

$$t1 = S[\text{Exp}[\text{Ar}[1, 3] + \text{Ar}[2, 3] + \text{Y}[1, 2, 3, \text{bch}]]]$$

$$S[\text{Ar}[0, 1] \rightarrow \text{Ar}[0, 1] + \text{Y}[0, 1, 3, \text{PH}[-1 + \left(\frac{x[1]}{2} + x[2]\right) z +$$

$$\left(-\frac{1}{6}x[1]^2 - \frac{1}{2}x[1]x[2] - \frac{x[2]^2}{2}\right)z^2 + \left(\frac{x[1]^3}{24} + \frac{1}{6}x[1]^2x[2] + \frac{1}{4}x[1]x[2]^2 + \frac{x[2]^3}{6}\right)z^3 +$$

$$\left(-\frac{1}{120}x[1]^4 - \frac{1}{24}x[1]^3x[2] - \frac{1}{12}x[1]^2x[2]^2 - \frac{1}{12}x[1]x[2]^3 - \frac{x[2]^4}{24}\right)z^4 + O[z]^5],$$

$$\text{Ar}[0, 2] \rightarrow \text{Ar}[0, 2] + \text{Y}[0, 2, 3, \text{PH}[-1 + \frac{1}{2}x[2]z - \frac{1}{6}x[2]^2z^2 + \frac{1}{24}x[2]^3z^3 - \frac{1}{120}x[2]^4z^4 + O[z]^5]],$$

$$\text{Ar}[0, 3] \rightarrow \text{Ar}[0, 3] + \text{Y}[0, 1, 3, \text{PH}\left[1 + \left(-\frac{x[1]}{2} - x[2]\right)z +$$

$$\left(\frac{x[1]^2}{6} + \frac{1}{2}x[1]x[2] + \frac{x[2]^2}{2}\right)z^2 + \left(-\frac{1}{24}x[1]^3 - \frac{1}{6}x[1]^2x[2] - \frac{1}{4}x[1]x[2]^2 - \frac{x[2]^3}{6}\right)z^3 +$$

$$\left(\frac{x[1]^4}{120} + \frac{1}{24}x[1]^3x[2] + \frac{1}{12}x[1]^2x[2]^2 + \frac{1}{12}x[1]x[2]^3 + \frac{x[2]^4}{24}\right)z^4 + O[z]^5] +$$

$$\text{Y}[0, 2, 3, \text{PH}\left[1 - \frac{1}{2}x[2]z + \frac{1}{6}x[2]^2z^2 - \frac{1}{24}x[2]^3z^3 + \frac{1}{120}x[2]^4z^4 + O[z]^5\right]],$$

$$\text{Ar}[3, 0] \rightarrow \text{Ar}[3, 0] + \text{Y}[1, 2, 0, \text{PH}[-x[3]z + \left(-\frac{1}{2}x[1]x[3] - \frac{1}{2}x[2]x[3]\right)z^2 +$$

$$\left(-\frac{1}{6}x[1]^2x[3] - \frac{1}{4}x[1]x[2]x[3] - \frac{1}{6}x[2]^2x[3]\right)z^3 +$$

$$\left(-\frac{1}{24}x[1]^3x[3] - \frac{1}{12}x[1]^2x[2]x[3] - \frac{1}{12}x[1]x[2]^2x[3] - \frac{1}{24}x[2]^3x[3]\right)z^4 + O[z]^5] +$$

$$\text{Y}[1, 3, 0, \text{PH}\left[1 + \left(\frac{x[1]}{2} + x[2]\right)z + \left(\frac{x[1]^2}{6} + \frac{1}{2}x[1]x[2] + \frac{x[2]^2}{2}\right)z^2 +$$

$$\left(\frac{x[1]^3}{24} + \frac{1}{6}x[1]^2x[2] + \frac{1}{4}x[1]x[2]^2 + \frac{x[2]^3}{6}\right)z^3 +$$

$$\left(\frac{x[1]^4}{120} + \frac{1}{24}x[1]^3x[2] + \frac{1}{12}x[1]^2x[2]^2 + \frac{1}{12}x[1]x[2]^3 + \frac{x[2]^4}{24}\right)z^4 + O[z]^5] +$$

$$\text{Y}[2, 3, 0, \text{PH}\left[1 + \frac{1}{2}x[2]z + \frac{1}{6}x[2]^2z^2 + \frac{1}{24}x[2]^3z^3 + \frac{1}{120}x[2]^4z^4 + O[z]^5\right]]]$$

t2 = CanonicalForm[ToPH[5, S[sigma[1, 3], sigma[2, 3]]]

$$\begin{aligned}
& S\left[\text{Ar}[0, 1] \rightarrow \text{Ar}[0, 1] + Y\left[0, 1, 3, \text{PH}\left[-1 + \left(\frac{x[1]}{2} + x[2]\right) z + \right.\right.\right. \\
& \quad \left.\left.\left(-\frac{1}{6} x[1]^2 - \frac{1}{2} x[1] x[2] - \frac{x[2]^2}{2}\right) z^2 + \left(\frac{x[1]^3}{24} + \frac{1}{6} x[1]^2 x[2] + \frac{1}{4} x[1] x[2]^2 + \frac{x[2]^3}{6}\right) z^3 + \right.\right. \\
& \quad \left.\left.\left(-\frac{1}{120} x[1]^4 - \frac{1}{24} x[1]^3 x[2] - \frac{1}{12} x[1]^2 x[2]^2 - \frac{1}{12} x[1] x[2]^3 - \frac{x[2]^4}{24}\right) z^4 + O[z]^5\right]\right], \\
& \text{Ar}[0, 2] \rightarrow \text{Ar}[0, 2] + Y\left[0, 2, 3, \text{PH}\left[-1 + \frac{1}{2} x[2] z - \frac{1}{6} x[2]^2 z^2 + \frac{1}{24} x[2]^3 z^3 - \frac{1}{120} x[2]^4 z^4 + O[z]^5\right]\right], \\
& \text{Ar}[0, 3] \rightarrow \text{Ar}[0, 3] + Y\left[0, 1, 3, \text{PH}\left[1 + \left(-\frac{x[1]}{2} - x[2]\right) z + \right.\right. \\
& \quad \left.\left.\left(\frac{x[1]^2}{6} + \frac{1}{2} x[1] x[2] + \frac{x[2]^2}{2}\right) z^2 + \left(-\frac{1}{24} x[1]^3 - \frac{1}{6} x[1]^2 x[2] - \frac{1}{4} x[1] x[2]^2 - \frac{x[2]^3}{6}\right) z^3 + \right.\right. \\
& \quad \left.\left.\left(\frac{x[1]^4}{120} + \frac{1}{24} x[1]^3 x[2] + \frac{1}{12} x[1]^2 x[2]^2 + \frac{1}{12} x[1] x[2]^3 + \frac{x[2]^4}{24}\right) z^4 + O[z]^5\right]\right] + \\
& Y\left[0, 2, 3, \text{PH}\left[1 - \frac{1}{2} x[2] z + \frac{1}{6} x[2]^2 z^2 - \frac{1}{24} x[2]^3 z^3 + \frac{1}{120} x[2]^4 z^4 + O[z]^5\right]\right], \\
& \text{Ar}[3, 0] \rightarrow \text{Ar}[3, 0] + Y\left[1, 2, 0, \text{PH}\left[-x[3] z + \left(-\frac{1}{2} x[1] x[3] - \frac{1}{2} x[2] x[3]\right) z^2 + \right.\right. \\
& \quad \left.\left.\left(-\frac{1}{6} x[1]^2 x[3] - \frac{1}{4} x[1] x[2] x[3] - \frac{1}{6} x[2]^2 x[3]\right) z^3 + \right.\right. \\
& \quad \left.\left.\left(-\frac{1}{24} x[1]^3 x[3] - \frac{1}{12} x[1]^2 x[2] x[3] - \frac{1}{12} x[1] x[2]^2 x[3] - \frac{1}{24} x[2]^3 x[3]\right) z^4 + O[z]^5\right]\right] + \\
& Y\left[1, 3, 0, \text{PH}\left[1 + \left(\frac{x[1]}{2} + x[2]\right) z + \left(\frac{x[1]^2}{6} + \frac{1}{2} x[1] x[2] + \frac{x[2]^2}{2}\right) z^2 + \right.\right. \\
& \quad \left.\left.\left(\frac{x[1]^3}{24} + \frac{1}{6} x[1]^2 x[2] + \frac{1}{4} x[1] x[2]^2 + \frac{x[2]^3}{6}\right) z^3 + \right.\right. \\
& \quad \left.\left.\left(\frac{x[1]^4}{120} + \frac{1}{24} x[1]^3 x[2] + \frac{1}{12} x[1]^2 x[2]^2 + \frac{1}{12} x[1] x[2]^3 + \frac{x[2]^4}{24}\right) z^4 + O[z]^5\right]\right] + \\
& Y\left[2, 3, 0, \text{PH}\left[1 + \frac{1}{2} x[2] z + \frac{1}{6} x[2]^2 z^2 + \frac{1}{24} x[2]^3 z^3 + \frac{1}{120} x[2]^4 z^4 + O[z]^5\right]\right]
\end{aligned}$$

MapThread[SameQ, {List @@ t1, List @@ t2}]

{True, True, True, True}

Last[t1]

$$\begin{aligned}
& \text{Ar}[3, 0] \rightarrow \text{Ar}[3, 0] + \text{Y}[1, 2, 0, \text{PH}[\\
& \quad -x[3] z + \left(-\frac{1}{2} x[1] x[3] - \frac{1}{2} x[2] x[3]\right) z^2 + \left(-\frac{1}{6} x[1]^2 x[3] - \frac{1}{4} x[1] x[2] x[3] - \frac{1}{6} x[2]^2 x[3]\right) z^3 + \\
& \quad \left(-\frac{1}{24} x[1]^3 x[3] - \frac{1}{12} x[1]^2 x[2] x[3] - \frac{1}{12} x[1] x[2]^2 x[3] - \frac{1}{24} x[2]^3 x[3]\right) z^4 + \text{O}[z]^5] + \\
& \text{Y}[1, 3, 0, \text{PH}[1 + \left(\frac{x[1]}{2} + x[2]\right) z + \left(\frac{x[1]^2}{6} + \frac{1}{2} x[1] x[2] + \frac{x[2]^2}{2}\right) z^2 + \\
& \quad \left(\frac{x[1]^3}{24} + \frac{1}{6} x[1]^2 x[2] + \frac{1}{4} x[1] x[2]^2 + \frac{x[2]^3}{6}\right) z^3 + \\
& \quad \left(\frac{x[1]^4}{120} + \frac{1}{24} x[1]^3 x[2] + \frac{1}{12} x[1]^2 x[2]^2 + \frac{1}{12} x[1] x[2]^3 + \frac{x[2]^4}{24}\right) z^4 + \text{O}[z]^5] + \\
& \text{Y}[2, 3, 0, \text{PH}[1 + \frac{1}{2} x[2] z + \frac{1}{6} x[2]^2 z^2 + \frac{1}{24} x[2]^3 z^3 + \frac{1}{120} x[2]^4 z^4 + \text{O}[z]^5]]
\end{aligned}$$

Last[t2]

$$\begin{aligned}
& \text{Ar}[3, 0] \rightarrow \text{Ar}[3, 0] + \text{Y}[1, 2, 0, \text{PH}[\\
& \quad -x[3] z + \left(-\frac{1}{2} x[1] x[3] - \frac{1}{2} x[2] x[3]\right) z^2 + \left(-\frac{1}{6} x[1]^2 x[3] - \frac{1}{4} x[1] x[2] x[3] - \frac{1}{6} x[2]^2 x[3]\right) z^3 + \\
& \quad \left(-\frac{1}{24} x[1]^3 x[3] - \frac{1}{12} x[1]^2 x[2] x[3] - \frac{1}{12} x[1] x[2]^2 x[3] - \frac{1}{24} x[2]^3 x[3]\right) z^4 + \text{O}[z]^5] + \\
& \text{Y}[1, 3, 0, \text{PH}[1 + \left(\frac{x[1]}{2} + x[2]\right) z + \left(\frac{x[1]^2}{6} + \frac{1}{2} x[1] x[2] + \frac{x[2]^2}{2}\right) z^2 + \\
& \quad \left(\frac{x[1]^3}{24} + \frac{1}{6} x[1]^2 x[2] + \frac{1}{4} x[1] x[2]^2 + \frac{x[2]^3}{6}\right) z^3 + \\
& \quad \left(\frac{x[1]^4}{120} + \frac{1}{24} x[1]^3 x[2] + \frac{1}{12} x[1]^2 x[2]^2 + \frac{1}{12} x[1] x[2]^3 + \frac{x[2]^4}{24}\right) z^4 + \text{O}[z]^5] + \\
& \text{Y}[2, 3, 0, \text{PH}[1 + \frac{1}{2} x[2] z + \frac{1}{6} x[2]^2 z^2 + \frac{1}{24} x[2]^3 z^3 + \frac{1}{120} x[2]^4 z^4 + \text{O}[z]^5]]
\end{aligned}$$

unknowns = DeclareSeries[bc[x[1], x[2]], 1]

{bc[0, 0]}

drules = Sequence@@ Der[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, PH[bc]]]

Sequence[Ar[0, 1] → Y[0, 1, 3, PH[1 - bc[0, 0] x[2] z + O[z]^2]],
Ar[0, 2] → Y[0, 2, 3, PH[1 + bc[0, 0] x[1] z + O[z]^2]], Ar[0, 3] →
Y[0, 1, 3, PH[-1 + bc[0, 0] x[2] z + O[z]^2]] + Y[0, 2, 3, PH[-1 - bc[0, 0] x[1] z + O[z]^2]],
Ar[3, 0] → Y[1, 2, 0, PH[bc[0, 0] x[3] z + O[z]^2]] + Y[1, 3, 0, AH[-1]] + Y[2, 3, 0, AH[-1]]]

ins = First /@ {drules}

{Ar[0, 1], Ar[0, 2], Ar[0, 3], Ar[3, 0]}

```

k0 = Length[ins]

4

outs = {};
For[k = 1, k ≤ Length[ins], ++k,
  AppendTo[outs, newout = Der[drules][ins[[k]]]];
  ins = ins ~Join~ Complement[
    Union[Cases[{newout}, Y[ijk_, _] ⇒ Y[ijk, AH[1]], Infinity]],
    ins
  ]
];
--k

9

ins
{Ar[0, 1], Ar[0, 2], Ar[0, 3], Ar[3, 0], Y[0, 1, 3, AH[1]],
 Y[0, 2, 3, AH[1]], Y[1, 2, 0, AH[1]], Y[1, 3, 0, AH[1]], Y[2, 3, 0, AH[1]]}

outs
{Y[0, 1, 3, PH[1 - bc[0, 0] x[2] z + O[z]^2]], Y[0, 2, 3, PH[1 + bc[0, 0] x[1] z + O[z]^2]],
 Y[0, 1, 3, PH[-1 + bc[0, 0] x[2] z + O[z]^2]] + Y[0, 2, 3, PH[-1 - bc[0, 0] x[1] z + O[z]^2]],
 Y[1, 2, 0, PH[bc[0, 0] x[3] z + O[z]^2]] + Y[1, 3, 0, AH[-1]] + Y[2, 3, 0, AH[-1]],
 Y[0, 1, 3, PH[(x[1] + x[2]) z + O[z]^3]], Y[0, 2, 3, PH[(x[1] + x[2]) z + O[z]^3]], 0,
 Y[1, 2, 0, PH[x[3] z + bc[0, 0] x[1] x[3] z^2 + O[z]^3]] + Y[1, 3, 0, AH[-x[1] - x[2]]],
 Y[1, 2, 0, PH[bc[0, 0] x[2] x[3] z^2 + O[z]^3]] + Y[1, 3, 0, AH[-x[2]]] + Y[2, 3, 0, AH[-x[2]]]}

zero = Table[0, {k}];
e[{{i_}}] := ReplacePart[zero, 1, i];
mat = Replace[
  outs /. Y[ijk_, h_] ⇒ -h e[Position[ins, Y[ijk, AH[1]]]],
  0 → zero,
  {1}
]

{{0, 0, 0, 0, PH[-1 + bc[0, 0] x[2] z + O[z]^2], 0, 0, 0, 0},
 {0, 0, 0, 0, 0, PH[-1 - bc[0, 0] x[1] z + O[z]^2], 0, 0, 0},
 {0, 0, 0, 0, PH[1 - bc[0, 0] x[2] z + O[z]^2], PH[1 + bc[0, 0] x[1] z + O[z]^2], 0, 0, 0},
 {0, 0, 0, 0, 0, 0, PH[-bc[0, 0] x[3] z + O[z]^2], AH[1], AH[1]},
 {0, 0, 0, 0, PH[-x[1] - x[2] z + O[z]^3], 0, 0, 0, 0},
 {0, 0, 0, 0, 0, PH[-x[1] - x[2] z + O[z]^3], 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, PH[-x[3] z - bc[0, 0] x[1] x[3] z^2 + O[z]^3], AH[x[1] + x[2]], 0},
 {0, 0, 0, 0, 0, 0, PH[-bc[0, 0] x[2] x[3] z^2 + O[z]^3], AH[x[2]], AH[x[2]]}}

```

mat // MatrixForm

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \text{PH}[-1 + \text{bc}[0, 0] \text{x}[2] \text{z} + \text{O}[\text{z}]^2] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{PH}[-1 - \text{bc}[0, 0] \text{x}[1] \text{z} + \text{O}[\text{z}]^2] & 0 \\ 0 & 0 & 0 & 0 & \text{PH}[1 - \text{bc}[0, 0] \text{x}[2] \text{z} + \text{O}[\text{z}]^2] & \text{PH}[1 + \text{bc}[0, 0] \text{x}[1] \text{z} + \text{O}[\text{z}]^2] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \text{PH}[-\text{bc}[0, 0] \text{x}[3]] \\ 0 & 0 & 0 & 0 & \text{PH}[(-\text{x}[1] - \text{x}[2]) \text{z} + \text{O}[\text{z}]^3] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{PH}[(-\text{x}[1] - \text{x}[2]) \text{z} + \text{O}[\text{z}]^3] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \text{PH}[-\text{x}[3] \text{z} - \text{bc}[0, 0] \text{x}[1]] \\ 0 & 0 & 0 & 0 & 0 & 0 & \text{PH}[-\text{bc}[0, 0] \text{x}[2] \text{x}[3]] \end{pmatrix}$$

drules

Sequence[Ar[0, 1] → Y[0, 1, 3, PH[1 - bc[0, 0] x[2] z + O[z]²]],
 Ar[0, 2] → Y[0, 2, 3, PH[1 + bc[0, 0] x[1] z + O[z]²]], Ar[0, 3] →
 Y[0, 1, 3, PH[-1 + bc[0, 0] x[2] z + O[z]²]] + Y[0, 2, 3, PH[-1 - bc[0, 0] x[1] z + O[z]²]],
 Ar[3, 0] → Y[1, 2, 0, PH[bc[0, 0] x[3] z + O[z]²]] + Y[1, 3, 0, AH[-1]] + Y[2, 3, 0, AH[-1]]]

drules = Sequence[

Ar[3, 0] → Y[1, 2, 0, PH[bc[0, 0] x[3] z + O[z]²]] + Y[1, 3, 0, AH[-1]] + Y[2, 3, 0, AH[-1]]]

Sequence[

Ar[3, 0] → Y[1, 2, 0, PH[bc[0, 0] x[3] z + O[z]²]] + Y[1, 3, 0, AH[-1]] + Y[2, 3, 0, AH[-1]]]

Y[1, 3, 0, AH[1]] // Der[drules]

Y[1, 2, 0, PH[x[3] z + bc[0, 0] x[1] x[3] z² + O[z]³]] + Y[1, 3, 0, AH[-x[1] - x[2]]]

Table[

ReducePrimitives[

Der[ToPH[3, Y[1, 2, 3, AH[1]]]@Ar[i, j] + Der[Ar[i, j]]@Y[1, 2, 3, ToPH[3, PH[1]]]
], {i, 4}, {j, 4}

]

{Y[1, 2, 3, PH[O[z]³]], Y[1, 2, 3, PH[O[z]³]], Y[1, 2, 3, PH[O[z]³]], Y[1, 2, 3, PH[O[z]³]]},
 {Y[1, 2, 3, PH[O[z]³]], Y[1, 2, 3, PH[O[z]³]], Y[1, 2, 3, PH[O[z]³]], Y[1, 2, 3, PH[O[z]³]]},
 {Y[1, 2, 1, PH[O[z]⁴]] + Y[1, 2, 3, PH[O[z]³]] + Y[1, 3, 1, PH[O[z]⁵]] + Y[1, 3, 3, PH[O[z]⁴]],
 Y[1, 2, 2, PH[O[z]⁴]] + Y[1, 2, 3, PH[O[z]³]] + Y[1, 3, 2, PH[O[z]⁵]] +
 Y[1, 3, 3, PH[O[z]⁴]] + Y[2, 3, 2, PH[O[z]⁵]] + Y[2, 3, 3, PH[O[z]⁴]],
 Y[1, 2, 3, PH[O[z]³]] + Y[1, 3, 3, PH[O[z]⁴]] + Y[2, 3, 3, PH[O[z]⁴]],
 Y[1, 2, 3, PH[O[z]³]] + Y[1, 2, 4, PH[O[z]⁴]]}, {Y[1, 2, 3, PH[O[z]³]] + Y[1, 4, 3, PH[O[z]⁴]],
 Y[1, 2, 3, PH[O[z]³]] + Y[1, 4, 3, PH[O[z]⁴]] + Y[2, 4, 3, PH[O[z]⁴]],
 Y[1, 2, 3, PH[O[z]³]] + Y[1, 4, 3, PH[O[z]⁴]] + Y[2, 4, 3, PH[O[z]⁴]], Y[1, 2, 3, PH[O[z]³]]}

```

Table[ReducePrimitives[Der[ToPH[3, Y[1, 2, 3, AH[1]]][Ar[i, j]] +
  Der[Ar[i, j]][Y[1, 2, 3, ToPH[3, PH[1]]]]], {i, 4}, {j, 4}]
{
  {Y[1, 2, 3, PH[O[z]^3]], Y[1, 2, 3, PH[O[z]^3]], Y[1, 2, 3, PH[O[z]^3]], Y[1, 2, 3, PH[O[z]^3]]},
  {Y[1, 2, 3, PH[O[z]^3]], Y[1, 2, 3, PH[O[z]^3]], Y[1, 2, 3, PH[O[z]^3]], Y[1, 2, 3, PH[O[z]^3]]},
  {Y[1, 2, 1, PH[O[z]^4]] + Y[1, 2, 3, PH[O[z]^3]] + Y[1, 3, 1, PH[O[z]^5]] + Y[1, 3, 3, PH[O[z]^4]],
  Y[1, 2, 2, PH[O[z]^4]] + Y[1, 2, 3, PH[O[z]^3]] + Y[1, 3, 2, PH[O[z]^5]] +
  Y[1, 3, 3, PH[O[z]^4]] + Y[2, 3, 2, PH[O[z]^5]] + Y[2, 3, 3, PH[O[z]^4]],
  Y[1, 2, 3, PH[O[z]^3]] + Y[1, 3, 3, PH[O[z]^4]] + Y[2, 3, 3, PH[O[z]^4]],
  Y[1, 2, 3, PH[O[z]^3]] + Y[1, 2, 4, PH[O[z]^4]]}, {Y[1, 2, 3, PH[O[z]^3]] + Y[1, 4, 3, PH[O[z]^4]],
  Y[1, 2, 3, PH[O[z]^3]] + Y[1, 4, 3, PH[O[z]^4]] + Y[2, 4, 3, PH[O[z]^4]],
  Y[1, 2, 3, PH[O[z]^3]] + Y[1, 4, 3, PH[O[z]^4]] + Y[2, 4, 3, PH[O[z]^4]], Y[1, 2, 3, PH[O[z]^3]]}}

```