

Scatter and Glow - Perturbative Testing

By Dror Bar-Natan with Kerene Chu, Zsuzsanna Dancso, Peter Lee and Louis Leung.

December 2008.

■ Declare Perturbative Hairstyle

```
$CutoffDegree = 5;  
H[h_] := PH[h /. x[i_] -> z * x[i]] + O[z]^$CutoffDegree;
```

■ The braid group on two strands is commutative:

```
Expect[Ar[1, 2],  
  Ar[1, 2] // S[sigma[1, 2]]  
]  
Ar[1, 2]
```

■ Reidemeister 2

```
Expect[SnG[S[], 0],  
  SnG[sigma[1, 2], sigbar[1, 2]]  
]  
SnG[S[], 0]
```

■ Locality in Scale (global over local)

```
S[sigma[3, 1], sigma[3, 2]]
```

$$\begin{aligned} & S\left[\text{Ar}[0, 1] \rightarrow \text{Ar}[0, 1] + Y\left[0, 3, 1, \text{PH}\left[1 - \frac{1}{2}x[3]z + \frac{1}{6}x[3]^2z^2 - \frac{1}{24}x[3]^3z^3 + \frac{1}{120}x[3]^4z^4 + O[z]^5\right]\right], \right. \\ & \text{Ar}[0, 2] \rightarrow \text{Ar}[0, 2] + Y\left[0, 3, 2, \text{PH}\left[1 - \frac{1}{2}x[3]z + \frac{1}{6}x[3]^2z^2 - \frac{1}{24}x[3]^3z^3 + \frac{1}{120}x[3]^4z^4 + O[z]^5\right]\right], \\ & \text{Ar}[0, 3] \rightarrow \text{Ar}[0, 3] + Y\left[0, 3, 1, \text{PH}\left[-1 + \frac{1}{2}x[3]z - \frac{1}{6}x[3]^2z^2 + \frac{1}{24}x[3]^3z^3 - \frac{1}{120}x[3]^4z^4 + O[z]^5\right]\right] + \\ & \quad Y\left[0, 3, 2, \text{PH}\left[-1 + \frac{1}{2}x[3]z - \frac{1}{6}x[3]^2z^2 + \frac{1}{24}x[3]^3z^3 - \frac{1}{120}x[3]^4z^4 + O[z]^5\right]\right], \\ & \text{Ar}[1, 0] \rightarrow \text{Ar}[1, 0] + Y\left[1, 3, 0, \text{PH}\left[-1 - \frac{1}{2}x[3]z - \frac{1}{6}x[3]^2z^2 - \frac{1}{24}x[3]^3z^3 - \frac{1}{120}x[3]^4z^4 + O[z]^5\right]\right], \\ & \left. \text{Ar}[2, 0] \rightarrow \text{Ar}[2, 0] + Y\left[2, 3, 0, \text{PH}\left[-1 - \frac{1}{2}x[3]z - \frac{1}{6}x[3]^2z^2 - \frac{1}{24}x[3]^3z^3 - \frac{1}{120}x[3]^4z^4 + O[z]^5\right]\right]\right] \end{aligned}$$

```
Expect[{Ar[1, 2], Ar[2, 1]},  
  {Ar[1, 2], Ar[2, 1]} // S[sigma[3, 1], sigma[3, 2]]  
]  
{Ar[1, 2], Ar[2, 1]}
```

Overcrossings Commute

```
oc1 = SnG[sigma[1, 2], sigma[1, 3]]
```

```
SnG[S[Ar[0, 1] →
```

$$\text{Ar}[0, 1] + \text{Y}\left[0, 1, 2, \text{PH}\left[-1 + \frac{1}{2} x[1] z - \frac{1}{6} x[1]^2 z^2 + \frac{1}{24} x[1]^3 z^3 - \frac{1}{120} x[1]^4 z^4 + \text{O}[z]^5\right]\right] +$$

$$\text{Y}\left[0, 1, 3, \text{PH}\left[-1 + \frac{1}{2} x[1] z - \frac{1}{6} x[1]^2 z^2 + \frac{1}{24} x[1]^3 z^3 - \frac{1}{120} x[1]^4 z^4 + \text{O}[z]^5\right]\right],$$

$$\text{Ar}[0, 2] \rightarrow \text{Ar}[0, 2] + \text{Y}\left[0, 1, 2, \text{PH}\left[1 - \frac{1}{2} x[1] z + \frac{1}{6} x[1]^2 z^2 - \frac{1}{24} x[1]^3 z^3 + \frac{1}{120} x[1]^4 z^4 + \text{O}[z]^5\right]\right],$$

$$\text{Ar}[0, 3] \rightarrow \text{Ar}[0, 3] + \text{Y}\left[0, 1, 3, \text{PH}\left[1 - \frac{1}{2} x[1] z + \frac{1}{6} x[1]^2 z^2 - \frac{1}{24} x[1]^3 z^3 + \frac{1}{120} x[1]^4 z^4 + \text{O}[z]^5\right]\right],$$

$$\text{Ar}[2, 0] \rightarrow \text{Ar}[2, 0] + \text{Y}\left[1, 2, 0, \text{PH}\left[1 + \frac{1}{2} x[1] z + \frac{1}{6} x[1]^2 z^2 + \frac{1}{24} x[1]^3 z^3 + \frac{1}{120} x[1]^4 z^4 + \text{O}[z]^5\right]\right],$$

$$\text{Ar}[3, 0] \rightarrow \text{Ar}[3, 0] + \text{Y}\left[1, 3, 0, \text{PH}\left[1 + \frac{1}{2} x[1] z + \frac{1}{6} x[1]^2 z^2 + \frac{1}{24} x[1]^3 z^3 + \frac{1}{120} x[1]^4 z^4 + \text{O}[z]^5\right]\right],$$

```
Ar[1, 2] + Ar[1, 3]]
```

```
oc2 = SnG[sigma[1, 3], sigma[1, 2]];

```

```
Test[oc1 == oc2]
```

```
True
```

Reidemeister 3

r31 = CanonicalForm[SnG[sigma[1, 2], sigma[1, 3], sigma[2, 3]]]

SnG[S[Ar[0, 1] →

$$\text{Ar}[0, 1] + \text{Y}\left[0, 1, 2, \text{PH}\left[-1 + \frac{1}{2} x[1] z - \frac{1}{6} x[1]^2 z^2 + \frac{1}{24} x[1]^3 z^3 - \frac{1}{120} x[1]^4 z^4 + \text{O}[z]^5\right]\right] +$$

$$\text{Y}\left[0, 1, 3, \text{PH}\left[-1 + \frac{1}{2} x[1] z - \frac{1}{6} x[1]^2 z^2 + \frac{1}{24} x[1]^3 z^3 - \frac{1}{120} x[1]^4 z^4 + \text{O}[z]^5\right]\right],$$

$$\text{Ar}[0, 2] \rightarrow \text{Ar}[0, 2] + \text{Y}\left[0, 1, 2, \text{PH}\left[1 - \frac{1}{2} x[1] z + \frac{1}{6} x[1]^2 z^2 - \frac{1}{24} x[1]^3 z^3 + \frac{1}{120} x[1]^4 z^4 + \text{O}[z]^5\right]\right] +$$

$$\text{Y}\left[0, 1, 3, \text{PH}\left[x[2] z + \left(-\frac{1}{2} x[1] x[2] - \frac{x[2]^2}{2}\right) z^2 + \left(\frac{1}{6} x[1]^2 x[2] + \frac{1}{4} x[1] x[2]^2 + \frac{x[2]^3}{6}\right) z^3 +$$

$$\left(-\frac{1}{24} x[1]^3 x[2] - \frac{1}{12} x[1]^2 x[2]^2 - \frac{1}{12} x[1] x[2]^3 - \frac{x[2]^4}{24}\right) z^4 + \text{O}[z]^5\right] +$$

$$\text{Y}\left[0, 2, 3, \text{PH}\left[-1 + \frac{1}{2} x[2] z - \frac{1}{6} x[2]^2 z^2 + \frac{1}{24} x[2]^3 z^3 - \frac{1}{120} x[2]^4 z^4 + \text{O}[z]^5\right]\right],$$

$$\text{Ar}[0, 3] \rightarrow \text{Ar}[0, 3] + \text{Y}\left[0, 1, 3, \text{PH}\left[1 + \left(-\frac{x[1]}{2} - x[2]\right) z + \left(\frac{x[1]^2}{6} + \frac{1}{2} x[1] x[2] + \frac{x[2]^2}{2}\right) z^2 +$$

$$\left(-\frac{1}{24} x[1]^3 - \frac{1}{6} x[1]^2 x[2] - \frac{1}{4} x[1] x[2]^2 - \frac{x[2]^3}{6}\right) z^3 +$$

$$\left(\frac{x[1]^4}{120} + \frac{1}{24} x[1]^3 x[2] + \frac{1}{12} x[1]^2 x[2]^2 + \frac{1}{12} x[1] x[2]^3 + \frac{x[2]^4}{24}\right) z^4 + \text{O}[z]^5\right] +$$

$$\text{Y}\left[0, 2, 3, \text{PH}\left[1 - \frac{1}{2} x[2] z + \frac{1}{6} x[2]^2 z^2 - \frac{1}{24} x[2]^3 z^3 + \frac{1}{120} x[2]^4 z^4 + \text{O}[z]^5\right]\right],$$

$$\text{Ar}[2, 0] \rightarrow \text{Ar}[2, 0] + \text{Y}\left[1, 2, 0, \text{PH}\left[1 + \frac{1}{2} x[1] z + \frac{1}{6} x[1]^2 z^2 + \frac{1}{24} x[1]^3 z^3 + \frac{1}{120} x[1]^4 z^4 + \text{O}[z]^5\right]\right],$$

$$\text{Ar}[3, 0] \rightarrow \text{Ar}[3, 0] + \text{Y}\left[1, 2, 0, \text{PH}\left[-x[3] z + \left(-\frac{1}{2} x[1] x[3] - \frac{1}{2} x[2] x[3]\right) z^2 +$$

$$\left(-\frac{1}{6} x[1]^2 x[3] - \frac{1}{4} x[1] x[2] x[3] - \frac{1}{6} x[2]^2 x[3]\right) z^3 +$$

$$\left(-\frac{1}{24} x[1]^3 x[3] - \frac{1}{12} x[1]^2 x[2] x[3] - \frac{1}{12} x[1] x[2]^2 x[3] - \frac{1}{24} x[2]^3 x[3]\right) z^4 + \text{O}[z]^5\right] +$$

$$\text{Y}\left[1, 3, 0, \text{PH}\left[1 + \left(\frac{x[1]}{2} + x[2]\right) z + \left(\frac{x[1]^2}{6} + \frac{1}{2} x[1] x[2] + \frac{x[2]^2}{2}\right) z^2 +$$

$$\left(\frac{x[1]^3}{24} + \frac{1}{6} x[1]^2 x[2] + \frac{1}{4} x[1] x[2]^2 + \frac{x[2]^3}{6}\right) z^3 +$$

$$\left(\frac{x[1]^4}{120} + \frac{1}{24} x[1]^3 x[2] + \frac{1}{12} x[1]^2 x[2]^2 + \frac{1}{12} x[1] x[2]^3 + \frac{x[2]^4}{24}\right) z^4 + \text{O}[z]^5\right] +$$

$$\text{Y}\left[2, 3, 0, \text{PH}\left[1 + \frac{1}{2} x[2] z + \frac{1}{6} x[2]^2 z^2 + \frac{1}{24} x[2]^3 z^3 + \frac{1}{120} x[2]^4 z^4 + \text{O}[z]^5\right]\right],$$

Ar[1, 2] + Ar[1, 3] + Ar[2, 3]

```

r32 = CanonicalForm[SnG[sigma[2, 3], sigma[1, 3], sigma[1, 2]]];
Test[r31 == r32]

True

```

■ Commutators Commute

```
cc11 = SnG[sigma[2, 1], sigma[3, 1], sigbar[2, 1], sigbar[3, 1]]
```

```

SnG[S[Ar[0, 1] → Ar[0, 1] +
Y[0, 2, 1, PH[-x[3] z + (-1/2 x[2] x[3] - x[3]^2/2) z^2 + (-1/6 x[2]^2 x[3] - 1/4 x[2] x[3]^2 - x[3]^3/6) z^3 +
(-1/24 x[2]^3 x[3] - 1/12 x[2]^2 x[3]^2 - 1/12 x[2] x[3]^3 - x[3]^4/24) z^4 + O[z]^5]] +
Y[0, 3, 1, PH[x[2] z + (x[2]^2/2 + 1/2 x[2] x[3]) z^2 + (x[2]^3/6 + 1/4 x[2]^2 x[3] + 1/6 x[2] x[3]^2) z^3 +
(x[2]^4/24 + 1/12 x[2]^3 x[3] + 1/12 x[2]^2 x[3]^2 + 1/24 x[2] x[3]^3) z^4 + O[z]^5]], Ar[0, 2] → Ar[0, 2] +
Y[0, 2, 1, PH[x[3] z + (1/2 x[2] x[3] + x[3]^2/2) z^2 + (1/6 x[2]^2 x[3] + 1/4 x[2] x[3]^2 + x[3]^3/6) z^3 +
(1/24 x[2]^3 x[3] + 1/12 x[2]^2 x[3]^2 + 1/12 x[2] x[3]^3 + x[3]^4/24) z^4 + O[z]^5]],
Ar[0, 3] → Ar[0, 3] + Y[0, 3, 1, PH[-x[2] z + (-1/2 x[2]^2 - 1/2 x[2] x[3]) z^2 +
(-1/6 x[2]^3 - 1/4 x[2]^2 x[3] - 1/6 x[2] x[3]^2) z^3 +
(-1/24 x[2]^4 - 1/12 x[2]^3 x[3] - 1/12 x[2]^2 x[3]^2 - 1/24 x[2] x[3]^3) z^4 + O[z]^5]], Ar[1, 0] →
Ar[1, 0] + Y[1, 2, 0, PH[x[3] z + (1/2 x[2] x[3] + x[3]^2/2) z^2 + (1/6 x[2]^2 x[3] + 1/4 x[2] x[3]^2 + x[3]^3/6)
z^3 + (1/24 x[2]^3 x[3] + 1/12 x[2]^2 x[3]^2 + 1/12 x[2] x[3]^3 + x[3]^4/24) z^4 + O[z]^5]] + Y[1, 3, 0,
PH[-x[2] z + (-1/2 x[2]^2 - 1/2 x[2] x[3]) z^2 + (-1/6 x[2]^3 - 1/4 x[2]^2 x[3] - 1/6 x[2] x[3]^2) z^3 +
(-1/24 x[2]^4 - 1/12 x[2]^3 x[3] - 1/12 x[2]^2 x[3]^2 - 1/24 x[2] x[3]^3) z^4 + O[z]^5]],
Y[2, 3, 1, PH[2 + (3 x[2]/2 + 3 x[3]/2) z + (2 x[2]^2/3 + x[2] x[3] + 2 x[3]^2/3) z^2 +
(5 x[2]^3/24 + 5/12 x[2]^2 x[3] + 5/12 x[2] x[3]^2 + 5 x[3]^3/24) z^3 +
(x[2]^4/20 + 1/8 x[2]^3 x[3] + 1/6 x[2]^2 x[3]^2 + 1/8 x[2] x[3]^3 + x[3]^4/20) z^4 + O[z]^5]]]]]

```

```
cc12 = SnG[sigma[4, 1], sigma[5, 1], sigbar[4, 1], sigbar[5, 1]];
Test[(cc11 ** cc12) == (cc12 ** cc11)]
```

True

```
cc21 = SnG[sigma[2, 1], sigma[3, 1], sigbar[2, 1], sigbar[3, 1]]
```

```
SnG[S[Ar[0, 1] → Ar[0, 1] +
Y[0, 2, 1, PH[-x[3] z + (-1/2 x[2] x[3] - x[3]^2/2) z^2 + (-1/6 x[2]^2 x[3] - 1/4 x[2] x[3]^2 - x[3]^3/6) z^3 +
(-1/24 x[2]^3 x[3] - 1/12 x[2]^2 x[3]^2 - 1/12 x[2] x[3]^3 - x[3]^4/24) z^4 + O[z]^5]] +
Y[0, 3, 1, PH[x[2] z + (x[2]^2/2 + 1/2 x[2] x[3]) z^2 + (x[2]^3/6 + 1/4 x[2]^2 x[3] + 1/6 x[2] x[3]^2) z^3 +
(x[2]^4/24 + 1/12 x[2]^3 x[3] + 1/12 x[2]^2 x[3]^2 + 1/24 x[2] x[3]^3) z^4 + O[z]^5]], Ar[0, 2] → Ar[0, 2] +
Y[0, 2, 1, PH[x[3] z + (1/2 x[2] x[3] + x[3]^2/2) z^2 + (1/6 x[2]^2 x[3] + 1/4 x[2] x[3]^2 + x[3]^3/6) z^3 +
(1/24 x[2]^3 x[3] + 1/12 x[2]^2 x[3]^2 + 1/12 x[2] x[3]^3 + x[3]^4/24) z^4 + O[z]^5]],
Ar[0, 3] → Ar[0, 3] + Y[0, 3, 1, PH[-x[2] z + (-1/2 x[2]^2 - 1/2 x[2] x[3]) z^2 +
(-1/6 x[2]^3 - 1/4 x[2]^2 x[3] - 1/6 x[2] x[3]^2) z^3 +
(-1/24 x[2]^4 - 1/12 x[2]^3 x[3] - 1/12 x[2]^2 x[3]^2 - 1/24 x[2] x[3]^3) z^4 + O[z]^5]], Ar[1, 0] →
Ar[1, 0] + Y[1, 2, 0, PH[x[3] z + (1/2 x[2] x[3] + x[3]^2/2) z^2 + (1/6 x[2]^2 x[3] + 1/4 x[2] x[3]^2 + x[3]^3/6)
z^3 + (1/24 x[2]^3 x[3] + 1/12 x[2]^2 x[3]^2 + 1/12 x[2] x[3]^3 + x[3]^4/24) z^4 + O[z]^5]] + Y[1, 3, 0,
PH[-x[2] z + (-1/2 x[2]^2 - 1/2 x[2] x[3]) z^2 + (-1/6 x[2]^3 - 1/4 x[2]^2 x[3] - 1/6 x[2] x[3]^2) z^3 +
(-1/24 x[2]^4 - 1/12 x[2]^3 x[3] - 1/12 x[2]^2 x[3]^2 - 1/24 x[2] x[3]^3) z^4 + O[z]^5]]],
Y[2, 3, 1, PH[2 + (3 x[2] + 3 x[3]) / 2 z + (2 x[2]^2 + x[2] x[3] + 2 x[3]^2) / 3 z^2 +
(5 x[2]^3 + 5 x[2]^2 x[3] + 5 x[2] x[3]^2 + 5 x[3]^3) / 24 z^3 +
(x[2]^4 + 1 x[2]^3 x[3] + 1 x[2]^2 x[3]^2 + 1 x[2] x[3]^3 + x[3]^4) / 20 z^4 + O[z]^5]]]]]
```

```
cc22 = SnG[sigma[3, 1], sigma[4, 1], sigbar[3, 1], sigbar[4, 1]];
Test[(cc21 ** cc22) == (cc22 ** cc21)]
```

True

```
cc31 = SnG[sigma[1, 2], sigma[3, 1], sigbar[1, 2], sigbar[3, 1]]
```

```
SnG[S[Ar[0, 1] → Ar[0, 1] +
```

$$Y[0, 1, 2, \text{PH}[-x[3] z + \left(-\frac{1}{2} x[1] x[3] + \frac{x[3]^2}{2}\right) z^2 + \left(-\frac{1}{6} x[1]^2 x[3] + \frac{1}{4} x[1] x[3]^2 - \frac{x[3]^3}{6}\right) z^3 + \left(-\frac{1}{24} x[1]^3 x[3] + \frac{1}{12} x[1]^2 x[3]^2 - \frac{1}{12} x[1] x[3]^3 + \frac{x[3]^4}{24}\right) z^4 + O[z]^5], \text{Ar}[0, 2] \rightarrow \text{Ar}[0, 2] +$$

$$Y[0, 1, 2, \text{PH}[x[3] z + \left(\frac{1}{2} x[1] x[3] - \frac{x[3]^2}{2}\right) z^2 + \left(\frac{1}{6} x[1]^2 x[3] - \frac{1}{4} x[1] x[3]^2 + \frac{x[3]^3}{6}\right) z^3 + \left(\frac{1}{24} x[1]^3 x[3] - \frac{1}{12} x[1]^2 x[3]^2 + \frac{1}{12} x[1] x[3]^3 - \frac{x[3]^4}{24}\right) z^4 + O[z]^5]] + Y[0, 3, 2,$$

$$\text{PH}[-x[1] z + \left(-\frac{1}{2} x[1]^2 + \frac{1}{2} x[1] x[3]\right) z^2 + \left(-\frac{1}{6} x[1]^3 + \frac{1}{4} x[1]^2 x[3] - \frac{1}{6} x[1] x[3]^2\right) z^3 + \left(-\frac{1}{24} x[1]^4 + \frac{1}{12} x[1]^3 x[3] - \frac{1}{12} x[1]^2 x[3]^2 + \frac{1}{24} x[1] x[3]^3\right) z^4 + O[z]^5], \text{Ar}[0, 3] \rightarrow$$

$$\text{Ar}[0, 3] + Y[0, 3, 2, \text{PH}[x[1] z + \left(\frac{x[1]^2}{2} - \frac{1}{2} x[1] x[3]\right) z^2 + \left(\frac{x[1]^3}{6} - \frac{1}{4} x[1]^2 x[3] + \frac{1}{6} x[1] x[3]^2\right) z^3 + \left(\frac{x[1]^4}{24} - \frac{1}{12} x[1]^3 x[3] + \frac{1}{12} x[1]^2 x[3]^2 - \frac{1}{24} x[1] x[3]^3\right) z^4 + O[z]^5]],$$

$$\text{Ar}[2, 0] \rightarrow \text{Ar}[2, 0] + Y[1, 3, 0, \text{PH}[x[2] z + \left(\frac{1}{2} x[1] x[2] - \frac{1}{2} x[2] x[3]\right) z^2 +$$

$$\left(\frac{1}{6} x[1]^2 x[2] - \frac{1}{4} x[1] x[2] x[3] + \frac{1}{6} x[2] x[3]^2\right) z^3 +$$

$$\left(\frac{1}{24} x[1]^3 x[2] - \frac{1}{12} x[1]^2 x[2] x[3] + \frac{1}{12} x[1] x[2] x[3]^2 - \frac{1}{24} x[2] x[3]^3\right) z^4 + O[z]^5]]],$$

$$Y[1, 3, 2, \text{PH}[-2 + \left(-\frac{3x[1]}{2} + \frac{3x[3]}{2}\right) z + \left(-\frac{2}{3} x[1]^2 + x[1] x[3] - \frac{2x[3]^2}{3}\right) z^2 +$$

$$\left(-\frac{5}{24} x[1]^3 + \frac{5}{12} x[1]^2 x[3] - \frac{5}{12} x[1] x[3]^2 + \frac{5x[3]^3}{24}\right) z^3 +$$

$$\left(-\frac{1}{20} x[1]^4 + \frac{1}{8} x[1]^3 x[3] - \frac{1}{6} x[1]^2 x[3]^2 + \frac{1}{8} x[1] x[3]^3 - \frac{x[3]^4}{20}\right) z^4 + O[z]^5]]]$$

```
cc32 = SnG[sigma[1, 4], sigma[5, 1], sigbar[1, 4], sigbar[5, 1]];
Test[(cc31 ** cc32) == (cc32 ** cc31)]
```

True

■ This last one we expect to fail:

```
cc41 = SnG[sigma[1, 2], sigma[3, 1], sigbar[1, 2], sigbar[3, 1]]
```

```
SnG[S[Ar[0, 1] → Ar[0, 1] +
```

$$Y[0, 1, 2, \text{PH}[-x[3] z + \left(-\frac{1}{2} x[1] x[3] + \frac{x[3]^2}{2}\right) z^2 + \left(-\frac{1}{6} x[1]^2 x[3] + \frac{1}{4} x[1] x[3]^2 - \frac{x[3]^3}{6}\right) z^3 + \left(-\frac{1}{24} x[1]^3 x[3] + \frac{1}{12} x[1]^2 x[3]^2 - \frac{1}{12} x[1] x[3]^3 + \frac{x[3]^4}{24}\right) z^4 + O[z^5]], \text{Ar}[0, 2] \rightarrow \text{Ar}[0, 2] +$$

$$Y[0, 1, 2, \text{PH}[x[3] z + \left(\frac{1}{2} x[1] x[3] - \frac{x[3]^2}{2}\right) z^2 + \left(\frac{1}{6} x[1]^2 x[3] - \frac{1}{4} x[1] x[3]^2 + \frac{x[3]^3}{6}\right) z^3 + \left(\frac{1}{24} x[1]^3 x[3] - \frac{1}{12} x[1]^2 x[3]^2 + \frac{1}{12} x[1] x[3]^3 - \frac{x[3]^4}{24}\right) z^4 + O[z^5]]] + Y[0, 3, 2,$$

$$\text{PH}[-x[1] z + \left(-\frac{1}{2} x[1]^2 + \frac{1}{2} x[1] x[3]\right) z^2 + \left(-\frac{1}{6} x[1]^3 + \frac{1}{4} x[1]^2 x[3] - \frac{1}{6} x[1] x[3]^2\right) z^3 + \left(-\frac{1}{24} x[1]^4 + \frac{1}{12} x[1]^3 x[3] - \frac{1}{12} x[1]^2 x[3]^2 + \frac{1}{24} x[1] x[3]^3\right) z^4 + O[z^5]], \text{Ar}[0, 3] \rightarrow$$

$$\text{Ar}[0, 3] + Y[0, 3, 2, \text{PH}[x[1] z + \left(\frac{x[1]^2}{2} - \frac{1}{2} x[1] x[3]\right) z^2 + \left(\frac{x[1]^3}{6} - \frac{1}{4} x[1]^2 x[3] + \frac{1}{6} x[1] x[3]^2\right) z^3 + \left(\frac{x[1]^4}{24} - \frac{1}{12} x[1]^3 x[3] + \frac{1}{12} x[1]^2 x[3]^2 - \frac{1}{24} x[1] x[3]^3\right) z^4 + O[z^5]]],$$

$$\text{Ar}[2, 0] \rightarrow \text{Ar}[2, 0] + Y[1, 3, 0, \text{PH}[x[2] z + \left(\frac{1}{2} x[1] x[2] - \frac{1}{2} x[2] x[3]\right) z^2 +$$

$$\left(\frac{1}{6} x[1]^2 x[2] - \frac{1}{4} x[1] x[2] x[3] + \frac{1}{6} x[2] x[3]^2\right) z^3 +$$

$$\left(\frac{1}{24} x[1]^3 x[2] - \frac{1}{12} x[1]^2 x[2] x[3] + \frac{1}{12} x[1] x[2] x[3]^2 - \frac{1}{24} x[2] x[3]^3\right) z^4 + O[z^5]]],$$

$$Y[1, 3, 2, \text{PH}[-2 + \left(-\frac{3 x[1]}{2} + \frac{3 x[3]}{2}\right) z + \left(-\frac{2}{3} x[1]^2 + x[1] x[3] - \frac{2 x[3]^2}{3}\right) z^2 +$$

$$\left(-\frac{5}{24} x[1]^3 + \frac{5}{12} x[1]^2 x[3] - \frac{5}{12} x[1] x[3]^2 + \frac{5 x[3]^3}{24}\right) z^3 +$$

$$\left(-\frac{1}{20} x[1]^4 + \frac{1}{8} x[1]^3 x[3] - \frac{1}{6} x[1]^2 x[3]^2 + \frac{1}{8} x[1] x[3]^3 - \frac{x[3]^4}{20}\right) z^4 + O[z^5]]]]]$$

```
cc42 = SnG[sigma[4, 1], sigma[5, 1], sigbar[4, 1], sigbar[5, 1]];
Expect[False,
```

```
(cc41 ** cc42) === (cc42 ** cc41)
```

```
]
False
```

■ Commutators Commutators are Central (along strand 1)

```
(ccc = SnG[
  sigma[1, 2], sigma[3, 1], sigbar[1, 2], sigbar[3, 1],
  sigma[4, 1], sigma[5, 1], sigbar[4, 1], sigbar[5, 1],
  sigma[3, 1], sigma[1, 2], sigbar[3, 1], sigbar[1, 2],
  sigma[5, 1], sigma[4, 1], sigbar[5, 1], sigbar[4, 1]
]) // Last

Y[1, 4, 2, PH[4 x[3] x[5] z^2 + (
  5/2 x[1] x[3] x[5] - 5/2 x[3]^2 x[5] + 5/2 x[3] x[4] x[5] + 5/2 x[3] x[5]^2
) z^3 +
  (
    x[1]^2 x[3] x[5] - 3/2 x[1] x[3]^2 x[5] + x[3]^3 x[5] + 3/2 x[1] x[3] x[4] x[5] -
    3/2 x[3]^2 x[4] x[5] + x[3] x[4]^2 x[5] + 3/2 x[1] x[3] x[5]^2 -
    3/2 x[3]^2 x[5]^2 + 3/2 x[3] x[4] x[5]^2 + x[3] x[5]^3
  ) z^4 + O[z]^5]]] + Y[1, 5, 2,
  PH[-4 (x[3] x[4]) z^2 + (
    -5/2 x[1] x[3] x[4] + 5/2 x[3]^2 x[4] - 5/2 x[3] x[4]^2 - 5/2 x[3] x[4] x[5]
  ) z^3 +
  (
    -x[1]^2 x[3] x[4] + 3/2 x[1] x[3]^2 x[4] - x[3]^3 x[4] -
    3/2 x[1] x[3] x[4]^2 + 3/2 x[3]^2 x[4]^2 - x[3] x[4]^3 - 3/2 x[1] x[3] x[4] x[5] +
    3/2 x[3]^2 x[4] x[5] - 3/2 x[3] x[4]^2 x[5] - x[3] x[4] x[5]^2
  ) z^4 + O[z]^5]]]

Test[ccc ** SnG[sigma[6, 1]] == SnG[sigma[6, 1]] ** ccc]
True

Test[ccc ** SnG[sigma[1, 6]] == SnG[sigma[1, 6]] ** ccc]
True
```

■ Tails Commute and 4T

```
Der[Ar[1, 2] + Ar[1, 3]][Ar[2, 4]]
Y[1, 2, 4, PH[-1 + O[z]^5]]
Expect[0, Der[Ar[1, 3]][Ar[1, 2]]]
0

Expect[0,
  Der[Ar[1, 2] + Ar[1, 3]][Ar[2, 3]]
]
0
```



```

Der[Ar[1, 2]][Ar[1, 3] + Ar[2, 3]]
Y[1, 2, 3, PH[-1 + O[z]^5]]

Expect[0,
  Der[Ar[1, 2]][Ar[3, 1] + Ar[3, 2]]
]
0

Expect[{0, 0},
  Der[Ar[1, 1]]@{Ar[1, 2], Ar[2, 1]}
]
{0, 0}

Expect[{0, 0},
  Der[Ar[1, 2]]@{Ar[1, 1], Ar[2, 2]}
]
{0, 0}

```

■ Antisymmetry of Der

```

Expect[{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
  Table[
    ReducePrimitives [
      Der[Y[1, 2, 3, H[1]]]@Ar[i, j] + Der[Ar[i, j]]@Y[1, 2, 3, H[1]]
    ], {i, 4}, {j, 4}
  ]
]
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Expect[{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  Table[
    ReducePrimitives [
      Der[Y[1, 2, 2, H[1]]]@Ar[i, j] + Der[Ar[i, j]]@Y[1, 2, 2, H[1]]
    ], {i, 3}, {j, 3}
  ]
]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

```

■ Scattering by Exponentials

```

S[Exp[Ar[1, 2]]]

S[Ar[0, 1] → Ar[0, 1] + Y[0, 1, 2, PH[-1 +  $\frac{1}{2}$  x[1] z -  $\frac{1}{6}$  x[1]^2 z^2 +  $\frac{1}{24}$  x[1]^3 z^3 -  $\frac{1}{120}$  x[1]^4 z^4 + O[z]^5]]],
  Ar[0, 2] → Ar[0, 2] + Y[0, 1, 2, PH[1 -  $\frac{1}{2}$  x[1] z +  $\frac{1}{6}$  x[1]^2 z^2 -  $\frac{1}{24}$  x[1]^3 z^3 +  $\frac{1}{120}$  x[1]^4 z^4 + O[z]^5]]],
  Ar[2, 0] → Ar[2, 0] + Y[1, 2, 0, PH[1 +  $\frac{1}{2}$  x[1] z +  $\frac{1}{6}$  x[1]^2 z^2 +  $\frac{1}{24}$  x[1]^3 z^3 +  $\frac{1}{120}$  x[1]^4 z^4 + O[z]^5]]]]

```

```

Test[
  CanonicalForm[S[Exp[Ar[1, 2]]]] == CanonicalForm[S[sigma[1, 2]]]
]
True

Test[CanonicalForm[
  S[Exp[-Ar[1, 2]]] == S[sigbar[1, 2]]
]]
True

```

■ The BCH Formula

```
S[sigma[1, 3], sigma[2, 3]] // Short
```

$$S\left[\text{Ar}[0, 1] \rightarrow \text{Ar}[0, 1] + Y\left[0, 1, 3, \text{PH}\left[-1 + \left(\frac{x[1]}{2} + x[2]\right) z + \ll 3 \gg + O[z]^5\right]\right], \ll 2 \gg, \text{Ar}[3, 0] \rightarrow \ll 1 \gg\right]$$

```
unknowns = DeclareSeries[bc[x[1], x[2]], $CutoffDegree - 1]
```

```
{bc[0, 0], bc[1, 0], bc[0, 1], bc[2, 0],
 bc[1, 1], bc[0, 2], bc[3, 0], bc[2, 1], bc[1, 2], bc[0, 3]}
```

```
PH[bc]
```

$$\text{PH}\left[\text{bc}[0, 0] + (\text{bc}[1, 0] x[1] + \text{bc}[0, 1] x[2]) z + \left(\frac{1}{2} \text{bc}[2, 0] x[1]^2 + \text{bc}[1, 1] x[1] x[2] + \frac{1}{2} \text{bc}[0, 2] x[2]^2\right) z^2 + \left(\frac{1}{6} \text{bc}[3, 0] x[1]^3 + \frac{1}{2} \text{bc}[2, 1] x[1]^2 x[2] + \frac{1}{2} \text{bc}[1, 2] x[1] x[2]^2 + \frac{1}{6} \text{bc}[0, 3] x[2]^3\right) z^3 + O[z]^4\right]$$

```
S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, PH[bc]]]] // Short[#, 5] &
```

$$S\left[\text{Ar}[0, 1] \rightarrow \text{Ar}[0, 1] + Y\left[0, 1, 3, \text{PH}\left[-1 + \left(\text{bc}[0, 0] x[2] + \frac{1}{2} (x[1] + x[2])\right) z + (\ll 1 \gg) z^2 + (\ll 1 \gg) z^3 + \left(-\frac{1}{120} (-x[1] - x[2])^4 + \frac{1}{6} \text{bc}[3, 0] x[1]^3 x[2] + \ll 4 \gg + \frac{1}{6} (-x[1] - x[2])^2 (\text{bc}[1, 0] x[1] x[2] + \text{bc}[0, 1] x[2]^2) + \frac{1}{2} (-x[1] - x[2]) \left(\frac{1}{2} \text{bc}[2, 0] x[1]^2 x[2] + \text{bc}[1, 1] x[1] x[2]^2 + \frac{1}{2} \text{bc}[0, 2] x[2]^3\right)\right]\right] z^4 + O[z]^5\right], \ll 2 \gg, \text{Ar}[3, 0] \rightarrow \ll 1 \gg\right]$$

$$\begin{aligned}
 & \text{eq} = \text{Coefficient}[\\
 & \quad \text{Ar}[0, 1] // \text{S}[\text{Exp}[\text{Ar}[1, 3] + \text{Ar}[2, 3] + \text{Y}[1, 2, 3, \text{PH}[\text{bc}]]]], \\
 & \quad \text{Y}[0, 1, 3] \\
 &] = \text{Coefficient}[\\
 & \quad \text{Ar}[0, 1] // \text{S}[\text{sigma}[1, 3], \text{sigma}[2, 3]], \\
 & \quad \text{Y}[0, 1, 3] \\
 &] \\
 & -1 + \left(\text{bc}[0, 0] x[2] + \frac{1}{2} (x[1] + x[2]) \right) z + \\
 & \left(-\frac{1}{6} (-x[1] - x[2])^2 + \text{bc}[1, 0] x[1] x[2] + \frac{1}{2} \text{bc}[0, 0] (-x[1] - x[2]) x[2] + \text{bc}[0, 1] x[2]^2 \right) z^2 + \\
 & \left(\frac{1}{2} \text{bc}[2, 0] x[1]^2 x[2] + \frac{1}{6} \text{bc}[0, 0] (-x[1] - x[2])^2 x[2] + \text{bc}[1, 1] x[1] x[2]^2 + \frac{1}{2} \text{bc}[0, 2] x[2]^3 + \right. \\
 & \quad \left. \frac{1}{24} (-x[1] - x[2])^2 (x[1] + x[2]) + \frac{1}{2} (-x[1] - x[2]) (\text{bc}[1, 0] x[1] x[2] + \text{bc}[0, 1] x[2]^2) \right) z^3 + \\
 & \left(-\frac{1}{120} (-x[1] - x[2])^4 + \frac{1}{6} \text{bc}[3, 0] x[1]^3 x[2] + \frac{1}{24} \text{bc}[0, 0] (-x[1] - x[2])^3 x[2] + \right. \\
 & \quad \frac{1}{2} \text{bc}[2, 1] x[1]^2 x[2]^2 + \frac{1}{2} \text{bc}[1, 2] x[1] x[2]^3 + \frac{1}{6} \text{bc}[0, 3] x[2]^4 + \\
 & \quad \frac{1}{6} (-x[1] - x[2])^2 (\text{bc}[1, 0] x[1] x[2] + \text{bc}[0, 1] x[2]^2) + \\
 & \quad \left. \frac{1}{2} (-x[1] - x[2]) \left(\frac{1}{2} \text{bc}[2, 0] x[1]^2 x[2] + \text{bc}[1, 1] x[1] x[2]^2 + \frac{1}{2} \text{bc}[0, 2] x[2]^3 \right) \right) z^4 + O[z]^5 = \\
 & -1 + \left(\frac{x[1]}{2} + x[2] \right) z + \left(-\frac{1}{6} x[1]^2 - \frac{1}{2} x[1] x[2] - \frac{x[2]^2}{2} \right) z^2 + \\
 & \left(\frac{x[1]^3}{24} + \frac{1}{6} x[1]^2 x[2] + \frac{1}{4} x[1] x[2]^2 + \frac{x[2]^3}{6} \right) z^3 + \\
 & \left(-\frac{1}{120} x[1]^4 - \frac{1}{24} x[1]^3 x[2] - \frac{1}{12} x[1]^2 x[2]^2 - \frac{1}{12} x[1] x[2]^3 - \frac{x[2]^4}{24} \right) z^4 + O[z]^5
 \end{aligned}$$

sol = First[PHSolve[eq, unknowns]]

$$\left\{ \text{bc}[0, 0] \rightarrow \frac{1}{2}, \text{bc}[1, 0] \rightarrow \frac{1}{12}, \text{bc}[0, 1] \rightarrow -\frac{1}{12}, \text{bc}[2, 0] \rightarrow 0, \text{bc}[1, 1] \rightarrow -\frac{1}{24}, \right. \\
 \left. \text{bc}[0, 2] \rightarrow 0, \text{bc}[3, 0] \rightarrow -\frac{1}{120}, \text{bc}[2, 1] \rightarrow -\frac{1}{90}, \text{bc}[1, 2] \rightarrow \frac{1}{90}, \text{bc}[0, 3] \rightarrow \frac{1}{120} \right\}$$

bch = PH[bc] /. sol

$$\begin{aligned}
 & \text{PH} \left[\frac{1}{2} + \left(\frac{x[1]}{12} - \frac{x[2]}{12} \right) z - \frac{1}{24} (x[1] x[2]) z^2 + \right. \\
 & \left. \left(-\frac{1}{720} x[1]^3 - \frac{1}{180} x[1]^2 x[2] + \frac{1}{180} x[1] x[2]^2 + \frac{x[2]^3}{720} \right) z^3 + O[z]^4 \right]
 \end{aligned}$$

```

Test[CanonicalForm[
  S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bch]]] == S[sigma[1, 3], sigma[2, 3]]
]]
True

```

■ Compare with Kurlin

```

Test[Simplify[(bch /. {x[1] → x, x[2] → y}) ==
  PH[[(1/y (1 - (e^x - 1)/x (x+y)/(e^{x+y} - 1)))/. {x → z x, y → z y} + O[z]^(CutoffDegree - 1)]]]]
True

```

■ Testing Code

```

SetAttributes[{Test, Expect}, {HoldAll}];
Test[expr_] := If[TrueQ[Check[expr, False]], True,
  If[Head[$FailLog] != List, $FailLog = {}];
  AppendTo[$FailLog,
    "On " <> ToString[Date[]] <> " failed in " <> ToString[HoldForm[expr]]];
  Print[Last[$FailLog]]
];
Expect[val_, expr_] := If[TrueQ[Test[val == expr]], val];

SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\ScatterAndGlow"]
<< ScatterAndGlow.m

C:\\drorbn\\AcademicPensieve\\Projects\\ScatterAndGlow

```

■ Test Test

```
Test[0 == 1]
```

```
On {2009, 1, 4, 11, 7, 33.6666000} failed in 0 == 1
```

■ Failed Tests

```
$FailLog
```

```
{On {2009, 1, 4, 11, 7, 23.2146000} failed in 0 == 1,
 On {2009, 1, 4, 11, 7, 33.6666000} failed in 0 == 1}
```