1]\,x[2]+3x[2]^2\right)z^2+\\
&\qquad\frac{1}{24}\left(x[1]^3+4x[1]^2x[2]+6x[1]\,x[2]^2+4x[2]^3\right)z^3+O[z]^4\Big]\Big]+\\
&\quad\mathrm{Y}\Big[2,3,0,\mathrm{PH}\Big[1+\frac{1}{2}x[2]\,z+\frac{1}{6}x[2]^2z^2+\frac{1}{24}x[2]^3z^3+O[z]^4\Big]\Big]\Big],\mathrm{Ar}[1,2]+\mathrm{Ar}[1,3]+\mathrm{Ar}[2,3]\Big]
\end{aligned}
$$

```
r32 = ToPH[4, CanonicalForm[SnG[sigma[2, 3], sigma[1, 3], sigma[1, 2]]]];
Test[r31 == r32]
```

```
True
```

Commutators Commute

```
cc11 = ToPH[3, SnG[sigma[2, 1], sigma[3, 1], sigbar[2, 1], sigbar[3, 1]]]
```

$$\text{SnG}\Big[\text{S}\Big[\text{Ar}[0,1] \to \text{Ar}[0,1] + \text{Y}\Big[0,2,1,\text{PH}\Big[-x[3]\,z + \left(-\frac{1}{2}x[2]\,x[3] - \frac{x[3]^2}{2}\right) z^2 + O[z]^3\Big]\Big] + \text{Y}\Big[0,3,1,\text{PH}\Big[x[2]\,z + \frac{1}{2}\left(x[2]^2 + x[2]\,x[3]\right) z^2 + O[z]^3\Big]\Big],$$

$$\text{Ar}[0,2] \to \text{Ar}[0,2] + \text{Y}\Big[0,2,1,\text{PH}\Big[x[3]\,z + \frac{1}{2}\left(x[2]\,x[3] + x[3]^2\right) z^2 + O[z]^3\Big]\Big],$$

$$\text{Ar}[0,3] \to \text{Ar}[0,3] + \text{Y}\Big[0,3,1,\text{PH}\Big[-x[2]\,z + \left(-\frac{1}{2}x[2]^2 - \frac{1}{2}x[2]\,x[3]\right) z^2 + O[z]^3\Big]\Big],$$

$$\text{Ar}[1,0] \to \text{Ar}[1,0] + \text{Y}\Big[1,2,0,\text{PH}\Big[x[3]\,z + \frac{1}{2}\left(x[2]\,x[3] + x[3]^2\right) z^2 + O[z]^3\Big]\Big] + \text{Y}\Big[1,3,0,\text{PH}\Big[-x[2]\,z + \left(-\frac{1}{2}x[2]^2 - \frac{1}{2}x[2]\,x[3]\right) z^2 + O[z]^3\Big]\Big]\Big],$$

$$\text{Y}\Big[2,3,1,\text{PH}\Big[2 + \frac{3}{2}\left(x[2] + x[3]\right) z + \left(\frac{2\,x[2]^2}{3} + x[2]\,x[3] + \frac{2\,x[3]^2}{3}\right) z^2 + O[z]^3\Big]\Big]\Big]$$

```
cc12 = ToPH[3, SnG[sigma[4, 1], sigma[5, 1], sigbar[4, 1], sigbar[5, 1]]];
Test[(cc11 ** cc12) == (cc12 ** cc11)]
```

```
True
```

```
cc21 = ToPH[4, SnG[sigma[2, 1], sigma[3, 1], sigbar[2, 1], sigbar[3, 1]]]
```

$$\mathrm{SnG}\Big[\mathrm{S}\Big[\mathrm{Ar}[0,1]\to\mathrm{Ar}[0,1]+\mathrm{Y}\Big[0,2,1,\mathrm{PH}\Big[-x[3]\,z+\left(-\frac{1}{2}x[2]\,x[3]-\frac{x[3]^2}{2}\right)z^2+\frac{1}{12}\left(-2x[2]^2x[3]-3x[2]\,x[3]^2-2x[3]^3\right)z^3+O[z]^4\Big]\Big]+\mathrm{Y}\Big[0,3,1,\mathrm{PH}\Big[x[2]\,z+\frac{1}{2}\left(x[2]^2+x[2]\,x[3]\right)z^2+\frac{1}{12}\left(2x[2]^3+3x[2]^2x[3]+2x[2]\,x[3]^2\right)z^3+O[z]^4\Big]\Big],$$

$$\mathrm{Ar}[0,2]\to\mathrm{Ar}[0,2]+\mathrm{Y}\Big[0,2,1,\mathrm{PH}\Big[x[3]\,z+\frac{1}{2}\left(x[2]\,x[3]+x[3]^2\right)z^2+\frac{1}{12}\left(2x[2]^2x[3]+3x[2]\,x[3]^2+2x[3]^3\right)z^3+O[z]^4\Big]\Big],$$

$$\mathrm{Ar}[0,3]\to\mathrm{Ar}[0,3]+\mathrm{Y}\Big[0,3,1,\mathrm{PH}\Big[-x[2]\,z+\left(-\frac{1}{2}x[2]^2-\frac{1}{2}x[2]\,x[3]\right)z^2+\frac{1}{12}\left(-2x[2]^3-3x[2]^2x[3]-2x[2]\,x[3]^2\right)z^3+O[z]^4\Big]\Big],$$

$$\mathrm{Ar}[1,0]\to\mathrm{Ar}[1,0]+\mathrm{Y}\Big[1,2,0,\mathrm{PH}\Big[x[3]\,z+\frac{1}{2}\left(x[2]\,x[3]+x[3]^2\right)z^2+\frac{1}{12}\left(2x[2]^2x[3]+3x[2]\,x[3]^2+2x[3]^3\right)z^3+O[z]^4\Big]\Big]+\mathrm{Y}\Big[1,3,0,\mathrm{PH}\Big[-x[2]\,z+\left(-\frac{1}{2}x[2]^2-\frac{1}{2}x[2]\,x[3]\right)z^2+\frac{1}{12}\left(-2x[2]^3-3x[2]^2x[3]-2x[2]\,x[3]^2\right)z^3+O[z]^4\Big]\Big]\Big],$$

$$\mathrm{Y}\Big[2,3,1,\mathrm{PH}\Big[2+\frac{3}{2}(x[2]+x[3])\,z+\left(\frac{2x[2]^2}{3}+x[2]\,x[3]+\frac{2x[3]^2}{3}\right)z^2+\frac{5}{24}\left(x[2]^3+2x[2]^2x[3]+2x[2]\,x[3]^2+x[3]^3\right)z^3+O[z]^4\Big]\Big]\Big]$$

```
cc22 = ToPH[4, SnG[sigma[3, 1], sigma[4, 1], sigbar[3, 1], sigbar[4, 1]]];
Test[(cc21 ** cc22) == (cc22 ** cc21)]
```

```
True
```

```
cc31 = ToPH[4, SnG[sigma[1, 2], sigma[3, 1]] ** SnG[sigbar[1, 2], sigbar[3, 1]]]
```

```
SnG[S[Ar[0, 1] → Ar[0, 1] + Y[0, 1, 2,
      PH[-x[3] z + 1/2 (-x[1] x[3] + x[3]^2) z^2 + 1/12 (-2 x[1]^2 x[3] + 3 x[1] x[3]^2 - 2 x[3]^3) z^3 + O[z]^4]],
   Ar[0, 2] → Ar[0, 2] + Y[0, 1, 2, PH[x[3] z + 1/2 (x[1] x[3] - x[3]^2) z^2 +
        1/12 (2 x[1]^2 x[3] - 3 x[1] x[3]^2 + 2 x[3]^3) z^3 + O[z]^4]] + Y[0, 3, 2,
      PH[-x[1] z + 1/2 (-x[1]^2 + x[1] x[3]) z^2 + 1/12 (-2 x[1]^3 + 3 x[1]^2 x[3] - 2 x[1] x[3]^2) z^3 + O[z]^4]],
   Ar[0, 3] → Ar[0, 3] + Y[0, 3, 2, PH[x[1] z + 1/2 (x[1]^2 - x[1] x[3]) z^2 +
        1/12 (2 x[1]^3 - 3 x[1]^2 x[3] + 2 x[1] x[3]^2) z^3 + O[z]^4]],
   Ar[2, 0] → Ar[2, 0] + Y[1, 3, 0, PH[x[2] z + 1/2 (x[1] x[2] - x[2] x[3]) z^2 +
        1/12 (2 x[1]^2 x[2] - 3 x[1] x[2] x[3] + 2 x[2] x[3]^2) z^3 + O[z]^4]]],
  Y[1, 3, 2, PH[-2 - 3/2 (x[1] - x[3]) z + 1/3 (-2 x[1]^2 + 3 x[1] x[3] - 2 x[3]^2) z^2 -
      5/24 (x[1]^3 - 2 x[1]^2 x[3] + 2 x[1] x[3]^2 - x[3]^3) z^3 + O[z]^4]]]
```

```
cc32 = ToPH[4, SnG[sigma[1, 4], sigma[5, 1], sigbar[1, 4], sigbar[5, 1]]];
Test[(cc31 ** cc32) == (cc32 ** cc31)]
```

```
True
```

■ Commutators Commutators are Central (along strand 1)

```
(ccc = SnG[
    sigma[1, 2], sigma[3, 1], sigbar[1, 2], sigbar[3, 1],
    sigma[4, 1], sigma[5, 1], sigbar[4, 1], sigbar[5, 1],
    sigma[3, 1], sigma[1, 2], sigbar[3, 1], sigbar[1, 2],
    sigma[5, 1], sigma[4, 1], sigbar[5, 1], sigbar[4, 1]
  ]) // Last
```

$$
\begin{aligned}
&\mathrm{Y}\Big[1,4,2,\\
&\quad \mathrm{AH}\Big[\frac{1}{x[1]\,x[4]}\,e^{-x[3]}\,\big(-e^{x[1]}x[1]+e^{x[1]+x[3]}x[1]+e^{x[1]+x[4]}x[1]-e^{x[1]+x[3]+x[4]}x[1]+e^{x[1]+x[5]}x[1]-\\
&\qquad e^{x[1]+x[3]+x[5]}x[1]-e^{x[1]+x[4]+x[5]}x[1]+e^{x[1]+x[3]+x[4]+x[5]}x[1]-x[3]+e^{x[1]}x[3]+\\
&\qquad e^{x[4]}x[3]-e^{x[1]+x[4]}x[3]+e^{x[5]}x[3]-e^{x[1]+x[5]}x[3]-e^{x[4]+x[5]}x[3]+\\
&\qquad e^{x[1]+x[4]+x[5]}x[3]-e^{x[4]}x[4]+e^{x[1]+x[4]}x[4]+e^{x[3]+x[4]}x[4]-e^{x[1]+x[3]+x[4]}x[4]+\\
&\qquad e^{x[4]+x[5]}x[4]-e^{x[1]+x[4]+x[5]}x[4]-e^{x[3]+x[4]+x[5]}x[4]+e^{x[1]+x[3]+x[4]+x[5]}x[4]-\\
&\qquad e^{x[5]}x[5]+e^{x[1]+x[5]}x[5]+e^{x[3]+x[5]}x[5]-e^{x[1]+x[3]+x[5]}x[5]+e^{x[4]+x[5]}x[5]-\\
&\qquad e^{x[1]+x[4]+x[5]}x[5]-e^{x[3]+x[4]+x[5]}x[5]+e^{x[1]+x[3]+x[4]+x[5]}x[5]\big)\Big]\Big]+\\
&\mathrm{Y}\Big[1,5,2,\mathrm{AH}\Big[-\frac{1}{x[1]\,x[5]}\,e^{-x[3]}\,\big(-e^{x[1]}x[1]+e^{x[1]+x[3]}x[1]+e^{x[1]+x[4]}x[1]-e^{x[1]+x[3]+x[4]}x[1]+\\
&\qquad e^{x[1]+x[5]}x[1]-e^{x[1]+x[3]+x[5]}x[1]-e^{x[1]+x[4]+x[5]}x[1]+e^{x[1]+x[3]+x[4]+x[5]}x[1]-\\
&\qquad x[3]+e^{x[1]}x[3]+e^{x[4]}x[3]-e^{x[1]+x[4]}x[3]+e^{x[5]}x[3]-e^{x[1]+x[5]}x[3]-\\
&\qquad e^{x[4]+x[5]}x[3]+e^{x[1]+x[4]+x[5]}x[3]-e^{x[4]}x[4]+e^{x[1]+x[4]}x[4]+e^{x[3]+x[4]}x[4]-\\
&\qquad e^{x[1]+x[3]+x[4]}x[4]+e^{x[4]+x[5]}x[4]-e^{x[1]+x[4]+x[5]}x[4]-e^{x[3]+x[4]+x[5]}x[4]+\\
&\qquad e^{x[1]+x[3]+x[4]+x[5]}x[4]-e^{x[5]}x[5]+e^{x[1]+x[5]}x[5]+e^{x[3]+x[5]}x[5]-e^{x[1]+x[3]+x[5]}x[5]+\\
&\qquad e^{x[4]+x[5]}x[5]-e^{x[1]+x[4]+x[5]}x[5]-e^{x[3]+x[4]+x[5]}x[5]+e^{x[1]+x[3]+x[4]+x[5]}x[5]\big)\Big]\Big]
\end{aligned}
$$

```
Test[ccc ** SnG[sigma[6, 1]] == SnG[sigma[6, 1]] ** ccc]
```

True

```
Test[ccc ** SnG[sigma[1, 6]] == SnG[sigma[1, 6]] ** ccc]
```

True

■ Tails Commute and 4T

```
Der[Ar[1, 2] + Ar[1, 3]][Ar[2, 4]]
```

Y[1, 2, 4, AH[-1]]

```
Expect[0, Der[Ar[1, 3]][Ar[1, 2]]]
```

0

```
Expect[0,
 Der[Ar[1, 2] + Ar[1, 3]][Ar[2, 3]]
]
```

0

```
Der[Ar[1, 2]][Ar[1, 3] + Ar[2, 3]]
```

```
Y[1, 2, 3, AH[-1]]
```

```
Expect[0,
 Der[Ar[1, 2]][Ar[3, 1] + Ar[3, 2]]
]
```

```
0
```

```
Expect[{0, 0},
 Der[Ar[1, 1]]@{Ar[1, 2], Ar[2, 1]}
]
```

```
{0, 0}
```

```
Expect[{0, 0},
 Der[Ar[1, 2]]@{Ar[1, 1], Ar[2, 2]}
]
```

```
{0, 0}
```

■ Antisymmetry of Der

```
Expect[{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 ToAH[Table[
   ReducePrimitives[
    Der[ToPH[3, Y[1, 2, 3, AH[1]]]]@Ar[i, j] + Der[Ar[i, j]]@Y[1, 2, 3, ToPH[3, PH[1]]]
   ], {i, 4}, {j, 4}
  ]]
]
```

```
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

■ Scattering by Exponentials

```
ToPH[3, S[Exp[Ar[1, 2]]]]
```

$$\mathrm{S}\Big[\mathrm{Ar}[0,1] \to \mathrm{Ar}[0,1] + \mathrm{Y}\Big[0, 1, 2, \mathrm{PH}\Big[-1 + \frac{1}{2}\,\mathrm{x}[1]\,\mathrm{z} - \frac{1}{6}\,\mathrm{x}[1]^2\,\mathrm{z}^2 + \mathrm{O}[\mathrm{z}]^3\Big]\Big],$$
$$\mathrm{Ar}[0,2] \to \mathrm{Ar}[0,2] + \mathrm{Y}\Big[0, 1, 2, \mathrm{PH}\Big[1 - \frac{1}{2}\,\mathrm{x}[1]\,\mathrm{z} + \frac{1}{6}\,\mathrm{x}[1]^2\,\mathrm{z}^2 + \mathrm{O}[\mathrm{z}]^3\Big]\Big],$$
$$\mathrm{Ar}[2,0] \to \mathrm{Ar}[2,0] + \mathrm{Y}\Big[1, 2, 0, \mathrm{PH}\Big[1 + \frac{1}{2}\,\mathrm{x}[1]\,\mathrm{z} + \frac{1}{6}\,\mathrm{x}[1]^2\,\mathrm{z}^2 + \mathrm{O}[\mathrm{z}]^3\Big]\Big]\Big]$$

```
Test[
 CanonicalForm[ToPH[9, S[Exp[Ar[1, 2]]]]] == CanonicalForm[ToPH[9, S[sigma[1, 2]]]]
]
```

```
True
```

The BCH Formula

```
ToPH[5, S[sigma[1, 3], sigma[2, 3]]] // Short
```

$$\mathrm{S}\Big[\mathrm{Ar}[0,1] \to \mathrm{Ar}[0,1] + \mathrm{Y}\big[0, 1, 3, \mathrm{PH}\big[-1 + \ll 4 \gg + \mathrm{O}[z]^5\big]\big], \ll 1 \gg, \ll 1 \gg, \mathrm{Ar}[3,0] \to \ll 1 \gg\Big]$$

```
unknowns = DeclareSeries[bc[x[1], x[2]], 4]
```

$$\{\mathrm{bc}[0,0], \mathrm{bc}[1,0], \mathrm{bc}[0,1], \mathrm{bc}[2,0], \mathrm{bc}[1,1], \mathrm{bc}[0,2], \mathrm{bc}[3,0], \mathrm{bc}[2,1], \mathrm{bc}[1,2], \mathrm{bc}[0,3]\}$$

```
PH[bc]
```

$$\mathrm{PH}\Big[\mathrm{bc}[0,0] + (\mathrm{bc}[1,0]\, x[1] + \mathrm{bc}[0,1]\, x[2])\, z + \left(\frac{1}{2}\mathrm{bc}[2,0]\, x[1]^2 + \mathrm{bc}[1,1]\, x[1]\, x[2] + \frac{1}{2}\mathrm{bc}[0,2]\, x[2]^2\right) z^2 + \left(\frac{1}{6}\mathrm{bc}[3,0]\, x[1]^3 + \frac{1}{2}\mathrm{bc}[2,1]\, x[1]^2\, x[2] + \frac{1}{2}\mathrm{bc}[1,2]\, x[1]\, x[2]^2 + \frac{1}{6}\mathrm{bc}[0,3]\, x[2]^3\right) z^3 + \mathrm{O}[z]^4\Big]$$

```
S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, PH[bc]]]] // Short[#, 5] &
```

$$\mathrm{S}\Big[\mathrm{Ar}[0,1] \to \mathrm{Ar}[0,1] + \mathrm{Y}\Big[0, \ll 2 \gg, \mathrm{PH}\Big[-1 + \left(\frac{x[1]}{2} + \frac{x[2]}{2} + \mathrm{bc}[0,0]\, x[2]\right) z + \left(\ll 10 \gg + \mathrm{bc}[0,1]\, x[2]^2\right) z^2 + \left(\frac{\ll 1 \gg^3}{24} + \ll 16 \gg + \frac{1}{2} \ll 1 \gg \ll 1 \gg^3\right) \ll 1 \gg \ll 1 \gg + \left(-\frac{1}{120} x[1]^4 - \frac{1}{30} x[1]^3\, x[2] - \frac{1}{24}\mathrm{bc}[0,0]\, x[1]^3\, x[2] + \ll 30 \gg + \frac{1}{6}\mathrm{bc}[0,1]\, x[2]^4 - \frac{1}{4}\mathrm{bc}[0,2]\, x[2]^4 + \frac{1}{6}\mathrm{bc}[0,3]\, x[2]^4\right) z^4 + \mathrm{O}[z]^5\Big]\Big], \ll 3 \gg\Big]$$

```
eq = Coefficient[
   Ar[0, 1] // S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, PH[bc]]]],
   Y[0, 1, 3]
  ] == Coefficient[
   Ar[0, 1] // S[sigma[1, 3], sigma[2, 3]],
   Y[0, 1, 3]
  ]
```

$$
\begin{aligned}
&\mathrm{PH}\Big[-1+\left(\frac{x[1]}{2}+\frac{x[2]}{2}+\mathrm{bc}[0,0]\,x[2]\right)z+\left(-\frac{1}{6}x[1]^2-\frac{1}{3}x[1]\,x[2]-\frac{1}{2}\mathrm{bc}[0,0]\,x[1]\,x[2]+\right.\\
&\quad\left.\mathrm{bc}[1,0]\,x[1]\,x[2]-\frac{x[2]^2}{6}-\frac{1}{2}\mathrm{bc}[0,0]\,x[2]^2+\mathrm{bc}[0,1]\,x[2]^2\right)z^2+\\
&\left(\frac{x[1]^3}{24}+\frac{1}{8}x[1]^2x[2]+\frac{1}{6}\mathrm{bc}[0,0]\,x[1]^2x[2]-\frac{1}{2}\mathrm{bc}[1,0]\,x[1]^2x[2]+\frac{1}{2}\mathrm{bc}[2,0]\,x[1]^2x[2]+\right.\\
&\quad\frac{1}{8}x[1]\,x[2]^2+\frac{1}{3}\mathrm{bc}[0,0]\,x[1]\,x[2]^2-\frac{1}{2}\mathrm{bc}[0,1]\,x[1]\,x[2]^2-\frac{1}{2}\mathrm{bc}[1,0]\,x[1]\,x[2]^2+\\
&\quad\left.\mathrm{bc}[1,1]\,x[1]\,x[2]^2+\frac{x[2]^3}{24}+\frac{1}{6}\mathrm{bc}[0,0]\,x[2]^3-\frac{1}{2}\mathrm{bc}[0,1]\,x[2]^3+\frac{1}{2}\mathrm{bc}[0,2]\,x[2]^3\right)z^3+\\
&\left(-\frac{1}{120}x[1]^4-\frac{1}{30}x[1]^3x[2]-\frac{1}{24}\mathrm{bc}[0,0]\,x[1]^3x[2]+\frac{1}{6}\mathrm{bc}[1,0]\,x[1]^3x[2]-\right.\\
&\quad\frac{1}{4}\mathrm{bc}[2,0]\,x[1]^3x[2]+\frac{1}{6}\mathrm{bc}[3,0]\,x[1]^3x[2]-\frac{1}{20}x[1]^2x[2]^2-\frac{1}{8}\mathrm{bc}[0,0]\,x[1]^2x[2]^2+\\
&\quad\frac{1}{6}\mathrm{bc}[0,1]\,x[1]^2x[2]^2+\frac{1}{3}\mathrm{bc}[1,0]\,x[1]^2x[2]^2-\frac{1}{2}\mathrm{bc}[1,1]\,x[1]^2x[2]^2-\\
&\quad\frac{1}{4}\mathrm{bc}[2,0]\,x[1]^2x[2]^2+\frac{1}{2}\mathrm{bc}[2,1]\,x[1]^2x[2]^2-\frac{1}{30}x[1]\,x[2]^3-\frac{1}{8}\mathrm{bc}[0,0]\,x[1]\,x[2]^3+\\
&\quad\frac{1}{3}\mathrm{bc}[0,1]\,x[1]\,x[2]^3-\frac{1}{4}\mathrm{bc}[0,2]\,x[1]\,x[2]^3+\frac{1}{6}\mathrm{bc}[1,0]\,x[1]\,x[2]^3-\\
&\quad\frac{1}{2}\mathrm{bc}[1,1]\,x[1]\,x[2]^3+\frac{1}{2}\mathrm{bc}[1,2]\,x[1]\,x[2]^3-\frac{x[2]^4}{120}-\frac{1}{24}\mathrm{bc}[0,0]\,x[2]^4+\frac{1}{6}\mathrm{bc}[0,1]\,x[2]^4-\\
&\quad\left.\frac{1}{4}\mathrm{bc}[0,2]\,x[2]^4+\frac{1}{6}\mathrm{bc}[0,3]\,x[2]^4\right)z^4+O[z]^5\Big]==\mathrm{AH}\Big[-\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big]
\end{aligned}
$$

```
sol = First[HSolve[eq, unknowns]]
```

$$
\begin{aligned}
&\Big\{\mathrm{bc}[0,0]\to\frac{1}{2},\ \mathrm{bc}[1,0]\to\frac{1}{12},\ \mathrm{bc}[0,1]\to-\frac{1}{12},\ \mathrm{bc}[2,0]\to0,\ \mathrm{bc}[1,1]\to-\frac{1}{24},\\
&\ \mathrm{bc}[0,2]\to0,\ \mathrm{bc}[3,0]\to-\frac{1}{120},\ \mathrm{bc}[2,1]\to-\frac{1}{90},\ \mathrm{bc}[1,2]\to\frac{1}{90},\ \mathrm{bc}[0,3]\to\frac{1}{120}\Big\}
\end{aligned}
$$

```
bch = PH[bc] /. sol
```

$$
\begin{aligned}
&\mathrm{PH}\Big[\frac{1}{2}+\left(\frac{x[1]}{12}-\frac{x[2]}{12}\right)z-\frac{1}{24}\left(x[1]\,x[2]\right)z^2+\\
&\quad\left(-\frac{1}{720}x[1]^3-\frac{1}{180}x[1]^2x[2]+\frac{1}{180}x[1]\,x[2]^2+\frac{x[2]^3}{720}\right)z^3+O[z]^4\Big]
\end{aligned}
$$

```
Test[CanonicalForm[
  S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bch]]] == ToPH[5, S[sigma[1, 3], sigma[2, 3]]]
 ]]
```

```
True
```

- **Compare with Kurlin**

```
Test[Simplify[(bch /. {x[1] → x, x[2] → y}) ==
   PH[(((1/y) (1 - (e^x - 1)/x (x + y)/(e^(x+y) - 1))) /. {x → z x, y → z y}) + O[z]^4]]]
```

$$\texttt{Test}\left[\texttt{Simplify}\left[(\texttt{bch} \mathbin{/.} \{x[1]\to x,\ x[2]\to y\}) == \texttt{PH}\left[\left(\left(\frac{1}{y}\left(1-\frac{e^x-1}{x}\,\frac{x+y}{e^{x+y}-1}\right)\right) \mathbin{/.} \{x\to z\,x,\ y\to z\,y\}\right)+O[z]^4\right]\right]\right]$$

```
True
```

Testing Code

```
SetAttributes[{Test, Expect}, {HoldAll}];
Test[expr_] := If[TrueQ[Check[expr, False]], True,
   If[Head[$FailLog] =!= List, $FailLog = {}];
   AppendTo[$FailLog,
    "On " <> ToString[Date[]] <> " failed in " <> ToString[HoldForm[expr]]];
   Print[Last[$FailLog]]
  ];
Expect[val_, expr_] := If[TrueQ[Test[val == expr]], val];

SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\ScatterAndGlow"]
<< ScatterAndGlow.m
```

```
C:\drorbn\AcademicPensieve\Projects\ScatterAndGlow
```

Test Test

```
Test[0 == 1]
```

```
On {2009, 1, 14, 11, 52, 12.9972000} failed in 0 == 1
```

Failed Tests

```
$FailLog
```

```
{On {2009, 1, 14, 11, 52, 6.3232000} failed in 0 == 1,
 On {2009, 1, 14, 11, 52, 12.9972000} failed in 0 == 1}
```