

CanonicalForm[S[sigma[1, 2]]]

$$\begin{aligned} &S\left[\text{Ar}[0, 1] \rightarrow \text{Ar}[0, 1] + Y\left[0, 1, 2, \text{PH}\left[-1 + \frac{1}{2} x[1] z - \frac{1}{6} x[1]^2 z^2 + \frac{1}{24} x[1]^3 z^3 - \frac{1}{120} x[1]^4 z^4 + O[z]^5\right]\right], \right. \\ &\quad \text{Ar}[0, 2] \rightarrow \text{Ar}[0, 2] + Y\left[0, 1, 2, \text{PH}\left[1 - \frac{1}{2} x[1] z + \frac{1}{6} x[1]^2 z^2 - \frac{1}{24} x[1]^3 z^3 + \frac{1}{120} x[1]^4 z^4 + O[z]^5\right]\right], \\ &\quad \left. \text{Ar}[2, 0] \rightarrow \text{Ar}[2, 0] + Y\left[1, 2, 0, \text{PH}\left[1 + \frac{1}{2} x[1] z + \frac{1}{6} x[1]^2 z^2 + \frac{1}{24} x[1]^3 z^3 + \frac{1}{120} x[1]^4 z^4 + O[z]^5\right]\right]\right] \end{aligned}$$

H[1] \* Ar[1, 2]

Ar[1, 2]

drules = Sequence @@ Der[Ar[1, 2]]

Sequence[Ar[2, 0] → Y[1, 2, 0, PH[-1 + O[z]^5]],  
Ar[0, 1] → Y[0, 1, 2, PH[1 + O[z]^5]], Ar[0, 2] → Y[1, 0, 2, PH[1 + O[z]^5]]]

k0 = Length[ins = First /@ {drules}];

outs = {};

For[k = 1, k ≤ Length[ins], ++k,

AppendTo[outs, newout = Der[drules][ins[[k]]];

ins = ins ~Join~ Complement[

Union[Cases[{newout}, Y[ijk\_, hs[\_]] ⇒ Y[ijk, hs[1]], Infinity]],

ins

]

];

--k;

hs = First[Cases[outs, Y[\_, hs[\_]] ⇒ hs, Infinity, 1];

zero = Table[0, {k}];

e[{{i\_}}] := ReplacePart[zero, 1, i];

mat = Replace[

outs /. Y[ijk\_, hs[h\_]] ⇒ -h e[Position[ins, Y[ijk, hs[1]]],

0 → zero,

{1}

];

expmat = MatrixExp[hs, mat]

$$\begin{aligned} &\left\{\left\{\text{PH}[1], 0, 0, \text{PH}\left[1 + \frac{1}{2} x[1] z + \frac{1}{6} x[1]^2 z^2 + \frac{1}{24} x[1]^3 z^3 + \frac{1}{120} x[1]^4 z^4 + O[z]^5\right], 0\right\}, \right. \\ &\quad \left\{0, \text{PH}[1], 0, 0, \text{PH}\left[-1 + \frac{1}{2} x[1] z - \frac{1}{6} x[1]^2 z^2 + \frac{1}{24} x[1]^3 z^3 - \frac{1}{120} x[1]^4 z^4 + O[z]^5\right]\right\}, \\ &\quad \left\{0, 0, \text{PH}[1], 0, \text{PH}\left[1 - \frac{1}{2} x[1] z + \frac{1}{6} x[1]^2 z^2 - \frac{1}{24} x[1]^3 z^3 + \frac{1}{120} x[1]^4 z^4 + O[z]^5\right]\right\}, \\ &\quad \left\{0, 0, 0, \text{PH}\left[1 + x[1] z + \frac{1}{2} x[1]^2 z^2 + \frac{1}{6} x[1]^3 z^3 + \frac{1}{24} x[1]^4 z^4 + O[z]^5\right], 0\right\}, \\ &\quad \left. \left\{0, 0, 0, 0, \text{PH}\left[1 - x[1] z + \frac{1}{2} x[1]^2 z^2 - \frac{1}{6} x[1]^3 z^3 + \frac{1}{24} x[1]^4 z^4 + O[z]^5\right]\right\}\right\} \end{aligned}$$

**mat**

{Der[Sequece[Ar[2, 0] → {0, 1 + O[z]<sup>5</sup>}, Ar[0, 1] → -e[{}] + O[z]<sup>5</sup>, Ar[0, 2] → -e[{}] + O[z]<sup>5</sup>][  
 Ar[2, 0] → {0, 1 + O[z]<sup>5</sup>}],  
 Der[Sequece[Ar[2, 0] → {0, 1 + O[z]<sup>5</sup>}, Ar[0, 1] → -e[{}] + O[z]<sup>5</sup>, Ar[0, 2] → -e[{}] + O[z]<sup>5</sup>][  
 {0, -1}]]

**S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, bc]]]**

S[Ar[0, 1] → Ar[0, 1] + Y[0, 1, 3,

$$\text{PH}\left[-1 + \left(\text{bc } x[2] + \frac{1}{2} (x[1] + x[2])\right) z + \left(-\frac{1}{6} (-x[1] - x[2])^2 + \frac{1}{2} \text{bc} (-x[1] - x[2]) x[2]\right) z^2 + \left(\frac{1}{6} \text{bc} (-x[1] - x[2])^2 x[2] + \frac{1}{24} (-x[1] - x[2])^2 (x[1] + x[2])\right) z^3 + \left(-\frac{1}{120} (-x[1] - x[2])^4 + \frac{1}{24} \text{bc} (-x[1] - x[2])^3 x[2]\right) z^4 + O[z]^5\right],$$

Ar[0, 2] → Ar[0, 2] + Y[0, 2, 3, PH[-1 + \left(-\text{bc } x[1] + \frac{1}{2} (x[1] + x[2])\right) z +

$$\left(-\frac{1}{2} \text{bc } x[1] (-x[1] - x[2]) - \frac{1}{6} (-x[1] - x[2])^2\right) z^2 + \left(-\frac{1}{6} \text{bc } x[1] (-x[1] - x[2])^2 + \frac{1}{24} (-x[1] - x[2])^2 (x[1] + x[2])\right) z^3 + \left(-\frac{1}{24} \text{bc } x[1] (-x[1] - x[2])^3 - \frac{1}{120} (-x[1] - x[2])^4\right) z^4 + O[z]^5],$$

Ar[0, 3] → Ar[0, 3] + Y[0, 1, 3, PH[1 + \left(\frac{1}{2} (-x[1] - x[2]) - \text{bc } x[2]\right) z +

$$\left(\frac{1}{6} (-x[1] - x[2])^2 - \frac{1}{2} \text{bc} (-x[1] - x[2]) x[2]\right) z^2 + \left(\frac{1}{24} (-x[1] - x[2])^3 - \frac{1}{6} \text{bc} (-x[1] - x[2])^2 x[2]\right) z^3 + \left(\frac{1}{120} (-x[1] - x[2])^4 - \frac{1}{24} \text{bc} (-x[1] - x[2])^3 x[2]\right) z^4 + O[z]^5]] + Y[0, 2, 3,$$

$$\text{PH}\left[1 + \left(\text{bc } x[1] + \frac{1}{2} (-x[1] - x[2])\right) z + \left(\frac{1}{2} \text{bc } x[1] (-x[1] - x[2]) + \frac{1}{6} (-x[1] - x[2])^2\right) z^2 + \left(\frac{1}{6} \text{bc } x[1] (-x[1] - x[2])^2 + \frac{1}{24} (-x[1] - x[2])^3\right) z^3 + \right]$$

$$\begin{aligned}
& \left( \frac{1}{24} bc x[1] (-x[1] - x[2])^3 + \frac{1}{120} (-x[1] - x[2])^4 \right) z^4 + O[z^5] \Big], \\
\text{Ar}[3, 0] & \rightarrow \text{Ar}[3, 0] + Y \left[ 1, 2, 0, \text{PH} \left[ \left( -\frac{x[3]}{2} - bc x[3] \right) z + \right. \right. \\
& \left. \left( \frac{1}{2} (-bc x[1] x[3] - bc x[2] x[3]) + \frac{1}{6} (-x[2] x[3] - (x[1] + x[2]) x[3]) \right) z^2 + \right. \\
& \left. \left( \frac{1}{6} (-bc x[1] x[2] x[3] - bc x[2]^2 x[3] - bc x[1] (x[1] + x[2]) x[3]) + \right. \right. \\
& \left. \left. \frac{1}{24} (-x[2]^2 x[3] - (x[1] + x[2]) (x[1] + 2x[2]) x[3]) \right) z^3 + \right. \\
& \left. \left( \frac{1}{120} (-x[2]^3 x[3] - (x[1] + x[2])^3 x[3] - (x[1] + x[2]) (x[2]^2 + x[2] (x[1] + x[2])) x[3]) + \right. \right. \\
& \left. \left. \frac{1}{24} (-bc x[1] (x[1] + x[2]) (x[1] + 2x[2]) x[3] + \right. \right. \\
& \left. \left. x[2] (-bc x[1] x[2] x[3] - bc x[2]^2 x[3]) \right) z^4 + O[z^5] \right] + \\
Y[1, 3, 0, \text{PH} & \left[ 1 + \frac{1}{2} (x[1] + 2x[2]) z + \frac{1}{6} (x[2]^2 + x[2] (x[1] + x[2]) + (x[1] + x[2])^2) z^2 + \right. \\
& \frac{1}{24} ((x[1] + x[2])^2 (x[1] + 2x[2]) + x[2] (x[2]^2 + x[2] (x[1] + x[2]))) z^3 + \\
& \frac{1}{120} ((x[1] + x[2])^4 + x[2]^2 (x[2]^2 + x[2] (x[1] + x[2]))) + \\
& \left. \left. (x[1] + x[2])^2 (x[2]^2 + x[2] (x[1] + x[2])) \right) z^4 + O[z^5] \right] + \\
Y[2, 3, 0, \text{PH} & \left[ 1 + \frac{1}{2} x[2] z + \frac{1}{6} x[2]^2 z^2 + \frac{1}{24} x[2]^3 z^3 + \frac{1}{120} x[2]^4 z^4 + O[z^5] \right] \Big]
\end{aligned}$$

**H[bc[x[1], x[2]]]**

$$\begin{aligned}
\text{PH} & \left[ bc[0, 0] + (x[2] bc^{(0,1)}[0, 0] + x[1] bc^{(1,0)}[0, 0]) z + \right. \\
& \frac{1}{2} (x[2]^2 bc^{(0,2)}[0, 0] + 2x[1] x[2] bc^{(1,1)}[0, 0] + x[1]^2 bc^{(2,0)}[0, 0]) z^2 + \\
& \frac{1}{6} (x[2]^3 bc^{(0,3)}[0, 0] + 3x[1] x[2]^2 bc^{(1,2)}[0, 0] + 3x[1]^2 x[2] bc^{(2,1)}[0, 0] + x[1]^3 bc^{(3,0)}[0, 0]) \\
& z^3 + \frac{1}{24} (x[2]^4 bc^{(0,4)}[0, 0] + 4x[1] x[2]^3 bc^{(1,3)}[0, 0] + 6x[1]^2 x[2]^2 bc^{(2,2)}[0, 0] + \\
& \left. 4x[1]^3 x[2] bc^{(3,1)}[0, 0] + x[1]^4 bc^{(4,0)}[0, 0]) z^4 + O[z^5] \right]
\end{aligned}$$

$$\begin{aligned}
& \text{lhs} = \text{First@CanonicalForm}[\text{PH}[\text{Coefficient} [ \\
& \quad \text{Ar}[0, 1] // \text{S}[\text{Exp}[\text{Ar}[1, 3] + \text{Ar}[2, 3] + \text{Y}[1, 2, 3, \text{H}[\text{bc}[\text{x}[1], \text{x}[2]]]]], \\
& \quad \text{Y}[0, 1, 3] \\
& \quad ]]] \\
& -1 + \left( \frac{\text{x}[1]}{2} + \frac{\text{x}[2]}{2} + \text{bc}[0, 0] \text{x}[2] \right) \text{z} + \\
& \left( -\frac{1}{6} \text{x}[1]^2 - \frac{1}{3} \text{x}[1] \text{x}[2] - \frac{1}{2} \text{bc}[0, 0] \text{x}[1] \text{x}[2] - \frac{\text{x}[2]^2}{6} - \frac{1}{2} \text{bc}[0, 0] \text{x}[2]^2 + \right. \\
& \quad \left. \text{x}[2]^2 \text{bc}^{(0,1)}[0, 0] + \text{x}[1] \text{x}[2] \text{bc}^{(1,0)}[0, 0] \right) \text{z}^2 + \\
& \left( \frac{\text{x}[1]^3}{24} + \frac{1}{8} \text{x}[1]^2 \text{x}[2] + \frac{1}{6} \text{bc}[0, 0] \text{x}[1]^2 \text{x}[2] + \frac{1}{8} \text{x}[1] \text{x}[2]^2 + \frac{1}{3} \text{bc}[0, 0] \text{x}[1] \text{x}[2]^2 + \right. \\
& \quad \frac{\text{x}[2]^3}{24} + \frac{1}{6} \text{bc}[0, 0] \text{x}[2]^3 - \frac{1}{2} \text{x}[1] \text{x}[2]^2 \text{bc}^{(0,1)}[0, 0] - \frac{1}{2} \text{x}[2]^3 \text{bc}^{(0,1)}[0, 0] + \\
& \quad \frac{1}{2} \text{x}[2]^3 \text{bc}^{(0,2)}[0, 0] - \frac{1}{2} \text{x}[1]^2 \text{x}[2] \text{bc}^{(1,0)}[0, 0] - \frac{1}{2} \text{x}[1] \text{x}[2]^2 \text{bc}^{(1,0)}[0, 0] + \\
& \quad \left. \text{x}[1] \text{x}[2]^2 \text{bc}^{(1,1)}[0, 0] + \frac{1}{2} \text{x}[1]^2 \text{x}[2] \text{bc}^{(2,0)}[0, 0] \right) \text{z}^3 + \\
& \left( -\frac{1}{120} \text{x}[1]^4 - \frac{1}{30} \text{x}[1]^3 \text{x}[2] - \frac{1}{24} \text{bc}[0, 0] \text{x}[1]^3 \text{x}[2] - \frac{1}{20} \text{x}[1]^2 \text{x}[2]^2 - \frac{1}{8} \text{bc}[0, 0] \text{x}[1]^2 \text{x}[2]^2 - \right. \\
& \quad \frac{1}{30} \text{x}[1] \text{x}[2]^3 - \frac{1}{8} \text{bc}[0, 0] \text{x}[1] \text{x}[2]^3 - \frac{\text{x}[2]^4}{120} - \frac{1}{24} \text{bc}[0, 0] \text{x}[2]^4 + \frac{1}{6} \text{x}[1]^2 \text{x}[2]^2 \text{bc}^{(0,1)}[0, 0] + \\
& \quad \frac{1}{3} \text{x}[1] \text{x}[2]^3 \text{bc}^{(0,1)}[0, 0] + \frac{1}{6} \text{x}[2]^4 \text{bc}^{(0,1)}[0, 0] - \frac{1}{4} \text{x}[1] \text{x}[2]^3 \text{bc}^{(0,2)}[0, 0] - \\
& \quad \frac{1}{4} \text{x}[2]^4 \text{bc}^{(0,2)}[0, 0] + \frac{1}{6} \text{x}[2]^4 \text{bc}^{(0,3)}[0, 0] + \frac{1}{6} \text{x}[1]^3 \text{x}[2] \text{bc}^{(1,0)}[0, 0] + \\
& \quad \frac{1}{3} \text{x}[1]^2 \text{x}[2]^2 \text{bc}^{(1,0)}[0, 0] + \frac{1}{6} \text{x}[1] \text{x}[2]^3 \text{bc}^{(1,0)}[0, 0] - \frac{1}{2} \text{x}[1]^2 \text{x}[2]^2 \text{bc}^{(1,1)}[0, 0] - \\
& \quad \frac{1}{2} \text{x}[1] \text{x}[2]^3 \text{bc}^{(1,1)}[0, 0] + \frac{1}{2} \text{x}[1] \text{x}[2]^3 \text{bc}^{(1,2)}[0, 0] - \frac{1}{4} \text{x}[1]^3 \text{x}[2] \text{bc}^{(2,0)}[0, 0] - \\
& \quad \left. \frac{1}{4} \text{x}[1]^2 \text{x}[2]^2 \text{bc}^{(2,0)}[0, 0] + \frac{1}{2} \text{x}[1]^2 \text{x}[2]^2 \text{bc}^{(2,1)}[0, 0] + \frac{1}{6} \text{x}[1]^3 \text{x}[2] \text{bc}^{(3,0)}[0, 0] \right) \text{z}^4 + \text{O}[\text{z}]^5
\end{aligned}$$

$$\begin{aligned}
& \text{rhs} = \text{First@CanonicalForm}[\text{PH}[\text{Coefficient} [ \\
& \quad \text{Ar}[0, 1] // \text{S}[\text{sigma}[1, 3], \text{sigma}[2, 3]], \\
& \quad \text{Y}[0, 1, 3] \\
& \quad ]]] \\
& -1 + \left( \frac{\text{x}[1]}{2} + \text{x}[2] \right) \text{z} + \left( -\frac{1}{6} \text{x}[1]^2 - \frac{1}{2} \text{x}[1] \text{x}[2] - \frac{\text{x}[2]^2}{2} \right) \text{z}^2 + \\
& \left( \frac{\text{x}[1]^3}{24} + \frac{1}{6} \text{x}[1]^2 \text{x}[2] + \frac{1}{4} \text{x}[1] \text{x}[2]^2 + \frac{\text{x}[2]^3}{6} \right) \text{z}^3 + \\
& \left( -\frac{1}{120} \text{x}[1]^4 - \frac{1}{24} \text{x}[1]^3 \text{x}[2] - \frac{1}{12} \text{x}[1]^2 \text{x}[2]^2 - \frac{1}{12} \text{x}[1] \text{x}[2]^3 - \frac{\text{x}[2]^4}{24} \right) \text{z}^4 + \text{O}[\text{z}]^5
\end{aligned}$$

**Solve**[lhs == rhs, bc]

{ {} }

**PH**[f] = **CanonicalForm**[**PH**[  
 (f[x[1], x[2]] /. x[i\_] => z \* x[i]) + O[z]^\$CutoffDegree  
 ]]

**PH**[f[0, 0] + (x[2] f<sup>(0,1)</sup>[0, 0] + x[1] f<sup>(1,0)</sup>[0, 0]) z +  
 (  $\frac{1}{2}$  x[2]<sup>2</sup> f<sup>(0,2)</sup>[0, 0] + x[1] x[2] f<sup>(1,1)</sup>[0, 0] +  $\frac{1}{2}$  x[1]<sup>2</sup> f<sup>(2,0)</sup>[0, 0] ) z<sup>2</sup> +  
 (  $\frac{1}{6}$  x[2]<sup>3</sup> f<sup>(0,3)</sup>[0, 0] +  $\frac{1}{2}$  x[1] x[2]<sup>2</sup> f<sup>(1,2)</sup>[0, 0] +  $\frac{1}{2}$  x[1]<sup>2</sup> x[2] f<sup>(2,1)</sup>[0, 0] +  $\frac{1}{6}$  x[1]<sup>3</sup> f<sup>(3,0)</sup>[0, 0] )  
 z<sup>3</sup> + (  $\frac{1}{24}$  x[2]<sup>4</sup> f<sup>(0,4)</sup>[0, 0] +  $\frac{1}{6}$  x[1] x[2]<sup>3</sup> f<sup>(1,3)</sup>[0, 0] +  $\frac{1}{4}$  x[1]<sup>2</sup> x[2]<sup>2</sup> f<sup>(2,2)</sup>[0, 0] +  
 $\frac{1}{6}$  x[1]<sup>3</sup> x[2] f<sup>(3,1)</sup>[0, 0] +  $\frac{1}{24}$  x[1]<sup>4</sup> f<sup>(4,0)</sup>[0, 0] ) z<sup>4</sup> + O[z]<sup>5</sup>]

**PH**[f] = **PH**[f] /. **Derivative**[ds\_\_\_][f][0...] => f[ds]

**PH**[  
 f[0, 0] + (f[1, 0] x[1] + f[0, 1] x[2]) z + (  $\frac{1}{2}$  f[2, 0] x[1]<sup>2</sup> + f[1, 1] x[1] x[2] +  $\frac{1}{2}$  f[0, 2] x[2]<sup>2</sup> ) z<sup>2</sup> +  
 (  $\frac{1}{6}$  f[3, 0] x[1]<sup>3</sup> +  $\frac{1}{2}$  f[2, 1] x[1]<sup>2</sup> x[2] +  $\frac{1}{2}$  f[1, 2] x[1] x[2]<sup>2</sup> +  $\frac{1}{6}$  f[0, 3] x[2]<sup>3</sup> ) z<sup>3</sup> +  
 (  $\frac{1}{24}$  f[4, 0] x[1]<sup>4</sup> +  $\frac{1}{6}$  f[3, 1] x[1]<sup>3</sup> x[2] +  $\frac{1}{4}$  f[2, 2] x[1]<sup>2</sup> x[2]<sup>2</sup> +  
 $\frac{1}{6}$  f[1, 3] x[1] x[2]<sup>3</sup> +  $\frac{1}{24}$  f[0, 4] x[2]<sup>4</sup> ) z<sup>4</sup> + O[z]<sup>5</sup>]

**Cases**[**PH**[f], f[\_\_\_], Infinity]

{f[0, 0], f[1, 0], f[0, 1], f[2, 0], f[1, 1], f[0, 2], f[3, 0],  
 f[2, 1], f[1, 2], f[0, 3], f[4, 0], f[3, 1], f[2, 2], f[1, 3], f[0, 4]}

**unknowns** = **DeclareSeries**[bc[x[1], x[2]], \$CutoffDegree - 1]

{bc[0, 0], bc[1, 0], bc[0, 1], bc[2, 0],  
 bc[1, 1], bc[0, 2], bc[3, 0], bc[2, 1], bc[1, 2], bc[0, 3]}

**PH**[bc]

**PH**[bc[0, 0] + (bc[1, 0] x[1] + bc[0, 1] x[2]) z +  
 (  $\frac{1}{2}$  bc[2, 0] x[1]<sup>2</sup> + bc[1, 1] x[1] x[2] +  $\frac{1}{2}$  bc[0, 2] x[2]<sup>2</sup> ) z<sup>2</sup> +  
 (  $\frac{1}{6}$  bc[3, 0] x[1]<sup>3</sup> +  $\frac{1}{2}$  bc[2, 1] x[1]<sup>2</sup> x[2] +  $\frac{1}{2}$  bc[1, 2] x[1] x[2]<sup>2</sup> +  $\frac{1}{6}$  bc[0, 3] x[2]<sup>3</sup> ) z<sup>3</sup> + O[z]<sup>4</sup>]

Coefficient[

Ar[0, 1] // S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, H[bc[x[1], x[2]]]]],  
Y[0, 1, 3]

]

$$\begin{aligned}
 & -1 + \left( bc[0, 0] x[2] + \frac{1}{2} (x[1] + x[2]) \right) z + \\
 & \left( -\frac{1}{6} (-x[1] - x[2])^2 + \frac{1}{2} bc[0, 0] (-x[1] - x[2]) x[2] + x[2]^2 bc^{(0,1)}[0, 0] + x[1] x[2] bc^{(1,0)}[0, 0] \right) \\
 & z^2 + \left( \frac{1}{6} bc[0, 0] (-x[1] - x[2])^2 x[2] + \frac{1}{24} (-x[1] - x[2])^2 (x[1] + x[2]) + \right. \\
 & \left. \frac{1}{2} x[2]^3 bc^{(0,2)}[0, 0] + \frac{1}{2} (-x[1] - x[2]) (x[2]^2 bc^{(0,1)}[0, 0] + x[1] x[2] bc^{(1,0)}[0, 0]) + \right. \\
 & \left. x[1] x[2]^2 bc^{(1,1)}[0, 0] + \frac{1}{2} x[1]^2 x[2] bc^{(2,0)}[0, 0] \right) z^3 + \\
 & \left( -\frac{1}{120} (-x[1] - x[2])^4 + \frac{1}{24} bc[0, 0] (-x[1] - x[2])^3 x[2] + \frac{1}{6} x[2]^4 bc^{(0,3)}[0, 0] + \right. \\
 & \left. \frac{1}{6} (-x[1] - x[2])^2 (x[2]^2 bc^{(0,1)}[0, 0] + x[1] x[2] bc^{(1,0)}[0, 0]) + \frac{1}{2} x[1] x[2]^3 bc^{(1,2)}[0, 0] + \right. \\
 & \left. \frac{1}{2} (-x[1] - x[2]) \left( \frac{1}{2} x[2]^3 bc^{(0,2)}[0, 0] + x[1] x[2]^2 bc^{(1,1)}[0, 0] + \frac{1}{2} x[1]^2 x[2] bc^{(2,0)}[0, 0] \right) + \right. \\
 & \left. \frac{1}{2} x[1]^2 x[2]^2 bc^{(2,1)}[0, 0] + \frac{1}{6} x[1]^3 x[2] bc^{(3,0)}[0, 0] \right) z^4 + O[z]^5
 \end{aligned}$$

```

eq = Coefficient[
  Ar[0, 1] // S[Exp[Ar[1, 3] + Ar[2, 3] + Y[1, 2, 3, PH[bc]]]],
  Y[0, 1, 3]
] == Coefficient[
  Ar[0, 1] // S[sigma[1, 3], sigma[2, 3]],
  Y[0, 1, 3]
]

```

$$\begin{aligned}
& -1 + \left( bc[0, 0] x[2] + \frac{1}{2} (x[1] + x[2]) \right) z + \\
& \left( -\frac{1}{6} (-x[1] - x[2])^2 + bc[1, 0] x[1] x[2] + \frac{1}{2} bc[0, 0] (-x[1] - x[2]) x[2] + bc[0, 1] x[2]^2 \right) z^2 + \\
& \left( \frac{1}{2} bc[2, 0] x[1]^2 x[2] + \frac{1}{6} bc[0, 0] (-x[1] - x[2])^2 x[2] + bc[1, 1] x[1] x[2]^2 + \frac{1}{2} bc[0, 2] x[2]^3 + \right. \\
& \quad \left. \frac{1}{24} (-x[1] - x[2])^2 (x[1] + x[2]) + \frac{1}{2} (-x[1] - x[2]) (bc[1, 0] x[1] x[2] + bc[0, 1] x[2]^2) \right) z^3 + \\
& \left( -\frac{1}{120} (-x[1] - x[2])^4 + \frac{1}{6} bc[3, 0] x[1]^3 x[2] + \frac{1}{24} bc[0, 0] (-x[1] - x[2])^3 x[2] + \right. \\
& \quad \frac{1}{2} bc[2, 1] x[1]^2 x[2]^2 + \frac{1}{2} bc[1, 2] x[1] x[2]^3 + \frac{1}{6} bc[0, 3] x[2]^4 + \\
& \quad \frac{1}{6} (-x[1] - x[2])^2 (bc[1, 0] x[1] x[2] + bc[0, 1] x[2]^2) + \\
& \quad \left. \frac{1}{2} (-x[1] - x[2]) \left( \frac{1}{2} bc[2, 0] x[1]^2 x[2] + bc[1, 1] x[1] x[2]^2 + \frac{1}{2} bc[0, 2] x[2]^3 \right) \right) z^4 + O[z]^5 = \\
& -1 + \left( \frac{x[1]}{2} + x[2] \right) z + \left( -\frac{1}{6} x[1]^2 - \frac{1}{2} x[1] x[2] - \frac{x[2]^2}{2} \right) z^2 + \\
& \left( \frac{x[1]^3}{24} + \frac{1}{6} x[1]^2 x[2] + \frac{1}{4} x[1] x[2]^2 + \frac{x[2]^3}{6} \right) z^3 + \\
& \left( -\frac{1}{120} x[1]^4 - \frac{1}{24} x[1]^3 x[2] - \frac{1}{12} x[1]^2 x[2]^2 - \frac{1}{12} x[1] x[2]^3 - \frac{x[2]^4}{24} \right) z^4 + O[z]^5
\end{aligned}$$

```
eq1 = Series[Normal[eq] /. z -> 1, Sequence @@ ({#, 0, $CutoffDegree} & /@ {x[1], x[2]})]
```

$$\begin{aligned} & \left( -1 + \left( \frac{1}{2} + bc[0, 0] \right) x[2] + \left( -\frac{1}{6} - \frac{1}{2} bc[0, 0] + bc[0, 1] \right) x[2]^2 + \right. \\ & \quad \left( \frac{1}{24} + \frac{1}{6} bc[0, 0] - \frac{1}{2} bc[0, 1] + \frac{1}{2} bc[0, 2] \right) x[2]^3 + \\ & \quad \left( -\frac{1}{120} - \frac{1}{24} bc[0, 0] + \frac{1}{6} bc[0, 1] - \frac{1}{4} bc[0, 2] + \frac{1}{6} bc[0, 3] \right) x[2]^4 + O[x[2]]^6 \Big) + \\ & \left( \frac{1}{2} + \left( -\frac{1}{3} - \frac{1}{2} bc[0, 0] + bc[1, 0] \right) x[2] + \left( \frac{1}{8} + \frac{1}{3} bc[0, 0] - \frac{1}{2} bc[0, 1] - \frac{1}{2} bc[1, 0] + bc[1, 1] \right) \right. \\ & \quad \left. x[2]^2 + \left( -\frac{1}{30} - \frac{1}{8} bc[0, 0] + \frac{1}{3} bc[0, 1] - \frac{1}{4} bc[0, 2] + \frac{1}{6} bc[1, 0] - \frac{1}{2} bc[1, 1] + \frac{1}{2} bc[1, 2] \right) \right. \\ & \quad \left. x[2]^3 + O[x[2]]^6 \right) x[1] + \\ & \left( -\frac{1}{6} + \left( \frac{1}{8} + \frac{1}{6} bc[0, 0] - \frac{1}{2} bc[1, 0] + \frac{1}{2} bc[2, 0] \right) x[2] + \left( -\frac{1}{20} - \frac{1}{8} bc[0, 0] + \frac{1}{6} bc[0, 1] + \right. \right. \\ & \quad \left. \left. \frac{1}{3} bc[1, 0] - \frac{1}{2} bc[1, 1] - \frac{1}{4} bc[2, 0] + \frac{1}{2} bc[2, 1] \right) x[2]^2 + O[x[2]]^6 \right) x[1]^2 + \\ & \left( \frac{1}{24} + \left( -\frac{1}{30} - \frac{1}{24} bc[0, 0] + \frac{1}{6} bc[1, 0] - \frac{1}{4} bc[2, 0] + \frac{1}{6} bc[3, 0] \right) x[2] + O[x[2]]^6 \right) x[1]^3 - \\ & \frac{x[1]^4}{120} + O[x[1]]^6 = \end{aligned}$$

$$\begin{aligned} & \left( -1 + x[2] - \frac{x[2]^2}{2} + \frac{x[2]^3}{6} - \frac{x[2]^4}{24} + O[x[2]]^6 \right) + \\ & \left( \frac{1}{2} - \frac{x[2]}{2} + \frac{x[2]^2}{4} - \frac{x[2]^3}{12} + O[x[2]]^6 \right) x[1] + \\ & \left( -\frac{1}{6} + \frac{x[2]}{6} - \frac{x[2]^2}{12} + O[x[2]]^6 \right) x[1]^2 + \\ & \left( \frac{1}{24} - \frac{x[2]}{24} + O[x[2]]^6 \right) x[1]^3 - \frac{x[1]^4}{120} + O[x[1]]^6 \end{aligned}$$

```
sol = First[Solve[eq1, unknowns]]
```

$$\left\{ bc[0, 0] \rightarrow \frac{1}{2}, bc[1, 0] \rightarrow \frac{1}{12}, bc[0, 1] \rightarrow -\frac{1}{12}, bc[2, 0] \rightarrow 0, bc[1, 1] \rightarrow -\frac{1}{24}, \right. \\ \left. bc[0, 2] \rightarrow 0, bc[3, 0] \rightarrow -\frac{1}{120}, bc[2, 1] \rightarrow -\frac{1}{90}, bc[1, 2] \rightarrow \frac{1}{90}, bc[0, 3] \rightarrow \frac{1}{120} \right\}$$

```
unknowns /. sol
```

$$\left\{ \frac{1}{2}, \frac{1}{12}, -\frac{1}{12}, 0, -\frac{1}{24}, 0, -\frac{1}{120}, -\frac{1}{90}, \frac{1}{90}, \frac{1}{120} \right\}$$



**bch = PH[bc] /. sol**

$$\text{PH}\left[\frac{1}{2} + \left(\frac{x[1]}{12} - \frac{x[2]}{12}\right) z - \frac{1}{24} (x[1] x[2]) z^2 + \left(-\frac{1}{720} x[1]^3 - \frac{1}{180} x[1]^2 x[2] + \frac{1}{180} x[1] x[2]^2 + \frac{x[2]^3}{720}\right) z^3 + O[z]^4\right]$$

$$\left(\left(\frac{1}{y} \left(1 - \frac{e^x - 1}{x} \frac{x+y}{e^{x+y} - 1}\right)\right)\right) /. \{x \rightarrow z x, y \rightarrow z y\} + O[z]^{\text{\$CutoffDegree} - 1}$$

$$\frac{1}{2} + \frac{1}{12} (x - y) z - \frac{1}{24} (x y) z^2 + \frac{1}{720} (-x^3 - 4 x^2 y + 4 x y^2 + y^3) z^3 + O[z]^4$$

**(bch /. {x[1] → x, x[2] → y})**

$$\text{PH}\left[\frac{1}{2} + \left(\frac{x}{12} - \frac{y}{12}\right) z - \frac{1}{24} (x y) z^2 + \left(-\frac{x^3}{720} - \frac{x^2 y}{180} + \frac{x y^2}{180} + \frac{y^3}{720}\right) z^3 + O[z]^4\right]$$