```
drules = Sequence @@ Der[a1 Ar[2, 1] + a2 Ar[1, 2]]
```

```
Sequence[Ar[0, 1] → Y[0, 1, 2, a2] + Y[2, 0, 1, a1],
 Ar[0, 2] → Y[0, 2, 1, a1] + Y[1, 0, 2, a2], Ar[1, 0] → Y[2, 1, 0, -a1], Ar[2, 0] → Y[1, 2, 0, -a2]]
```

```
k0 = Length[ins = First /@ {drules}]
```

```
4
```

```
ins
```

```
{Ar[0, 1], Ar[0, 2], Ar[1, 0], Ar[2, 0]}
```

```
k0 = Length[ins = Take[ins, 1]]
```

```
1
```

```
outs = {};
For[k = 1, k ≤ Length[ins], ++k,
  AppendTo[outs, newout = Der[drules][ins[[k]]]];
  ins = ins ~Join~ Complement[
      Union[Cases[{newout}, Y[ijk___, _] :↦ Y[ijk, 1], Infinity]],
      ins
    ]
 ];
--k;
```

```
Print /@ Thread[ins → outs];
```

```
Ar[0, 1] → Y[0, 1, 2, a2] + Y[0, 2, 1, -a1]

Y[0, 1, 2, 1] → Y[0, 1, 1, -a1 x[2]] + Y[0, 1, 2, a2 x[1] - a1 x[2]] + Y[0, 2, 2, a1 x[1]]

Y[0, 2, 1, 1] → Y[0, 1, 1, a2 x[2]] + Y[0, 2, 1, -a2 x[1] + a1 x[2]] + Y[0, 2, 2, -a2 x[1]]

Y[0, 1, 1, 1] → Y[0, 1, 2, -a2 x[1]] + Y[0, 2, 1, a1 x[1]]

Y[0, 2, 2, 1] → Y[0, 1, 2, a2 x[2]] + Y[0, 2, 1, -a1 x[2]]
```

```
Print /@ (Thread[
    Table[s[i], {i, Length[ins]}] == (ins /. Y[ijk__, _] :↦ Y[ijk])
   ]);
```

```
s[1] == Ar[0, 1]

s[2] == Y[0, 1, 2]

s[3] == Y[0, 2, 1]

s[4] == Y[0, 1, 1]

s[5] == Y[0, 2, 2]
```

```
Print /@ (Thread[ins → outs] /. Y[ijk__, h_] :↦ h Y[ijk] /. Thread[
     (ins /. Y[ijk__, _] :↦ Y[ijk]) → Table[s[i], {i, Length[ins]}]
    ]);
```

$$s[1] \to a2\, s[2] - a1\, s[3]$$

$$s[2] \to a1\, s[5]\, x[1] - a1\, s[4]\, x[2] + s[2]\,(a2\, x[1] - a1\, x[2])$$

$$s[3] \to -a2\, s[5]\, x[1] + a2\, s[4]\, x[2] + s[3]\,(-a2\, x[1] + a1\, x[2])$$

$$s[4] \to -a2\, s[2]\, x[1] + a1\, s[3]\, x[1]$$

$$s[5] \to a2\, s[2]\, x[2] - a1\, s[3]\, x[2]$$

```
zero = Table[0, {k}];
e[{{i_}}] := ReplacePart[zero, 1, i];
mat = Replace[
   outs /. Y[ijk__, h_] :> -h e[Position[ins, Y[ijk, 1]]],
   0 → zero,
   {1}
  ];
```

```
mat // MatrixForm
```

$$\begin{pmatrix} 0 & -a2 & a1 & 0 & 0 \\ 0 & -a2\, x[1] + a1\, x[2] & 0 & a1\, x[2] & -a1\, x[1] \\ 0 & 0 & a2\, x[1] - a1\, x[2] & -a2\, x[2] & a2\, x[1] \\ 0 & a2\, x[1] & -a1\, x[1] & 0 & 0 \\ 0 & -a2\, x[2] & a1\, x[2] & 0 & 0 \end{pmatrix}$$

```
ReducePrimitives[Take[SH[MatrixExp[mat /. {a1 → 1, a2 → 1}]].ins, k0]]
```

$$\left\{Ar[0, 1] + Y\left[0, 1, 1, -\frac{e^{-\sqrt{(x[1]+x[2])^2}}\left(-1 + e^{\sqrt{(x[1]+x[2])^2}}\right)^2 x[2]}{(x[1] + x[2])^2}\right] + \right.$$

$$Y\left[0, 1, 2, \frac{1}{2\,(x[1] + x[2])^2}\, e^{-\sqrt{(x[1]+x[2])^2}}\left(-1 + e^{\sqrt{(x[1]+x[2])^2}}\right)\left(-x[1] + e^{\sqrt{(x[1]+x[2])^2}}\, x[1] + \right.\right.$$

$$\left.\left. x[2] - e^{\sqrt{(x[1]+x[2])^2}}\, x[2] - \sqrt{(x[1] + x[2])^2} - e^{\sqrt{(x[1]+x[2])^2}}\sqrt{(x[1] + x[2])^2}\right)\right] +$$

$$Y\left[0, 2, 1, \frac{1}{2\,(x[1] + x[2])^2}\, e^{-\sqrt{(x[1]+x[2])^2}}\left(-1 + e^{\sqrt{(x[1]+x[2])^2}}\right)\left(-x[1] + e^{\sqrt{(x[1]+x[2])^2}}\, x[1] + \right.\right.$$

$$\left.\left. x[2] - e^{\sqrt{(x[1]+x[2])^2}}\, x[2] + \sqrt{(x[1] + x[2])^2} + e^{\sqrt{(x[1]+x[2])^2}}\sqrt{(x[1] + x[2])^2}\right)\right] +$$

$$\left. Y\left[0, 2, 2, \frac{e^{-\sqrt{(x[1]+x[2])^2}}\left(-1 + e^{\sqrt{(x[1]+x[2])^2}}\right)^2 x[1]}{(x[1] + x[2])^2}\right]\right\}$$

```
expmat = SH[PowerExpand[MatrixExp[mat]]];
```

```
Union[Cases[expmat, Sqrt[y_] :> y, Infinity]]
```

$$\left\{(a2\, x[1] + a1\, x[2])^2\right\}$$

```
PowerExpand[Eigenvectors[Transpose[mat]]] // Simplify // Transpose // MatrixForm
```

$$\begin{pmatrix} -x[2] & x[1] & -\frac{a2\,x[1]}{a1}+x[2] & 0 & 0 \\ 0 & 0 & \frac{a2}{a1} & \frac{a2}{a1} & -\frac{x[2]}{x[1]} \\ 0 & 0 & 1 & -\frac{a1\,x[2]}{a2\,x[1]} & 1 \\ 0 & 1 & 0 & -\frac{x[2]}{x[1]} & -\frac{x[2]}{x[1]} \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$