

Ar[0, 1] // S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, PH[bc]]]]

$$\begin{aligned}
& \text{Ar}\left[0, (1 + Y[0, 1, 3, \text{AH}[-1]]) + Y\left[0, 1, 3, \text{AH}\left[\text{bc}[0, 0] x[2] + \frac{1}{2} (x[1] + x[2])\right]\right] z + Y\left[0, 1, 3, \right. \\
& \quad \left. \text{AH}\left[-\frac{1}{6} (-x[1] - x[2])^2 + \text{bc}[1, 0] x[1] x[2] + \frac{1}{2} \text{bc}[0, 0] (-x[1] - x[2]) x[2] + \text{bc}[0, 1] x[2]^2\right]\right] \\
& \quad z^2 + Y\left[0, 1, 3, \text{AH}\left[\frac{1}{2} \text{bc}[2, 0] x[1]^2 x[2] + \frac{1}{6} \text{bc}[0, 0] (-x[1] - x[2])^2 x[2] + \right. \right. \\
& \quad \left. \left. \text{bc}[1, 1] x[1] x[2]^2 + \frac{1}{2} \text{bc}[0, 2] x[2]^3 + \frac{1}{24} (-x[1] - x[2])^2 (x[1] + x[2]) + \right. \right. \\
& \quad \left. \left. \frac{1}{2} (-x[1] - x[2]) (\text{bc}[1, 0] x[1] x[2] + \text{bc}[0, 1] x[2]^2)\right]\right] z^3 + \\
& \quad Y\left[0, 1, 3, \text{AH}\left[-\frac{1}{120} (-x[1] - x[2])^4 + \frac{1}{6} \text{bc}[3, 0] x[1]^3 x[2] + \frac{1}{24} \text{bc}[0, 0] (-x[1] - x[2])^3 x[2] + \right. \right. \\
& \quad \left. \left. \frac{1}{2} \text{bc}[2, 1] x[1]^2 x[2]^2 + \frac{1}{2} \text{bc}[1, 2] x[1] x[2]^3 + \frac{1}{6} \text{bc}[0, 3] x[2]^4 + \right. \right. \\
& \quad \left. \left. \frac{1}{6} (-x[1] - x[2])^2 (\text{bc}[1, 0] x[1] x[2] + \text{bc}[0, 1] x[2]^2) + \right. \right. \\
& \quad \left. \left. \frac{1}{2} (-x[1] - x[2]) \left(\frac{1}{2} \text{bc}[2, 0] x[1]^2 x[2] + \text{bc}[1, 1] x[1] x[2]^2 + \frac{1}{2} \text{bc}[0, 2] x[2]^3\right)\right]\right] z^4 + O[z]^5]
\end{aligned}$$

```
drules = Sequence @@Der[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, PH[bc]]]
```

```
Sequence[Ar[0, 1] → Y[0, 1, 3, PH[1 - bc[0, 0] x[2] z + (-bc[1, 0] x[1] x[2] - bc[0, 1] x[2]2) z2 +  

  (- $\frac{1}{2}$  bc[2, 0] x[1]2 x[2] - bc[1, 1] x[1] x[2]2 -  $\frac{1}{2}$  bc[0, 2] x[2]3) z3 + (- $\frac{1}{6}$  bc[3, 0] x[1]3 x[2] -  

   $\frac{1}{2}$  bc[2, 1] x[1]2 x[2]2 -  $\frac{1}{2}$  bc[1, 2] x[1] x[2]3 -  $\frac{1}{6}$  bc[0, 3] x[2]4) z4 + O[z]5],
```

```
Ar[0, 2] → Y[0, 2, 3, PH[1 + bc[0, 0] x[1] z + (bc[1, 0] x[1]2 + bc[0, 1] x[1] x[2]) z2 +  

  ( $\frac{1}{2}$  bc[2, 0] x[1]3 + bc[1, 1] x[1]2 x[2] +  $\frac{1}{2}$  bc[0, 2] x[1] x[2]2) z3 +  

  ( $\frac{1}{6}$  bc[3, 0] x[1]4 +  $\frac{1}{2}$  bc[2, 1] x[1]3 x[2] +  $\frac{1}{2}$  bc[1, 2] x[1]2 x[2]2 +  $\frac{1}{6}$  bc[0, 3] x[1] x[2]3) z4 +  

  O[z]5],
```

```
Ar[0, 3] → Y[0, 1, 3, PH[-1 + bc[0, 0] x[2] z + (bc[1, 0] x[1] x[2] + bc[0, 1] x[2]2) z2 +  

  ( $\frac{1}{2}$  bc[2, 0] x[1]2 x[2] + bc[1, 1] x[1] x[2]2 +  $\frac{1}{2}$  bc[0, 2] x[2]3) z3 + ( $\frac{1}{6}$  bc[3, 0] x[1]3 x[2] +  

   $\frac{1}{2}$  bc[2, 1] x[1]2 x[2]2 +  $\frac{1}{2}$  bc[1, 2] x[1] x[2]3 +  $\frac{1}{6}$  bc[0, 3] x[2]4) z4 + O[z]5]] +
```

```
Y[0, 2, 3, PH[-1 - bc[0, 0] x[1] z + (-bc[1, 0] x[1]2 - bc[0, 1] x[1] x[2]) z2 +  

  (- $\frac{1}{2}$  bc[2, 0] x[1]3 - bc[1, 1] x[1]2 x[2] -  $\frac{1}{2}$  bc[0, 2] x[1] x[2]2) z3 + (- $\frac{1}{6}$  bc[3, 0] x[1]4 -  

   $\frac{1}{2}$  bc[2, 1] x[1]3 x[2] -  $\frac{1}{2}$  bc[1, 2] x[1]2 x[2]2 -  $\frac{1}{6}$  bc[0, 3] x[1] x[2]3) z4 + O[z]5],
```

```
Ar[3, 0] → Y[1, 2, 0, PH[bc[0, 0] x[3] z + (bc[1, 0] x[1] x[3] + bc[0, 1] x[2] x[3]) z2 +  

  ( $\frac{1}{2}$  bc[2, 0] x[1]2 x[3] + bc[1, 1] x[1] x[2] x[3] +  $\frac{1}{2}$  bc[0, 2] x[2]2 x[3]) z3 +  

  ( $\frac{1}{6}$  bc[3, 0] x[1]3 x[3] +  $\frac{1}{2}$  bc[2, 1] x[1]2 x[2] x[3] +  $\frac{1}{2}$  bc[1, 2] x[1] x[2]2 x[3] +  

   $\frac{1}{6}$  bc[0, 3] x[2]3 x[3]) z4 + O[z]5]] + Y[1, 3, 0, AH[-1]] + Y[2, 3, 0, AH[-1]]]
```

```
ins = First /@ {drules}
```

```
{Ar[0, 1], Ar[0, 2], Ar[0, 3], Ar[3, 0]}
```

```
outs = {};
```

```
For[k = 1, k ≤ Length[ins], ++k,
```

```
  AppendTo[outs, newout = Der[drules][ins[[k]]];
```

```
  ins = ins ~Join~ Complement[
```

```
    Union[Cases[{newout}, Y[ijk_, _] → Y[ijk, AH[1], Infinity]],
```

```
    ins
```

```
  ]
```

```
];
```

```
--k
```

outs

$$\begin{aligned}
& \left\{ Y\left[0, 1, 3, \text{PH}\left[1 - \text{bc}[0, 0] x[2] z + \left(-\text{bc}[1, 0] x[1] x[2] - \text{bc}[0, 1] x[2]^2\right) z^2 + \right. \right. \right. \\
& \quad \left. \left. \left(-\frac{1}{2} \text{bc}[2, 0] x[1]^2 x[2] - \text{bc}[1, 1] x[1] x[2]^2 - \frac{1}{2} \text{bc}[0, 2] x[2]^3\right) z^3 + \right. \right. \\
& \quad \left. \left. \left(-\frac{1}{6} \text{bc}[3, 0] x[1]^3 x[2] - \frac{1}{2} \text{bc}[2, 1] x[1]^2 x[2]^2 - \frac{1}{2} \text{bc}[1, 2] x[1] x[2]^3 - \frac{1}{6} \text{bc}[0, 3] x[2]^4\right) z^4 + \right. \right. \\
& \quad \left. \left. O[z]^5\right]\right], Y\left[0, 2, 3, \text{PH}\left[1 + \text{bc}[0, 0] x[1] z + \left(\text{bc}[1, 0] x[1]^2 + \text{bc}[0, 1] x[1] x[2]\right) z^2 + \right. \right. \\
& \quad \left. \left. \left(\frac{1}{2} \text{bc}[2, 0] x[1]^3 + \text{bc}[1, 1] x[1]^2 x[2] + \frac{1}{2} \text{bc}[0, 2] x[1] x[2]^2\right) z^3 + \right. \right. \\
& \quad \left. \left. \left(\frac{1}{6} \text{bc}[3, 0] x[1]^4 + \frac{1}{2} \text{bc}[2, 1] x[1]^3 x[2] + \frac{1}{2} \text{bc}[1, 2] x[1]^2 x[2]^2 + \frac{1}{6} \text{bc}[0, 3] x[1] x[2]^3\right) z^4 + \right. \right. \\
& \quad \left. \left. O[z]^5\right]\right], Y\left[0, 1, 3, \text{PH}\left[-1 + \text{bc}[0, 0] x[2] z + \left(\text{bc}[1, 0] x[1] x[2] + \text{bc}[0, 1] x[2]^2\right) z^2 + \right. \right. \\
& \quad \left. \left. \left(\frac{1}{2} \text{bc}[2, 0] x[1]^2 x[2] + \text{bc}[1, 1] x[1] x[2]^2 + \frac{1}{2} \text{bc}[0, 2] x[2]^3\right) z^3 + \right. \right. \\
& \quad \left. \left. \left(\frac{1}{6} \text{bc}[3, 0] x[1]^3 x[2] + \frac{1}{2} \text{bc}[2, 1] x[1]^2 x[2]^2 + \frac{1}{2} \text{bc}[1, 2] x[1] x[2]^3 + \frac{1}{6} \text{bc}[0, 3] x[2]^4\right) z^4 + \right. \right. \\
& \quad \left. \left. O[z]^5\right]\right] + Y\left[0, 2, 3, \text{PH}\left[-1 - \text{bc}[0, 0] x[1] z + \left(-\text{bc}[1, 0] x[1]^2 - \text{bc}[0, 1] x[1] x[2]\right) z^2 + \right. \right. \\
& \quad \left. \left. \left(-\frac{1}{2} \text{bc}[2, 0] x[1]^3 - \text{bc}[1, 1] x[1]^2 x[2] - \frac{1}{2} \text{bc}[0, 2] x[1] x[2]^2\right) z^3 + \left(-\frac{1}{6} \text{bc}[3, 0] x[1]^4 - \right. \right. \\
& \quad \left. \left. \frac{1}{2} \text{bc}[2, 1] x[1]^3 x[2] - \frac{1}{2} \text{bc}[1, 2] x[1]^2 x[2]^2 - \frac{1}{6} \text{bc}[0, 3] x[1] x[2]^3\right) z^4 + O[z]^5\right]\right], \\
& Y\left[1, 2, 0, \text{PH}\left[\text{bc}[0, 0] x[3] z + \left(\text{bc}[1, 0] x[1] x[3] + \text{bc}[0, 1] x[2] x[3]\right) z^2 + \right. \right. \\
& \quad \left. \left. \left(\frac{1}{2} \text{bc}[2, 0] x[1]^2 x[3] + \text{bc}[1, 1] x[1] x[2] x[3] + \frac{1}{2} \text{bc}[0, 2] x[2]^2 x[3]\right) z^3 + \right. \right. \\
& \quad \left. \left. \left(\frac{1}{6} \text{bc}[3, 0] x[1]^3 x[3] + \frac{1}{2} \text{bc}[2, 1] x[1]^2 x[2] x[3] + \right. \right. \\
& \quad \left. \left. \frac{1}{2} \text{bc}[1, 2] x[1] x[2]^2 x[3] + \frac{1}{6} \text{bc}[0, 3] x[2]^3 x[3]\right) z^4 + O[z]^5\right]\right] + \\
& Y\left[1, 3, 0, \text{AH}[-1]\right] + Y\left[2, 3, 0, \text{AH}[-1]\right], Y\left[0, 1, 3, \text{PH}\left[(x[1] + x[2]) z + O[z]^6\right]\right], \\
& Y\left[0, 2, 3, \text{PH}\left[(x[1] + x[2]) z + O[z]^6\right]\right], \\
& 0, \\
& Y\left[1, 2, 0, \text{PH}\left[x[3] + \text{bc}[0, 0] x[1] x[3] z^2 + \left(\text{bc}[1, 0] x[1]^2 x[3] + \text{bc}[0, 1] x[1] x[2] x[3]\right) z^3 + \right. \right.
\end{aligned}$$


```
expmat = MatrixExp[hs, mat];
expmat // MatrixForm
```

$$\begin{pmatrix} 1 + O[z]^5 & 0 & 0 & 0 & -1 + (bc[0, 0] x[2] + \frac{1}{2} (x[1] + x[2])) z + (-\frac{1}{6} (-x[1] - x[2])^2 + \\ 0 & 1 + O[z]^5 & 0 & 0 & \\ 0 & 0 & 1 + O[z]^5 & 0 & 1 + (\frac{1}{2} (-x[1] - x[2]) - bc[0, 0] x[2]) z + (\frac{1}{6} (-x[1] - x[2] \\ 0 & 0 & 0 & 1 + O[z]^5 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \end{pmatrix}$$