Pensieve Header: Solving the two F equations with perturbative hair, where F is written as an exponential. This time with a corrected F21 and with correct grading on the hair. Working, but needs cleaning.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\ScatterAndGlow"];
<< ScatterAndGlow.m
```

```
$CutoffDegree = 4;
```

```
f1coeffs = DeclareSeries[f1[x[1], x[2]], $CutoffDegree];
f2coeffs = DeclareSeries[f2[x[1], x[2]], $CutoffDegree];
PH[f1]
```

$$\mathrm{PH}\Big[\mathrm{f1}[0,0] + (\mathrm{f1}[1,0]\,x[1] + \mathrm{f1}[0,1]\,x[2])\,z + \left(\frac{1}{2}\mathrm{f1}[2,0]\,x[1]^2 + \mathrm{f1}[1,1]\,x[1]\,x[2] + \frac{1}{2}\mathrm{f1}[0,2]\,x[2]^2\right) z^2 + \left(\frac{1}{6}\mathrm{f1}[3,0]\,x[1]^3 + \frac{1}{2}\mathrm{f1}[2,1]\,x[1]^2\,x[2] + \frac{1}{2}\mathrm{f1}[1,2]\,x[1]\,x[2]^2 + \frac{1}{6}\mathrm{f1}[0,3]\,x[2]^3\right) z^3 + O[z]^4\Big]$$

```
F = S[Exp[a1 Ar[2, 1] + a2 Ar[1, 2] + Y[1, 2, 1, PH[f1]] + Y[1, 2, 2, PH[f2]]]];
F21 = S[Exp[(
       a1 Ar[1, 2] + a2 Ar[2, 1] + Y[2, 1, 2, PH[f1]] + Y[2, 1, 1, PH[f2]]
      ) /. {x[1] → x[2], x[2] → x[1]}
   ]];
```

```
Short[
 lhs1 = ToPH[$CutoffDegree + 2, Ar[3, 0] // S[Exp[Ar[1, 3] + Ar[2, 3]]] // F]
]
```

$$\mathrm{Ar}[3,0] + \ll 1 \gg + \ll 1 \gg + \mathrm{Y}\Big[2,3,0,\mathrm{PH}\Big[1 + \frac{1}{2}x[2]\,z + \frac{1}{6}x[2]^2 z^2 + \ll 1 \gg + \frac{1}{120}\ll 1 \gg^4 z^4 + \frac{1}{720}x[2]^5 z^5 + O[z]^6\Big]\Big]$$

```
rhs1 = ToPH[$CutoffDegree + 2, Ar[3, 0] // F // S[sigma[1, 3], sigma[2, 3]]]
```

$$\mathrm{Ar}[3,0] + \mathrm{Y}\Big[1,2,0,\mathrm{PH}\Big[-x[3]\,z + \left(-\frac{1}{2}x[1]\,x[3] - \frac{1}{2}x[2]\,x[3]\right) z^2 + \frac{1}{12}\left(-2x[1]^2 x[3] - 3x[1]\,x[2]\,x[3] - 2x[2]^2 x[3]\right) z^3 + \frac{1}{24}\left(-x[1]^3 x[3] - 2x[1]^2 x[2]\,x[3] - 2x[1]\,x[2]^2 x[3] - x[2]^3 x[3]\right) z^4 + \frac{1}{720}\left(-6x[1]^4 x[3] - 15x[1]^3 x[2]\,x[3] - 20x[1]^2 x[2]^2 x[3] - 15x[1]\,x[2]^3 x[3] - 6x[2]^4 x[3]\right) z^5 + O[z]^6\Big]\Big] + \mathrm{Y}\Big[1,3,0,\mathrm{PH}\Big[1 + \frac{1}{2}(x[1] + 2x[2])\,z + \frac{1}{6}\left(x[1]^2 + 3x[1]\,x[2] + 3x[2]^2\right) z^2 + \frac{1}{24}\left(x[1]^3 + 4x[1]^2 x[2] + 6x[1]\,x[2]^2 + 4x[2]^3\right) z^3 + \frac{1}{120}\left(x[1]^4 + 5x[1]^3 x[2] + 10x[1]^2 x[2]^2 + 10x[1]\,x[2]^3 + 5x[2]^4\right) z^4 + \frac{1}{720}\left(x[1]^5 + 6x[1]^4 x[2] + 15x[1]^3 x[2]^2 + 20x[1]^2 x[2]^3 + 15x[1]\,x[2]^4 + 6x[2]^5\right) z^5 + O[z]^6\Big]\Big] + \mathrm{Y}\Big[2,3,0,\mathrm{PH}\Big[1 + \frac{1}{2}x[2]\,z + \frac{1}{6}x[2]^2 z^2 + \frac{1}{24}x[2]^3 z^3 + \frac{1}{120}x[2]^4 z^4 + \frac{1}{720}x[2]^5 z^5 + O[z]^6\Big]\Big]$$

```
Short[
 lhs2 = ToPH[$CutoffDegree + 1,
   Ar[1, 0] // F21 // S[sigma[1, 2]] // S[Exp[1 / 2 Ar[1, 1]]] // S[Exp[1 / 2 Ar[2, 2]]]
  ]
]
```

$$\text{Ar}[1,0]+\text{Y}\left[1,2,0,\text{PH}\left[-\text{a2}+(\ll 1\gg)\,z+(\ll 1\gg)\,z^2+(\ll 1\gg)\,z^3+(\ll 1\gg)\,z^4+O[z]^5\right]\right]$$

```
Short[
 rhs2 = ToPH[$CutoffDegree + 1,
   Ar[1, 0] // S[Exp[Expand[1 / 2 (Ar[1, 1] + Ar[1, 2] + Ar[2, 1] + Ar[2, 2])]]] // F
  ]
]
```

$$\text{Ar}[1,0]+\text{Y}\left[1,2,0,\text{PH}\left[\left(-\frac{1}{2}-\text{a1}\right)+(\ll 1\gg)\,z+(\ll 1\gg)\,z^2+(\ll 1\gg)\,z^3+(\ll 1\gg)\,z^4+O[z]^5\right]\right]$$

```
Short[
 eqs = {
   Coefficient[lhs1, Y[1, 2, 0]] == Coefficient[rhs1, Y[1, 2, 0]],
   Coefficient[lhs2, Y[1, 2, 0]] == Coefficient[rhs2, Y[1, 2, 0]]
  }
]
```

$$\left\{\text{PH}\left[\left(-\frac{x[3]}{2}+\text{a1}\,x[3]-\text{a2}\,x[3]\right)z+\left(\frac{1}{2}\,\text{a1}\,x[1]\,x[3]+\ll 21\gg\right)z^2+\ll 1\gg+\ll 1\gg+(\ll 1\gg)\,z^5+O[z]^6\right]==\ll 1\gg,\ \ll 1\gg\right\}$$

```
sol = Union[HSolve[
   eqs,
   Join[{a1, a2}, f1coeffs, f2coeffs]
  ]]
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

$$\left\{\left\{\text{f1}[3,0]\to\frac{1}{240}\,\text{a2}\left(-1+\text{a2}^3\right),\right.\right.$$
$$\text{f1}[2,1]\to\frac{1}{34\,560}\left(29-120\,\text{a2}-408\,\text{a2}^2+192\,\text{a2}^3+288\,\text{a2}^4-11\,520\,\text{f2}[1,0]+\right.$$
$$\left.11\,520\,\text{a2}\,\text{f2}[1,0]-34\,560\,\text{f2}[1,1]-34\,560\,\text{f2}[1,2]\right),$$
$$\text{f1}[1,2]\to\frac{1}{34\,560}\left(29-120\,\text{a2}-408\,\text{a2}^2+192\,\text{a2}^3+288\,\text{a2}^4-11\,520\,\text{f2}[1,0]+\right.$$
$$\left.\left.11\,520\,\text{a2}\,\text{f2}[1,0]-34\,560\,\text{f2}[1,1]-34\,560\,\text{f2}[2,1]\right),\right.$$

$$\mathrm{f1}[0,3] \to \frac{1}{2880}\left(-4-16\,\mathrm{a2}-9\,\mathrm{a2}^2-12\,\mathrm{a2}^3+48\,\mathrm{a2}^4-1080\,\mathrm{f2}[1,0]+1440\,\mathrm{a2}\,\mathrm{f2}[1,0]-4320\,\mathrm{f2}[1,1]-8640\,\mathrm{f2}[1,2]\right),$$

$$\mathrm{f2}[3,0] \to \frac{1}{11\,520}\left(-5+40\,\mathrm{a2}+108\,\mathrm{a2}^2-48\,\mathrm{a2}^3-96\,\mathrm{a2}^4+4320\,\mathrm{f2}[1,0]-5760\,\mathrm{a2}\,\mathrm{f2}[1,0]+17\,280\,\mathrm{f2}[1,1]+34\,560\,\mathrm{f2}[1,2]\right),$$

$$\mathrm{f2}[0,3] \to \frac{-7+8\,\mathrm{a2}+24\,\mathrm{a2}^2-32\,\mathrm{a2}^3+16\,\mathrm{a2}^4}{3840},\ \mathrm{f1}[2,0] \to \frac{1}{360}\left(1-5\,\mathrm{a2}^2\right),$$

$$\mathrm{f1}[0,2] \to \frac{9-20\,\mathrm{a2}-10\,\mathrm{a2}^2-720\,\mathrm{f2}[1,0]-2880\,\mathrm{f2}[1,1]}{1440},$$

$$\mathrm{f1}[1,1] \to \frac{1}{192}\left(1-4\,\mathrm{a2}^2-96\,\mathrm{f2}[1,0]-192\,\mathrm{f2}[1,1]\right),$$

$$\mathrm{f2}[2,0] \to \frac{1}{720}\left(-2+5\,\mathrm{a2}^2+360\,\mathrm{f2}[1,0]+1440\,\mathrm{f2}[1,1]\right),$$

$$\mathrm{f2}[0,2] \to \frac{1-20\,\mathrm{a2}+20\,\mathrm{a2}^2}{1440},\ \mathrm{f1}[0,1] \to \frac{1}{96}\left(1+4\,\mathrm{a2}-8\,\mathrm{a2}^2-96\,\mathrm{f2}[1,0]\right),$$

$$\mathrm{f1}[1,0] \to \frac{1}{24}\left(\mathrm{a2}-\mathrm{a2}^2\right),\ \mathrm{f2}[0,1] \to \frac{1}{96}\left(1-4\,\mathrm{a2}^2\right),$$

$$\mathrm{f1}[0,0] \to \frac{1}{12}\,(-1+3\,\mathrm{a2}),\ \mathrm{f2}[0,0] \to \frac{1}{24}\,(-1+6\,\mathrm{a2}),\ \mathrm{a1} \to \frac{1}{2}\,(-1+2\,\mathrm{a2})\Big\},$$

$$\Big\{\mathrm{f1}[3,0] \to 0,\ \mathrm{f1}[2,1] \to \frac{29-11\,520\,\mathrm{f2}[1,0]-34\,560\,\mathrm{f2}[1,1]-34\,560\,\mathrm{f2}[1,2]}{34\,560},$$

$$\mathrm{f1}[1,2] \to \frac{29-11\,520\,\mathrm{f2}[1,0]-34\,560\,\mathrm{f2}[1,1]-34\,560\,\mathrm{f2}[2,1]}{34\,560},$$

$$\mathrm{f1}[0,3] \to \frac{1}{720}\,(-1-270\,\mathrm{f2}[1,0]-1080\,\mathrm{f2}[1,1]-2160\,\mathrm{f2}[1,2]),$$

$$\mathrm{f2}[3,0] \to \frac{-1+864\,\mathrm{f2}[1,0]+3456\,\mathrm{f2}[1,1]+6912\,\mathrm{f2}[1,2]}{2304},\ \mathrm{f2}[0,3] \to -\frac{7}{3840},$$

$$\mathrm{f1}[2,0] \to \frac{1}{360},\ \mathrm{f1}[0,2] \to \frac{1}{160}\,(1-80\,\mathrm{f2}[1,0]-320\,\mathrm{f2}[1,1]),$$

$$\mathrm{f1}[1,1] \to \frac{1}{192}\,(1-96\,\mathrm{f2}[1,0]-192\,\mathrm{f2}[1,1]),$$

$$\mathrm{f2}[2,0] \to \frac{1}{360}\,(-1+180\,\mathrm{f2}[1,0]+720\,\mathrm{f2}[1,1]),\ \mathrm{f2}[0,2] \to \frac{1}{1440},$$

$$\mathrm{f1}[0,1] \to \frac{1}{96}\,(1-96\,\mathrm{f2}[1,0]),\ \mathrm{f1}[1,0] \to 0,\ \mathrm{f2}[0,1] \to \frac{1}{96},$$

$$\mathrm{f1}[0,0] \to -\frac{1}{12},\ \mathrm{f2}[0,0] \to -\frac{1}{24},\ \mathrm{a1} \to -\frac{1}{2},\ \mathrm{a2} \to 0\Big\}\Big\}$$

Length[sol]

2

```
verify = CanonicalForm[ToAH[
   S[Exp[Ar[1, 3] + Ar[2, 3]]] ** F == F ** S[sigma[1, 3], sigma[2, 3]] /. sol
  ]]
```

A very large output was generated. Here is a sample of it:

$$\Big\{S[\ll 1\gg] == S[\ll 1\gg],\ S\Big[Ar[0,1] \to Ar[0,1] + Y\Big[0,1,1,AH\Big[\frac{x[2]^2\,(240+40\,x[1]-5760\,f2[1,0]\,x[1]+\ll 12\gg+3\,x[2]^2)}{11520}\Big]\Big] + Y\Big[0,1,2,AH\Big[\frac{x[2]\,(-2880+\ll 29\gg+18\,\ll 1\gg^3)}{69120}\Big]\Big] + Y[0,\ll 3\gg] + Y\Big[0,2,1,AH\Big[\frac{\ll 1\gg}{69120}\Big]\Big] + Y\Big[0,2,2,AH\Big[\frac{x[1]\,x[2]\,(\ll 1\gg)}{11520}\Big]\Big] + Y\Big[0,2,3,AH\Big[-\frac{x[1]\,(34560-11520\,x[1]+2880\,x[1]^2-576\,x[1]^3+\ll 32\gg+34560\,f2[1,2]\,x[1]\,x[2]^3+996\,x[2]^4)}{69120}\Big]\Big],\ \ll 5\gg\Big] == S[\ll 1\gg]\Big\}$$

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```
elements = Cases[verify, _Ar, Infinity] // Union
```

```
{Ar[0, 1], Ar[0, 2], Ar[0, 3], Ar[1, 0], Ar[2, 0], Ar[3, 0]}
```

```
ReducePrimitives[
    (# // S[Exp[Ar[1, 3] + Ar[2, 3]]] // (F /. sol[[1]])) -
     (# // (F /. sol[[1]]) // S[sigma[1, 3], sigma[2, 3]])
  ] & /@
 elements
```

$$\{Y[0,1,3,PH[O[z]^6]] + Y[0,2,3,PH[O[z]^6]] + Y[1,2,3,PH[O[z]^6]],\ Y[0,1,3,PH[O[z]^6]] + Y[0,2,3,PH[O[z]^6]] + Y[1,2,3,PH[O[z]^6]],\ Y[0,1,3,PH[O[z]^6]] + Y[0,2,3,PH[O[z]^6]] + Y[1,2,3,PH[O[z]^6]], 0, 0, Y[1,2,0,PH[O[z]^6]]\}$$

```
ReducePrimitives[
    (# // (F21 /. sol[[1]]) // S[sigma[1, 2]] //
        S[Exp[1 / 2 Ar[1, 1]]] // S[Exp[1 / 2 Ar[2, 2]]]) -
     (# // S[Exp[Expand[1 / 2 (Ar[1, 1] + Ar[1, 2] + Ar[2, 1] + Ar[2, 2])]]] // (F /. sol[[1]]))
  ] & /@
 elements
```

$$\{Y[0,1,1,PH[O[z]^5]] + Y[0,1,2,PH[O[z]^5]] + Y[0,2,1,PH[O[z]^5]] + Y[0,2,2,PH[O[z]^5]],\ Y[0,1,1,PH[O[z]^5]] + Y[0,1,2,PH[O[z]^5]] + Y[0,2,1,PH[O[z]^5]] + Y[0,2,2,PH[O[z]^5]],\ 0, Y[1,2,0,PH[O[z]^5]], Y[1,2,0,PH[O[z]^5]], 0\}$$