Scatter and Glow - Experiments

December 29, 2008

- **Cleaning Exponentiation**
- **The Two F Equations**

```
F0 = S[Exp[a1 Ar[2, 1] + a2 Ar[1, 2]]] /. √(a2 x[1] + a1 x[2])^2 → (a2 x[1] + a1 x[2])
```

$$
\begin{aligned}
&S\Big[Ar[0,1] \to Ar[0,1] + Y\Big[0,1,1,-\frac{a1\,a2\,e^{-a2\,x[1]-a1\,x[2]}\left(-1+e^{a2\,x[1]+a1\,x[2]}\right)^2 x[2]}{(a2\,x[1]+a1\,x[2])^2}\Big] + \\
&\quad Y\Big[0,1,2,\Big(a2\,e^{-a2\,x[1]-a1\,x[2]}\left(-1+e^{a2\,x[1]+a1\,x[2]}\right)\left(-a2^2\,x[1]^2-a2^2\,e^{a2\,x[1]+a1\,x[2]}\,x[1]^2-\right. \\
&\qquad 2\,a1\,a2\,x[1]\,x[2]-2\,a1\,a2\,e^{a2\,x[1]+a1\,x[2]}\,x[1]\,x[2]-a1^2\,x[2]^2-a1^2\,e^{a2\,x[1]+a1\,x[2]}\,x[2]^2- \\
&\qquad a2\,x[1]\,(a2\,x[1]+a1\,x[2])+a2\,e^{a2\,x[1]+a1\,x[2]}\,x[1]\,(a2\,x[1]+a1\,x[2])+ \\
&\qquad \left.a1\,x[2]\,(a2\,x[1]+a1\,x[2])-a1\,e^{a2\,x[1]+a1\,x[2]}\,x[2]\,(a2\,x[1]+a1\,x[2])\right)\Big)\Big/ \\
&\qquad \left(2\,(a2\,x[1]+a1\,x[2])^2\sqrt{(a2\,x[1]+a1\,x[2])^2}\right)\Big] + Y\Big[0,2,1, \\
&\quad \Big(a1\,e^{-a2\,x[1]-a1\,x[2]}\left(-1+e^{a2\,x[1]+a1\,x[2]}\right)\left(a2^2\,x[1]^2+a2^2\,e^{a2\,x[1]+a1\,x[2]}\,x[1]^2+\right. \\
&\qquad 2\,a1\,a2\,x[1]\,x[2]+2\,a1\,a2\,e^{a2\,x[1]+a1\,x[2]}\,x[1]\,x[2]+a1^2\,x[2]^2+a1^2\,e^{a2\,x[1]+a1\,x[2]}\,x[2]^2- \\
&\qquad a2\,x[1]\,(a2\,x[1]+a1\,x[2])+a2\,e^{a2\,x[1]+a1\,x[2]}\,x[1]\,(a2\,x[1]+a1\,x[2])+ \\
&\qquad \left.a1\,x[2]\,(a2\,x[1]+a1\,x[2])-a1\,e^{a2\,x[1]+a1\,x[2]}\,x[2]\,(a2\,x[1]+a1\,x[2])\right)\Big)\Big/ \\
&\qquad \left(2\,(a2\,x[1]+a1\,x[2])^2\sqrt{(a2\,x[1]+a1\,x[2])^2}\right)\Big] + Y\Big[0,2,2, \\
&\quad \frac{a1\,a2\,e^{-a2\,x[1]-a1\,x[2]}\left(-1+e^{a2\,x[1]+a1\,x[2]}\right)^2 x[1]}{(a2\,x[1]+a1\,x[2])^2}\Big], \\
&Ar[0,2] \to Ar[0,2] + Y\Big[0,1,1,\frac{a1\,a2\,e^{-a2\,x[1]-a1\,x[2]}\left(-1+e^{a2\,x[1]+a1\,x[2]}\right)^2 x[2]}{(a2\,x[1]+a1\,x[2])^2}\Big] + \\
&\quad Y\Big[0,1,2, \\
&\quad -\Big(a2\,e^{-a2\,x[1]-a1\,x[2]}\left(-1+e^{a2\,x[1]+a1\,x[2]}\right)\left(-a2^2\,x[1]^2-a2^2\,e^{a2\,x[1]+a1\,x[2]}\,x[1]^2-\right. \\
&\qquad 2\,a1\,a2\,x[1]\,x[2]-2\,a1\,a2\,e^{a2\,x[1]+a1\,x[2]}\,x[1]\,x[2]-a1^2\,x[2]^2-a1^2\,e^{a2\,x[1]+a1\,x[2]}\,x[2]^2- \\
&\qquad a2\,x[1]\,(a2\,x[1]+a1\,x[2])+a2\,e^{a2\,x[1]+a1\,x[2]}\,x[1]\,(a2\,x[1]+a1\,x[2])+ \\
&\qquad \left.a1\,x[2]\,(a2\,x[1]+a1\,x[2])-a1\,e^{a2\,x[1]+a1\,x[2]}\,x[2]\,(a2\,x[1]+a1\,x[2])\right)\Big)\Big/ \\
&\qquad \left(2\,(a2\,x[1]+a1\,x[2])^2\sqrt{(a2\,x[1]+a1\,x[2])^2}\right)\Big] + Y\Big[0,2,1, \\
&\quad -\Big(a1\,e^{-a2\,x[1]-a1\,x[2]}\left(-1+e^{a2\,x[1]+a1\,x[2]}\right)\left(a2^2\,x[1]^2+a2^2\,e^{a2\,x[1]+a1\,x[2]}\,x[1]^2+\right. \\
&\qquad 2\,a1\,a2\,x[1]\,x[2]+2\,a1\,a2\,e^{a2\,x[1]+a1\,x[2]}\,x[1]\,x[2]+a1^2\,x[2]^2+a1^2\,e^{a2\,x[1]+a1\,x[2]}\,x[2]^2- \\
&\qquad a2\,x[1]\,(a2\,x[1]+a1\,x[2])+a2\,e^{a2\,x[1]+a1\,x[2]}\,x[1]\,(a2\,x[1]+a1\,x[2])+ \\
&\qquad \left.a1\,x[2]\,(a2\,x[1]+a1\,x[2])-a1\,e^{a2\,x[1]+a1\,x[2]}\,x[2]\,(a2\,x[1]+a1\,x[2])\right)\Big)\Big/ \\
&\qquad \left(2\,(a2\,x[1]+a1\,x[2])^2\sqrt{(a2\,x[1]+a1\,x[2])^2}\right)\Big] + \\
&\quad Y\Big[0,2,2,-\frac{a1\,a2\,e^{-a2\,x[1]-a1\,x[2]}\left(-1+e^{a2\,x[1]+a1\,x[2]}\right)^2 x[1]}{(a2\,x[1]+a1\,x[2])^2}\Big], Ar[\\
&\quad 1, \\
&\quad 0] \to \\
&\quad Ar[1,0] + Y\Big[1,2,0,-\frac{a1\left(-1+e^{a2\,x[1]+a1\,x[2]}\right)}{a2\,x[1]+a1\,x[2]}\Big], \\
&Ar[\\
&\quad 2, \\
&\quad 0] \to \\
&\quad Ar[2,0] + Y\Big[1,2,0,\frac{a2\left(-1+e^{a2\,x[1]+a1\,x[2]}\right)}{a2\,x[1]+a1\,x[2]}\Big]\Big]
\end{aligned}
$$

```
eq1 = (
  Coefficient[Ar[3, 0] // S[Exp[Ar[1, 3] + Ar[2, 3]]] // F, Y[1, 2, 0]]
   == Coefficient[Ar[3, 0] // F // S[sigma[1, 3], sigma[2, 3]], Y[1, 2, 0]]
 )
```

```
F21 = S[Exp[a1 Ar[1, 2] + a2 Ar[2, 1] + Y[2, 1, 2, f1] + Y[2, 1, 1, f2]]]
```

```
F = S[Exp[a1 Ar[2, 1] + a2 Ar[1, 2] + Y[1, 2, 1, f1] + Y[1, 2, 2, f2]]]
```

```
$Aborted
```

December 28, 2008

```
Ar[2, 1] // S[Exp[Ar[2, 1]]]
```

```
Ar[2, 1]
```

```
Ar[2, 1] // S[Exp[Ar[2, 1] + Ar[3, 1]]]
```

$$\text{Ar}[2, 1] + \text{Y}\left[2, 3, 1, \frac{e^{-x[2]-x[3]}\left(-1 + e^{x[2]+x[3]}\right)}{x[2] + x[3]}\right]$$

```
S[Exp[Ar[2, 1]]]
```

$$\text{S}\left[\text{Ar}[0, 1] \to \text{Ar}[0, 1] + \text{Y}\left[0, 2, 1, \frac{e^{-x[2]}\left(-1 + e^{x[2]}\right)}{x[2]}\right],\ \text{Ar}[0, 2] \to \text{Ar}[0, 2] + \text{Y}\left[0, 2, 1, -\frac{e^{-x[2]}\left(-1 + e^{x[2]}\right)}{x[2]}\right],\ \text{Ar}[1, 0] \to \text{Ar}[1, 0] + \text{Y}\left[1, 2, 0, -\frac{-1 + e^{x[2]}}{x[2]}\right]\right]$$

```
S[Exp[Ar[2, 1] + Ar[3, 1]]]
```

$$\text{S}\left[\text{Ar}[0, 1] \to \text{Ar}[0, 1] + \text{Y}\left[0, 2, 1, \frac{e^{-x[2]-x[3]}\left(-1 + e^{x[2]+x[3]}\right)}{x[2] + x[3]}\right] + \text{Y}\left[0, 3, 1, \frac{e^{-x[2]-x[3]}\left(-1 + e^{x[2]+x[3]}\right)}{x[2] + x[3]}\right],\ \text{Ar}[0, 2] \to \text{Ar}[0, 2] + \text{Y}\left[0, 2, 1, -\frac{e^{-x[2]-x[3]}\left(-1 + e^{x[2]+x[3]}\right)}{x[2] + x[3]}\right],\ \text{Ar}[0, 3] \to \text{Ar}[0, 3] + \text{Y}\left[0, 3, 1, -\frac{e^{-x[2]-x[3]}\left(-1 + e^{x[2]+x[3]}\right)}{x[2] + x[3]}\right],\ \text{Ar}[1, 0] \to \text{Ar}[1, 0] + \text{Y}\left[1, 2, 0, -\frac{-1 + e^{x[2]+x[3]}}{x[2] + x[3]}\right] + \text{Y}\left[1, 3, 0, -\frac{-1 + e^{x[2]+x[3]}}{x[2] + x[3]}\right]\right]$$

December 27, 2008

- **R4 Work**

```
F = S[Exp[(-1 / 2) Ar[2, 1] + (0) Ar[1, 2] + Y[1, 2, 1, f1] + Y[1, 2, 2, 0]]];
```

```
Ar[3, 0] // S[Exp[Ar[1, 3] + Ar[2, 3]]] // F
```

$$\text{Ar}[3,0] + Y\left[1, 2, 0, \frac{1}{x[1]\,x[2]\,(x[1]+x[2])}\left(e^{x[2]}\,x[1] - e^{\left(-\frac{1}{2}+f1\,x[1]\right)x[2]}\,x[1] - e^{x[1]+x[2]}\,x[1] + e^{x[1]+x[2]+\left(-\frac{1}{2}+f1\,x[1]\right)x[2]}\,x[1] + e^{x[2]}\,x[2] - e^{x[1]+x[2]}\,x[2]\right)x[3]\right] + Y\left[1, 3, 0, \frac{e^{x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\right] + Y\left[2, 3, 0, \frac{-1+e^{x[2]}}{x[2]}\right]$$

```
Ar[3, 0] // F // S[sigma[1, 3], sigma[2, 3]]
```

$$\text{Ar}[3,0] + Y\left[1, 2, 0, -\frac{\left(-1+e^{x[1]}\right)\left(-1+e^{x[2]}\right)x[3]}{x[1]\,x[2]}\right] + Y\left[1, 3, 0, \frac{e^{x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\right] + Y\left[2, 3, 0, \frac{-1+e^{x[2]}}{x[2]}\right]$$

```
Solve[
 Coefficient[Ar[3, 0] // S[Exp[Ar[1, 3] + Ar[2, 3]]] // F, Y[1, 2, 0]]
  == Coefficient[Ar[3, 0] // F // S[sigma[1, 3], sigma[2, 3]], Y[1, 2, 0]],
 f1
]
```

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$$\left\{\left\{f1 \to \frac{\text{Log}\left[\frac{e^{\frac{x[2]}{2}}\left(-1+e^{x[1]}\right)(x[1]+x[2])}{\left(-1+e^{x[1]+x[2]}\right)x[1]}\right]}{x[1]\,x[2]}\right\}\right\}$$

```
Series[e^(x[2]/2) (-1 + e^x[1]) (x[1] + x[2]) / ((-1 + e^(x[1] + x[2])) x[1]) /. x[i_] :> h x[i], {h, 0, 2}]
```

$$1 + \frac{1}{24}\left(-2\,x[1]\,x[2] - x[2]^2\right)h^2 + O[h]^3$$

```
Der[Y[1, 1, 1, 1]]
```

```
Der[]
```

```
Der[Y[1, 1, 2, 1]]
```

```
Der[]
```

```
Der[Y[1, 2, 1, 1]]
```

```
Der[Ar[1, 0] → Y[2, 1, 0, -x[1]], Ar[0, 2] → Y[2, 0, 1, -x[1]], Ar[0, 1] → Y[2, 0, 1, x[1]]]
```

```
Der[Y[2, 1, 1, 1]]
```

```
Der[Ar[1, 0] → Y[2, 1, 0, x[1]], Ar[0, 2] → Y[2, 0, 1, x[1]], Ar[0, 1] → Y[2, 0, 1, -x[1]]]
```

```
Der[Y[1, 2, 3, 2]]
```

```
Der[Ar[0, 1] → Y[0, 1, 3, -2 x[2]], Ar[0, 2] → Y[0, 2, 3, 2 x[1]],
 Ar[0, 3] → Y[0, 1, 3, 2 x[2]] + Y[0, 2, 3, -2 x[1]], Ar[3, 0] → Y[1, 2, 0, 2 x[3]]]
```

```
{Der[Ar[1, 3]][Y[1, 2, 2, 1]], Der[Ar[2, 3]][Y[1, 2, 2, 1]],
 Der[Ar[3, 1]][Y[1, 2, 2, 1]], Der[Ar[3, 2]][Y[1, 2, 2, 1]]}
```

```
{0, Y[1, 2, 3, -x[2]], Y[1, 3, 2, -x[2]], Y[1, 3, 2, x[2]]}
```

```
Table[
  ReducePrimitives[
   Der[Y[1, 2, 2, 1]]@Ar[i, j] + Der[Ar[i, j]]@Y[1, 2, 2, 1]
  ], {i, 3}, {j, 3}
 ] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
Der[Y[1, 2, 1, f1] + Y[1, 2, 2, f2]]
```

```
Der[Ar[0, 1] → Y[1, 0, 2, f2 x[2]] + Y[2, 0, 1, f1 x[1]],
 Ar[0, 2] → Y[1, 0, 2, -f2 x[2]] + Y[2, 0, 1, -f1 x[1]],
 Ar[1, 0] → Y[2, 1, 0, -f1 x[1]], Ar[2, 0] → Y[1, 2, 0, f2 x[2]]]
```

```
S[Exp[Y[1, 2, 1, f1] + Y[1, 2, 2, f2]]]
```

$$S\left[\text{Ar}[0,1] \to \text{Ar}[0,1] + Y\left[0,1,1,\frac{e^{-\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}}\right)^2 f1\, f2}{(f1-f2)^2 x[1]}\right] + Y\left[0,1,2,\left(e^{-\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}}\right) f2 \left(f1^2 x[1] x[2] + e^{\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}} f1^2 x[1] x[2] - 2 f1 f2 x[1] x[2] - 2 e^{\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}} f1 f2 x[1] x[2] + f2^2 x[1] x[2] + e^{\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}} f2^2 x[1] x[2] - f1\sqrt{(f1-f2)^2 x[1]^2 x[2]^2} + e^{\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}} f1\sqrt{(f1-f2)^2 x[1]^2 x[2]^2} - f2\sqrt{(f1-f2)^2 x[1]^2 x[2]^2} + e^{\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}} f2\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}\right)\Big/\left(2(f1-f2)^2 x[1]\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}\right)\right] + Y\left[0,2,1,-\left(e^{-\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}}\right) f1 \left(-f1^2 x[1] x[2] - e^{\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}} f1^2 x[1] x[2] + 2 f1 f2 x[1] x[2] + 2 e^{\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}} f1 f2 x[1] x[2] - f2^2 x[1] x[2] - e^{\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}} f2^2 x[1] x[2] - f1\sqrt{(f1-f2)^2 x[1]^2 x[2]^2} + e^{\sqrt{(f1-f2)^2 x[1]^2 x[2]^2}} f1\sqrt{(f1-f2)^2 x[1]^2 x[2]^2} -\right.\right.\right.$$

$$f2\sqrt{(f1-f2)^2x[1]^2x[2]^2}+e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}\,f2\sqrt{(f1-f2)^2x[1]^2x[2]^2}\Big)\Big/\Big(2(f1-f2)^2x[2]\sqrt{(f1-f2)^2x[1]^2x[2]^2}\Big)\Big]+Y\Big[0,2,2,-\frac{e^{-\sqrt{(f1-f2)^2x[1]^2x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}\right)^2f1\,f2}{(f1-f2)^2x[2]}\Big],$$

$$\text{Ar}[0,2]\to\text{Ar}[0,2]+Y\Big[0,1,1,-\frac{e^{-\sqrt{(f1-f2)^2x[1]^2x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}\right)^2f1\,f2}{(f1-f2)^2x[1]}\Big]+$$

$$Y\Big[0,1,2,-\Big(e^{-\sqrt{(f1-f2)^2x[1]^2x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}\right)f2\Big(f1^2x[1]x[2]+e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}f1^2x[1]x[2]-2f1\,f2\,x[1]x[2]-2e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}f1\,f2\,x[1]x[2]+f2^2x[1]x[2]+e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}f2^2x[1]x[2]-f1\sqrt{(f1-f2)^2x[1]^2x[2]^2}+e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}f1\sqrt{(f1-f2)^2x[1]^2x[2]^2}-f2\sqrt{(f1-f2)^2x[1]^2x[2]^2}+e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}f2\sqrt{(f1-f2)^2x[1]^2x[2]^2}\Big)\Big)\Big/\Big(2(f1-f2)^2x[1]\sqrt{(f1-f2)^2x[1]^2x[2]^2}\Big)\Big]+Y\Big[0,2,1,\Big(e^{-\sqrt{(f1-f2)^2x[1]^2x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}\right)f1\Big(-f1^2x[1]x[2]-e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}f1^2x[1]x[2]+2f1\,f2\,x[1]x[2]+2e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}f1\,f2\,x[1]x[2]-f2^2x[1]x[2]-e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}f2^2x[1]x[2]-f1\sqrt{(f1-f2)^2x[1]^2x[2]^2}+e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}f1\sqrt{(f1-f2)^2x[1]^2x[2]^2}-f2\sqrt{(f1-f2)^2x[1]^2x[2]^2}+e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}f2\sqrt{(f1-f2)^2x[1]^2x[2]^2}\Big)\Big)\Big/\Big(2(f1-f2)^2x[2]\sqrt{(f1-f2)^2x[1]^2x[2]^2}\Big)\Big]+Y\Big[0,2,2,\frac{e^{-\sqrt{(f1-f2)^2x[1]^2x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2x[1]^2x[2]^2}}\right)^2f1\,f2}{(f1-f2)^2x[2]}\Big],$$

$$\text{Ar}[1,0]\to\text{Ar}[1,0]+Y\Big[1,2,0,\frac{e^{-f2\,x[1]\,x[2]}\left(-e^{f1\,x[1]\,x[2]}+e^{f2\,x[1]\,x[2]}\right)f1}{(f1-f2)\,x[2]}\Big],$$

```
Ar[
   2,
   0] →
  Ar[2, 0] + Y[1, 2, 0, (ℯ^(-f2 x[1] x[2]) (-ℯ^(f1 x[1] x[2]) + ℯ^(f2 x[1] x[2])) f2)/((f1 - f2) x[1])]]
```

```
drules = Sequence @@ Der[Y[1, 2, 1, f1] + Y[1, 2, 2, f2]]
```

```
Sequence[Ar[0, 1] → Y[1, 0, 2, f2 x[2]] + Y[2, 0, 1, f1 x[1]],
 Ar[0, 2] → Y[1, 0, 2, -f2 x[2]] + Y[2, 0, 1, -f1 x[1]],
 Ar[1, 0] → Y[2, 1, 0, -f1 x[1]], Ar[2, 0] → Y[1, 2, 0, f2 x[2]]]
```

```
k0 = Length[ins = First /@ {drules}]
```

```
4
```

```
ins
```

```
{Ar[0, 1], Ar[0, 2], Ar[1, 0], Ar[2, 0]}
```

```
k0 = Length[ins = Take[ins, -2]]
```

```
2
```

```
outs = {};
For[k = 1, k ≤ Length[ins], ++k,
  AppendTo[outs, newout = Der[drules][ins[[k]]]];
  ins = ins ~Join~ Complement[
     Union[Cases[{newout}, Y[ijk___, _] :> Y[ijk, 1], Infinity]],
     ins
    ]
 ];
--k;
```

```
ins
```

```
{Ar[1, 0], Ar[2, 0], Y[1, 2, 0, 1]}
```

```
outs
```

```
{Y[1, 2, 0, f1 x[1]], Y[1, 2, 0, f2 x[2]], Y[1, 2, 0, -f1 x[1] x[2] + f2 x[1] x[2]]}
```

```
zero = Table[0, {k}];
e[{{i_}}] := ReplacePart[zero, 1, i];
mat = Replace[
   outs /. Y[ijk__, h_] :> -h e[Position[ins, Y[ijk, 1]]],
   0 → zero,
   {1}
  ];
```

```
mat // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & -f1\,x[1] \\ 0 & 0 & -f2\,x[2] \\ 0 & 0 & f1\,x[1]\,x[2] - f2\,x[1]\,x[2] \end{pmatrix}$$

```
ReducePrimitives[Take[SH[MatrixExp[mat]].ins, k0]]
```

$$\left\{\text{Ar}[1,0]+\text{Y}\left[1,2,0,\frac{e^{-\text{f2}\,x[1]\,x[2]}\left(-e^{\text{f1}\,x[1]\,x[2]}+e^{\text{f2}\,x[1]\,x[2]}\right)\text{f1}}{(\text{f1}-\text{f2})\,x[2]}\right],\right.$$
$$\left.\text{Ar}[2,0]+\text{Y}\left[1,2,0,\frac{e^{-\text{f2}\,x[1]\,x[2]}\left(-e^{\text{f1}\,x[1]\,x[2]}+e^{\text{f2}\,x[1]\,x[2]}\right)\text{f2}}{(\text{f1}-\text{f2})\,x[1]}\right]\right\}$$

```
expmat = SH[PowerExpand[MatrixExp[mat]]]
```

$$\left\{\left\{1,\frac{1}{2\,(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]}e^{-\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\right)\text{f2}\right.\right.$$
$$\left(-\text{f1}\,x[1]\,x[2]+e^{\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\,\text{f1}\,x[1]\,x[2]-\text{f2}\,x[1]\,x[2]+e^{\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\,\text{f2}\,x[1]\,x[2]+\right.$$
$$\left.\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}+e^{\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}\right),$$
$$-\frac{1}{2\,(\text{f1}-\text{f2})^2\,x[1]\,x[2]^2}e^{-\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\right)\text{f1}$$
$$\left(-\text{f1}\,x[1]\,x[2]+e^{\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\,\text{f1}\,x[1]\,x[2]-\text{f2}\,x[1]\,x[2]+e^{\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\,\text{f2}\,x[1]\,x[2]-\right.$$
$$\left.\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}-e^{\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}\right),$$
$$\frac{e^{-\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\right)^2\text{f1}\,\text{f2}}{(\text{f1}-\text{f2})^2\,x[1]},$$
$$\left.-\frac{e^{-\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\right)^2\text{f1}\,\text{f2}}{(\text{f1}-\text{f2})^2\,x[2]}\right\},$$
$$\left\{0,\frac{1}{2\,(\text{f1}-\text{f2})^2\,x[1]\,x[2]}e^{-\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\left(\text{f1}^2\,x[1]\,x[2]+e^{2\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\,\text{f1}^2\,x[1]\,x[2]-\right.\right.$$
$$4\,e^{\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\,\text{f1}\,\text{f2}\,x[1]\,x[2]+\text{f2}^2\,x[1]\,x[2]+e^{2\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\,\text{f2}^2\,x[1]\,x[2]-$$
$$\text{f1}\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}+e^{2\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\,\text{f1}\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}-$$
$$\left.\text{f2}\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}+e^{2\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\,\text{f2}\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}\right),$$
$$-\frac{e^{-\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(\text{f1}-\text{f2})^2\,x[1]^2\,x[2]^2}}\right)^2\text{f1}^2\,x[1]}{(\text{f1}-\text{f2})^2\,x[2]},$$

$$\frac{1}{2\,(f1-f2)^2\,x[1]\,x[2]}\,e^{-\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\right)f1$$
$$\left(-f1\,x[1]\,x[2]+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f1\,x[1]\,x[2]-f2\,x[1]\,x[2]+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f2\,x[1]\,x[2]+\right.$$
$$\left.\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}\right),$$
$$-\frac{1}{2\,(f1-f2)^2\,x[2]^2}\,e^{-\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\right)f1$$
$$\left(-f1\,x[1]\,x[2]+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f1\,x[1]\,x[2]-f2\,x[1]\,x[2]+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f2\,x[1]\,x[2]+\right.$$
$$\left.\left.\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}\right)\right\},$$
$$\left\{0,-\frac{e^{-\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\right)^2 f2^2\,x[2]}{(f1-f2)^2\,x[1]}\right.,$$
$$\frac{1}{2\,(f1-f2)^2\,x[1]\,x[2]}\,e^{-\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\left(f1^2\,x[1]\,x[2]+e^{2\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f1^2\,x[1]\,x[2]-\right.$$
$$4\,e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f1\,f2\,x[1]\,x[2]+f2^2\,x[1]\,x[2]+e^{2\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f2^2\,x[1]\,x[2]+$$
$$f1\,\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}-e^{2\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f1\,\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}+$$
$$\left.f2\,\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}-e^{2\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f2\,\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}\right),$$
$$-\frac{1}{2\,(f1-f2)^2\,x[1]^2}\,e^{-\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\right)f2$$
$$\left(-f1\,x[1]\,x[2]+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f1\,x[1]\,x[2]-f2\,x[1]\,x[2]+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f2\,x[1]\,x[2]-\right.$$
$$\left.\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}-e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}\right),$$
$$\frac{1}{2\,(f1-f2)^2\,x[1]\,x[2]}\,e^{-\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\right)f2$$
$$\left(-f1\,x[1]\,x[2]+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f1\,x[1]\,x[2]-f2\,x[1]\,x[2]+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f2\,x[1]\,x[2]-\right.$$
$$\left.\left.\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}-e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}\right)\right\},$$
$$\left\{0,-\frac{1}{2\,(f1-f2)^2\,x[1]\,x[2]}\,e^{-\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\right)f2\right.$$
$$\left(-f1\,x[1]\,x[2]+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f1\,x[1]\,x[2]-f2\,x[1]\,x[2]+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f2\,x[1]\,x[2]+\right.$$

$$\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}\Big),$$

$$-\frac{1}{2\,(f1-f2)^2\,x[2]^2}e^{-\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\right)f1$$
$$\Big(f1\,x[1]\,x[2]-e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f1\,x[1]\,x[2]+f2\,x[1]\,x[2]-e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f2\,x[1]\,x[2]+$$
$$\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}\Big),$$

$$\frac{e^{-\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\left(e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f1-f2\right)\left(f1-e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f2\right)}{(f1-f2)^2},$$

$$\frac{e^{-\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\right)^2 f1\,f2\,x[1]}{(f1-f2)^2\,x[2]}\Big\},$$

$$\Big\{0,\ \frac{1}{2\,(f1-f2)^2\,x[1]^2}e^{-\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\right)f2$$
$$\Big(-f1\,x[1]\,x[2]+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f1\,x[1]\,x[2]-f2\,x[1]\,x[2]+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f2\,x[1]\,x[2]+$$
$$\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}\Big),$$

$$-\frac{1}{2\,(f1-f2)^2\,x[1]\,x[2]}e^{-\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\right)f1$$
$$\Big(-f1\,x[1]\,x[2]+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f1\,x[1]\,x[2]-f2\,x[1]\,x[2]+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f2\,x[1]\,x[2]-$$
$$\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}-e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}\Big),$$

$$\frac{e^{-\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\left(-1+e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\right)^2 f1\,f2\,x[2]}{(f1-f2)^2\,x[1]},$$

$$\frac{e^{-\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\left(e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f1-f2\right)\left(f1-e^{\sqrt{(f1-f2)^2\,x[1]^2\,x[2]^2}}\,f2\right)}{(f1-f2)^2}\Big\}\Big\}$$

```
ReducePrimitives[Take[SH[MatrixExp[mat]].ins, k0]] /.
 Sqrt[(f1 - f2)^2 x[1]^2 x[2]^2] → (f1 - f2) x[1] x[2]
```

$$\left\{\text{Ar}[0,1]+\text{Y}\left[0,1,1,\frac{e^{-(f1-f2)\,x[1]\,x[2]}\left(-1+e^{(f1-f2)\,x[1]\,x[2]}\right)^2 f1\,f2}{(f1-f2)^2\,x[1]}\right]+\right.$$
$$\text{Y}\left[0,1,2,\frac{1}{2\,(f1-f2)^2\,x[1]^2\,x[2]}e^{-(f1-f2)\,x[1]\,x[2]}\left(-1+e^{(f1-f2)\,x[1]\,x[2]}\right)\right.$$
$$f2\left(-f1\,x[1]\,x[2]+e^{(f1-f2)\,x[1]\,x[2]}\,f1\,x[1]\,x[2]+(f1-f2)\,x[1]\,x[2]+\right.$$
$$\left.e^{(f1-f2)\,x[1]\,x[2]}\,(f1-f2)\,x[1]\,x[2]-f2\,x[1]\,x[2]+e^{(f1-f2)\,x[1]\,x[2]}\,f2\,x[1]\,x[2]\right)\Big]+$$
$$\text{Y}\left[0,2,1,-\frac{1}{2\,(f1-f2)^2\,x[1]\,x[2]^2}e^{-(f1-f2)\,x[1]\,x[2]}\left(-1+e^{(f1-f2)\,x[1]\,x[2]}\right)f1\right.$$
$$\left(-f1\,x[1]\,x[2]+e^{(f1-f2)\,x[1]\,x[2]}\,f1\,x[1]\,x[2]-(f1-f2)\,x[1]\,x[2]-\right.$$
$$\left.e^{(f1-f2)\,x[1]\,x[2]}\,(f1-f2)\,x[1]\,x[2]-f2\,x[1]\,x[2]+e^{(f1-f2)\,x[1]\,x[2]}\,f2\,x[1]\,x[2]\right)\Big]+$$
$$\left.\text{Y}\left[0,2,2,-\frac{e^{-(f1-f2)\,x[1]\,x[2]}\left(-1+e^{(f1-f2)\,x[1]\,x[2]}\right)^2 f1\,f2}{(f1-f2)^2\,x[2]}\right]\right\}$$

```
Union[Cases[expmat, Sqrt[y_] :> y, Infinity]]
```

$$\left\{(f1-f2)^2\,x[1]^2\,x[2]^2\right\}$$

■ BCH Work

```
S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bc]]]
```

$$\text{S}\left[\text{Ar}[0,1]\to\text{Ar}[0,1]+\text{Y}\left[0,1,3,\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]+x[2]}\right)(-1+\text{bc}\,x[2])}{x[1]+x[2]}\right],\right.$$
$$\text{Ar}[0,2]\to\text{Ar}[0,2]+\text{Y}\left[0,2,3,-\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]+x[2]}\right)(1+\text{bc}\,x[1])}{x[1]+x[2]}\right],$$
$$\text{Ar}[0,3]\to\text{Ar}[0,3]+\text{Y}\left[0,1,3,-\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]+x[2]}\right)(-1+\text{bc}\,x[2])}{x[1]+x[2]}\right]+$$
$$\text{Y}\left[0,2,3,\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]+x[2]}\right)(1+\text{bc}\,x[1])}{x[1]+x[2]}\right],\ \text{Ar}[3,0]\to\text{Ar}[3,0]+\text{Y}\left[1,2,0,\right.$$
$$\left.-\frac{\left(x[1]-e^{x[2]}\,x[1]-e^{x[2]}\,x[2]+e^{x[1]+x[2]}\,x[2]-\text{bc}\,x[1]\,x[2]+\text{bc}\,e^{x[1]+x[2]}\,x[1]\,x[2]\right)x[3]}{x[1]\,x[2]\,(x[1]+x[2])}\right]+$$
$$\left.\text{Y}\left[1,3,0,\frac{e^{x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\right]+\text{Y}\left[2,3,0,\frac{-1+e^{x[2]}}{x[2]}\right]\right]$$

```
Coefficient[
 Ar[0, 1] // S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bc]]],
 Y[0, 1, 3]
]
```

$$\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]+x[2]}\right)(-1+\text{bc}\,x[2])}{x[1]+x[2]}$$

```
bch = bc /. First[Solve[
    Coefficient[
      Ar[0, 1] // S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bc]]],
      Y[0, 1, 3]
     ] == Coefficient[
      Ar[0, 1] // S[sigma[1, 3], sigma[2, 3]],
      Y[0, 1, 3]
     ],
    bc
   ]]
```

$$\frac{-e^{x[1]}\,x[1] + e^{x[1]+x[2]}\,x[1] + x[2] - e^{x[1]}\,x[2]}{\left(-1 + e^{x[1]+x[2]}\right)\,x[1]\,x[2]}$$

```
S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bch]]]
```

$$S\Big[Ar[0,1] \to Ar[0,1] + Y\Big[0,1,3,-\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big],$$
$$Ar[0,2] \to Ar[0,2] + Y\Big[0,2,3,-\frac{e^{-x[2]}\left(-1+e^{x[2]}\right)}{x[2]}\Big],$$
$$Ar[0,3] \to Ar[0,3] + Y\Big[0,1,3,\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big] + Y\Big[0,2,3,\frac{e^{-x[2]}\left(-1+e^{x[2]}\right)}{x[2]}\Big],$$
$$Ar[3,0] \to Ar[3,0] + Y\Big[1,2,0,-\frac{\left(-1+e^{x[1]}\right)\left(-1+e^{x[2]}\right)x[3]}{x[1]\,x[2]}\Big] +$$
$$Y\Big[1,3,0,\frac{e^{x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big] + Y\Big[2,3,0,\frac{-1+e^{x[2]}}{x[2]}\Big]\Big]$$

```
S[sigma[1, 3], sigma[2, 3]]
```

$$S\Big[Ar[0,1] \to Ar[0,1] + Y\Big[0,1,3,-\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big],$$
$$Ar[0,2] \to Ar[0,2] + Y\Big[0,2,3,-\frac{e^{-x[2]}\left(-1+e^{x[2]}\right)}{x[2]}\Big],$$
$$Ar[0,3] \to Ar[0,3] + Y\Big[0,1,3,\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big] + Y\Big[0,2,3,\frac{e^{-x[2]}\left(-1+e^{x[2]}\right)}{x[2]}\Big],$$
$$Ar[3,0] \to Ar[3,0] + Y\Big[1,2,0,-\frac{\left(-1+e^{x[1]}\right)\left(-1+e^{x[2]}\right)x[3]}{x[1]\,x[2]}\Big] +$$
$$Y\Big[1,3,0,\frac{e^{x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big] + Y\Big[2,3,0,\frac{-1+e^{x[2]}}{x[2]}\Big]\Big]$$

```
S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bch]]] == S[sigma[1, 3], sigma[2, 3]]
```

```
True
```

```
1
```

```
drules = Sequence @@ Der[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bch]]
```

$$\text{Sequence}\left[\text{Ar}[0,1] \to \text{Y}[0,1,3,1] + \text{Y}\left[0,1,3,-\frac{-e^{x[1]}\,x[1] + e^{x[1]+x[2]}\,x[1] + x[2] - e^{x[1]}\,x[2]}{\left(-1+e^{x[1]+x[2]}\right)x[1]}\right],\right.$$

$$\text{Ar}[0,2] \to \text{Y}[0,2,3,1] + \text{Y}\left[0,2,3,\frac{-e^{x[1]}\,x[1] + e^{x[1]+x[2]}\,x[1] + x[2] - e^{x[1]}\,x[2]}{\left(-1+e^{x[1]+x[2]}\right)x[2]}\right],$$

$$\text{Ar}[0,3] \to \text{Y}\left[0,1,3,\frac{-e^{x[1]}\,x[1] + e^{x[1]+x[2]}\,x[1] + x[2] - e^{x[1]}\,x[2]}{\left(-1+e^{x[1]+x[2]}\right)x[1]}\right] + \text{Y}\left[0,2,3,\right.$$

$$\left.-\frac{-e^{x[1]}\,x[1] + e^{x[1]+x[2]}\,x[1] + x[2] - e^{x[1]}\,x[2]}{\left(-1+e^{x[1]+x[2]}\right)x[2]}\right] + \text{Y}[1,0,3,1] + \text{Y}[2,0,3,1],\ \text{Ar}[3,0] \to$$

$$\left.\text{Y}\left[1,2,0,\frac{\left(-e^{x[1]}\,x[1] + e^{x[1]+x[2]}\,x[1] + x[2] - e^{x[1]}\,x[2]\right)x[3]}{\left(-1+e^{x[1]+x[2]}\right)x[1]\,x[2]}\right] + \text{Y}[1,3,0,-1] + \text{Y}[2,3,0,-1]\right]$$

ins

```
{Ar[0, 1], Ar[0, 2], Ar[0, 3], Ar[3, 0]}
```

k0 = Length[ins = First /@ {drules}]

```
4
```

```
outs = {};
For[k = 1, k ≤ Length[ins], ++k,
  AppendTo[outs, newout = Der[drules][ins[[k]]]];
  ins = ins ~Join~ Complement[
     Union[Cases[{newout}, Y[ijk___, _] :> Y[ijk, 1], Infinity]],
     ins
    ]
 ];
--k;
```

ins

```
{Ar[0, 1], Ar[0, 2], Ar[0, 3], Ar[3, 0], Y[0, 1, 3, 1],
 Y[0, 2, 3, 1], Y[1, 2, 0, 1], Y[1, 3, 0, 1], Y[2, 3, 0, 1]}
```

Print /@ outs;

$$Y\left[0,\ 1,\ 3,\ \frac{\left(-1+e^{x[1]}\right)\ (x[1]+x[2])}{\left(-1+e^{x[1]+x[2]}\right)\ x[1]}\right]$$

$$Y\left[0,\ 2,\ 3,\ \frac{e^{x[1]}\ \left(-1+e^{x[2]}\right)\ (x[1]+x[2])}{\left(-1+e^{x[1]+x[2]}\right)\ x[2]}\right]$$

$$Y\left[0,\ 1,\ 3,\ -\frac{\left(-1+e^{x[1]}\right)\ (x[1]+x[2])}{\left(-1+e^{x[1]+x[2]}\right)\ x[1]}\right]+Y\left[0,\ 2,\ 3,\ -\frac{e^{x[1]}\ \left(-1+e^{x[2]}\right)\ (x[1]+x[2])}{\left(-1+e^{x[1]+x[2]}\right)\ x[2]}\right]$$

$$Y\left[1,\ 2,\ 0,\ \frac{\left(-e^{x[1]}\ x[1]+e^{x[1]+x[2]}\ x[1]+x[2]-e^{x[1]}\ x[2]\right)\ x[3]}{\left(-1+e^{x[1]+x[2]}\right)\ x[1]\ x[2]}\right]+Y[1,\ 3,\ 0,\ -1]+Y[2,\ 3,\ 0,\ -1]$$

$$Y[0,\ 1,\ 3,\ x[1]+x[2]]$$

$$Y[0,\ 2,\ 3,\ x[1]+x[2]]$$

$$0$$

$$Y\left[1,\ 2,\ 0,\ \frac{e^{x[1]}\ \left(-1+e^{x[2]}\right)\ (x[1]+x[2])\ x[3]}{\left(-1+e^{x[1]+x[2]}\right)\ x[2]}\right]+Y[1,\ 3,\ 0,\ -x[1]-x[2]]$$

$$Y\left[1,\ 2,\ 0,\ \frac{\left(-e^{x[1]}\ x[1]+e^{x[1]+x[2]}\ x[1]+x[2]-e^{x[1]}\ x[2]\right)\ x[3]}{\left(-1+e^{x[1]+x[2]}\right)\ x[1]}\right]+Y[1,\ 3,\ 0,\ -x[2]]+Y[2,\ 3,\ 0,\ -x[2]]$$

```
zero = Table[0, {k}];
e[{{i_}}] := ReplacePart[zero, 1, i];
mat = Replace[
  outs /. Y[ijk__, h_] :> -h e[Position[ins, Y[ijk, 1]]],
  0 → zero,
  {1}
 ]
```

$$\left\{\left\{0,\ 0,\ 0,\ 0,\ -\frac{\left(-1+e^{x[1]}\right)\ (x[1]+x[2])}{\left(-1+e^{x[1]+x[2]}\right)\ x[1]},\ 0,\ 0,\ 0,\ 0\right\},\right.$$

$$\left\{0,\ 0,\ 0,\ 0,\ 0,\ -\frac{e^{x[1]}\ \left(-1+e^{x[2]}\right)\ (x[1]+x[2])}{\left(-1+e^{x[1]+x[2]}\right)\ x[2]},\ 0,\ 0,\ 0\right\},$$

$$\left\{0,\ 0,\ 0,\ 0,\ \frac{\left(-1+e^{x[1]}\right)\ (x[1]+x[2])}{\left(-1+e^{x[1]+x[2]}\right)\ x[1]},\ \frac{e^{x[1]}\ \left(-1+e^{x[2]}\right)\ (x[1]+x[2])}{\left(-1+e^{x[1]+x[2]}\right)\ x[2]},\ 0,\ 0,\ 0\right\},$$

$$\left\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -\frac{\left(-e^{x[1]}\ x[1]+e^{x[1]+x[2]}\ x[1]+x[2]-e^{x[1]}\ x[2]\right)\ x[3]}{\left(-1+e^{x[1]+x[2]}\right)\ x[1]\ x[2]},\ 1,\ 1\right\},$$

$$\{0,\ 0,\ 0,\ 0,\ -x[1]-x[2],\ 0,\ 0,\ 0,\ 0\},$$

$$\{0,\ 0,\ 0,\ 0,\ 0,\ -x[1]-x[2],\ 0,\ 0,\ 0\},\ \{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\},$$

$$\left\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -\frac{e^{x[1]}\ \left(-1+e^{x[2]}\right)\ (x[1]+x[2])\ x[3]}{\left(-1+e^{x[1]+x[2]}\right)\ x[2]},\ x[1]+x[2],\ 0\right\},$$

$$\left.\left\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -\frac{\left(-e^{x[1]}\ x[1]+e^{x[1]+x[2]}\ x[1]+x[2]-e^{x[1]}\ x[2]\right)\ x[3]}{\left(-1+e^{x[1]+x[2]}\right)\ x[1]},\ x[2],\ x[2]\right\}\right\}$$

MatrixExp[mat] // MatrixForm

$$\begin{pmatrix} 1 & 0 & 0 & & & & 0 \\ 0 & 1 & 0 & & & & 0 \\ 0 & 0 & 1 & & & & 0 \\ 0 & 0 & 0 & \frac{x[1]}{\left(-1+e^{x[1]+x[2]}\right)^2 (x[1]+x[2])} - \frac{2\, e^{x[1]+x[2]}\, x[1]}{\left(-1+e^{x[1]+x[2]}\right)^2 (x[1]+x[2])} + \frac{e^{2x[1]+2x[2]}\, x[1]}{\left(-1+e^{x[1]+x[2]}\right)^2 (x[1]+x[2])} + \frac{x[2]}{\left(-1+e^{x[1]+x[2]}\right)^2 (x[1]+x[2])} - \frac{}{\left(-1+e^{}\right.} & & & 0 \\ 0 & 0 & 0 & & & & 0 \\ 0 & 0 & 0 & & & & 0 \\ 0 & 0 & 0 & & & & 0 \\ 0 & 0 & 0 & & & & 0 \\ 0 & 0 & 0 & & & & 0 \end{pmatrix}$$

SH[MatrixExp[mat]] // MatrixForm

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]}\right)}{x[1]} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{e^{-x[2]}\left(-1+e^{x[2]}\right)}{x[2]} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]}\right)}{x[1]} & \frac{e^{-x[2]}\left(-1+e^{x[2]}\right)}{x[2]} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{\left(-1+e^{x[1]}\right)\left(-1+e^{x[2]}\right) x[3]}{x[1]\, x[2]} & \frac{e^{x[2]}\left(-1+e^{x[1]}\right)}{x[1]} & \frac{-1+e^{x[2]}}{x[2]} \\ 0 & 0 & 0 & 0 & e^{-x[1]-x[2]} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-x[1]-x[2]} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{e^{x[1]}\left(-1+e^{x[2]}\right) x[3]}{x[2]} & e^{x[1]+x[2]} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\left(-1+e^{x[1]}\right)\left(-1+e^{x[2]}\right) x[3]}{x[1]} & \frac{e^{x[2]}\left(-1+e^{x[1]}\right) x[2]}{x[1]} & e^{x[2]} \end{pmatrix}$$

(1 - (E^x - 1) (x + y) / x / (E^(x + y) - 1)) / y

$$\frac{1 - \frac{\left(-1+e^{x}\right)(x+y)}{\left(-1+e^{x+y}\right) x}}{y}$$

Simplify[1/y (1 - (e^x - 1)/x (x + y)/(e^(x+y) - 1))]

$$\frac{1 - \frac{\left(-1+e^{x}\right)(x+y)}{\left(-1+e^{x+y}\right) x}}{y}$$

bch

$$\frac{-e^{x[1]}\, x[1] + e^{x[1]+x[2]}\, x[1] + x[2] - e^{x[1]}\, x[2]}{\left(-1 + e^{x[1]+x[2]}\right) x[1]\, x[2]}$$

Simplify[(bch /. {x[1] → x, x[2] → y})]

$$\frac{e^{x+y}\, x + y - e^{x}\, (x + y)}{\left(-1 + e^{x+y}\right) x\, y}$$

```
Test[Simplify[(bch /. {x[1] → x, x[2] → y}) == 1/y (1 - (E^x - 1)/x (x + y)/(E^(x+y) - 1))]]
```

```
True
```

December 26, 2008

```
drules = Sequence @@ Der[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, f[x[1], x[2]]]]
```

```
Sequence[Ar[0, 1] → Y[0, 1, 3, 1] + Y[0, 1, 3, -f[x[1], x[2]] x[2]],
 Ar[0, 2] → Y[0, 2, 3, 1] + Y[0, 2, 3, f[x[1], x[2]] x[1]], Ar[0, 3] →
  Y[0, 1, 3, f[x[1], x[2]] x[2]] + Y[0, 2, 3, -f[x[1], x[2]] x[1]] + Y[1, 0, 3, 1] + Y[2, 0, 3, 1],
 Ar[3, 0] → Y[1, 2, 0, f[x[1], x[2]] x[3]] + Y[1, 3, 0, -1] + Y[2, 3, 0, -1]]
```

```
drules = Sequence @@ Der[Ar[1, 2]]
```

```
Sequence[Ar[2, 0] → Y[1, 2, 0, -1], Ar[0, 1] → Y[0, 1, 2, 1], Ar[0, 2] → Y[1, 0, 2, 1]]
```

```
k0 = Length[ins = First /@ {drules}];
outs = {};
For[k = 1, k ≤ Length[ins], ++k,
  AppendTo[outs, newout = Der[drules][ins[[k]]]];
  ins = ins ~Join~ Complement[
     Union[Cases[{newout}, Y[ijk___, _] :> Y[ijk, 1], Infinity]],
     ins
    ]
 ];
--k;
zero = Table[0, {k}];
e[{{i_}}] := ReplacePart[zero, 1, i];
mat = Replace[
   outs /. Y[ijk__, h_] :> -h e[Position[ins, Y[ijk, 1]]],
   0 → zero,
   {1}
  ];
S @@ Sort[Thread[
   Take[ins, k0] →
    ReducePrimitives[Take[MatrixExp[mat].ins, k0]]
  ]]
```

$$S\left[Ar[0, 1] \to Ar[0, 1] + Y\left[0, 1, 2, -\frac{1}{x[1]} + \frac{e^{-x[1]}}{x[1]}\right],\right.$$
$$\left. Ar[0, 2] \to Ar[0, 2] + Y\left[0, 1, 2, \frac{1}{x[1]} - \frac{e^{-x[1]}}{x[1]}\right], Ar[2, 0] \to Ar[2, 0] + Y\left[1, 2, 0, -\frac{1}{x[1]} + \frac{e^{x[1]}}{x[1]}\right]\right]$$

```
S[sigma[1, 2]]
```

$$S\left[Ar[0, 1] \to Ar[0, 1] + Y\left[0, 1, 2, -\frac{1}{x[1]} + \frac{e^{-x[1]}}{x[1]}\right],\right.$$
$$\left. Ar[0, 2] \to Ar[0, 2] + Y\left[0, 1, 2, \frac{1}{x[1]} - \frac{e^{-x[1]}}{x[1]}\right], Ar[2, 0] \to Ar[2, 0] + Y\left[1, 2, 0, -\frac{1}{x[1]} + \frac{e^{x[1]}}{x[1]}\right]\right]$$

MatrixExp[mat] // Simplify // MatrixForm

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1-e^{-x[1]}}{x[1]} & 0 \\ 0 & 1 & 0 & 0 & \frac{-1+e^{x[1]}}{x[1]} \\ 0 & 0 & 1 & 0 & \frac{1-e^{x[1]}}{x[1]} \\ 0 & 0 & 0 & e^{-x[1]} & 0 \\ 0 & 0 & 0 & 0 & e^{x[1]} \end{pmatrix}$$

ins

```
{Ar[2, 0], Ar[0, 1], Ar[0, 2], Y[1, 2, 0, 1], Y[0, 1, 2, 1]}
```

outs

```
{Y[1, 2, 0, -1], Y[0, 1, 2, 1], Y[0, 1, 2, -1], Y[1, 2, 0, -x[1]], Y[0, 1, 2, x[1]]}
```

k0

```
3
```

Cases[outs, _f, Infinity]

```
{f[x[1], x[2]], f[x[1], x[2]], f[x[1], x[2]],
 f[x[1], x[2]], f[x[1], x[2]], f[x[1], x[2]], f[x[1], x[2]]}
```

Length[ins]

```
9
```

Length[outs]

```
9
```

■ BCH Work

S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bch]]] /. Y[ijk__, h_] :> Y[ijk, Factor[h]]

$$S\Big[Ar[0,1] \to Ar[0,1] + Y\Big[0,1,3,\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]+x[2]}\right)(-1+\text{bch}\,x[2])}{x[1]+x[2]}\Big],$$
$$Ar[0,2] \to Ar[0,2] + Y\Big[0,2,3,-\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]+x[2]}\right)(1+\text{bch}\,x[1])}{x[1]+x[2]}\Big],$$
$$Ar[0,3] \to Ar[0,3] + Y\Big[0,1,3,-\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]+x[2]}\right)(-1+\text{bch}\,x[2])}{x[1]+x[2]}\Big] +$$
$$Y\Big[0,2,3,\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]+x[2]}\right)(1+\text{bch}\,x[1])}{x[1]+x[2]}\Big], Ar[3,0] \to Ar[3,0] + Y\Big[1,2,0,$$
$$-\left(\left(x[1]-e^{x[2]}x[1]-e^{x[2]}x[2]+e^{x[1]+x[2]}x[2]-\text{bch}\,x[1]x[2]+\text{bch}\,e^{x[1]+x[2]}x[1]x[2]\right)x[3]\right)/$$
$$\left(x[1]x[2](x[1]+x[2])\right)\Big] + Y\Big[1,3,0,\frac{e^{x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big] + Y\Big[2,3,0,\frac{-1+e^{x[2]}}{x[2]}\Big]\Big]$$

Ar[0, 1] // S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bch]]]

$$Ar[0,1] + Y\Big[0,1,3,-\frac{1}{x[1]+x[2]}+\frac{e^{-x[1]-x[2]}}{x[1]+x[2]}+\frac{\text{bch}\,x[2]}{x[1]+x[2]}-\frac{\text{bch}\,e^{-x[1]-x[2]}x[2]}{x[1]+x[2]}\Big]$$

```
Coefficient[
 Ar[0, 1] // S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bch]]],
 Y[0, 1, 3]
]
```

$$-\frac{1}{x[1]+x[2]}+\frac{e^{-x[1]-x[2]}}{x[1]+x[2]}+\frac{\text{bch}\,x[2]}{x[1]+x[2]}-\frac{\text{bch}\,e^{-x[1]-x[2]}\,x[2]}{x[1]+x[2]}$$

```
Ar[0, 1] // S[sigma[2, 3], sigma[1, 3]]
```

$$\text{Ar}[0,1]+\text{Y}\left[0,1,3,-\frac{1}{x[1]}+\frac{e^{-x[1]}}{x[1]}\right]$$

```
Coefficient[
 Ar[0, 1] // S[sigma[1, 3], sigma[2, 3]],
 Y[0, 1, 3]
]
```

$$\frac{e^{-x[1]-x[2]}}{x[1]}-\frac{e^{-x[2]}}{x[1]}$$

```
bch = bch /. First[Solve[
    Coefficient[
      Ar[0, 1] // S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bch]]],
      Y[0, 1, 3]
     ] == Coefficient[
      Ar[0, 1] // S[sigma[1, 3], sigma[2, 3]],
      Y[0, 1, 3]
     ],
    bch
   ]]
```

$$\frac{-e^{x[1]}\,x[1]+e^{x[1]+x[2]}\,x[1]+x[2]-e^{x[1]}\,x[2]}{\left(-1+e^{x[1]+x[2]}\right)x[1]\,x[2]}$$

```
S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bch]]] /. Y[ijk__, h_] :> Y[ijk, Simplify[h]]
```

$$S\Big[Ar[0,1] \to Ar[0,1] + Y\Big[0,1,3,-\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big],$$

$$Ar[0,2] \to Ar[0,2] + Y\Big[0,2,3,\frac{-1+e^{-x[2]}}{x[2]}\Big],$$

$$Ar[0,3] \to Ar[0,3] + Y\Big[0,1,3,\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big] + Y\Big[0,2,3,\frac{1-e^{-x[2]}}{x[2]}\Big],$$

$$Ar[3,0] \to Ar[3,0]\left(\frac{x[1]}{\left(-1+e^{x[1]+x[2]}\right)^2(x[1]+x[2])} - \frac{2e^{x[1]+x[2]}x[1]}{\left(-1+e^{x[1]+x[2]}\right)^2(x[1]+x[2])} + \frac{e^{2x[1]+2x[2]}x[1]}{\left(-1+e^{x[1]+x[2]}\right)^2(x[1]+x[2])} + \frac{x[2]}{\left(-1+e^{x[1]+x[2]}\right)^2(x[1]+x[2])} - \frac{2e^{x[1]+x[2]}x[2]}{\left(-1+e^{x[1]+x[2]}\right)^2(x[1]+x[2])} + \frac{e^{2x[1]+2x[2]}x[2]}{\left(-1+e^{x[1]+x[2]}\right)^2(x[1]+x[2])}\right) +$$

$$Y\Big[1,2,0,-\frac{\left(-1+e^{x[1]}\right)\left(-1+e^{x[2]}\right)x[3]}{x[1]\,x[2]}\Big] + Y\Big[1,3,0,\frac{e^{x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big] + Y\Big[2,3,0,\frac{-1+e^{x[2]}}{x[2]}\Big]\Big]$$

```
S[sigma[1, 3], sigma[2, 3]] /. Y[ijk__, h_] :> Y[ijk, Simplify[h]]
```

$$S\Big[Ar[0,1] \to Ar[0,1] + Y\Big[0,1,3,-\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big],$$

$$Ar[0,2] \to Ar[0,2] + Y\Big[0,2,3,\frac{-1+e^{-x[2]}}{x[2]}\Big],$$

$$Ar[0,3] \to Ar[0,3] + Y\Big[0,1,3,\frac{e^{-x[1]-x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big] + Y\Big[0,2,3,\frac{1-e^{-x[2]}}{x[2]}\Big],$$

$$Ar[3,0] \to Ar[3,0] + Y\Big[1,2,0,-\frac{\left(-1+e^{x[1]}\right)\left(-1+e^{x[2]}\right)x[3]}{x[1]\,x[2]}\Big] +$$

$$Y\Big[1,3,0,\frac{e^{x[2]}\left(-1+e^{x[1]}\right)}{x[1]}\Big] + Y\Big[2,3,0,\frac{-1+e^{x[2]}}{x[2]}\Big]\Big]$$

```
Factor[
```

$$\left(\frac{x[1]}{\left(-1+e^{x[1]+x[2]}\right)^2(x[1]+x[2])} - \frac{2e^{x[1]+x[2]}x[1]}{\left(-1+e^{x[1]+x[2]}\right)^2(x[1]+x[2])} + \frac{e^{2x[1]+2x[2]}x[1]}{\left(-1+e^{x[1]+x[2]}\right)^2(x[1]+x[2])} + \frac{x[2]}{\left(-1+e^{x[1]+x[2]}\right)^2(x[1]+x[2])} - \frac{2e^{x[1]+x[2]}x[2]}{\left(-1+e^{x[1]+x[2]}\right)^2(x[1]+x[2])} + \frac{e^{2x[1]+2x[2]}x[2]}{\left(-1+e^{x[1]+x[2]}\right)^2(x[1]+x[2])}\right)]$$

```
1
```

- **Compare with Kurlin**

```
Simplify[(bch /. {x[1] → x, x[2] → y}) == (1 - (E^x - 1) (x + y) / x / (E^(x + y) - 1)) / y]
```

```
True
```