Scatter and Glow

By Dror Bar-Natan, Kerene Chu, Zsuzsanna Dancso, Peter Lee and Louis Leung, December 2008.

Project goals

- Verify R3, OC, Locallity, R2, CC on scatter level.
- Same, on glow level.
- Recover the Alexander polynomial of all knots.
- Recover the multi-variable Alexander polynomial of all links.
- Implement strand deletion and doubling.
- Find an explicit BCH formula.
- The scatter and glow of an arbitrary exponential.
- Solve R4 for F at the scatter level.
- Verify the pentagon.
- Solve for F at the glow level.
- Check the Hexagons.
- Solve the \theta-R-F equation.
- Verify the Hexagons.
- Recover the Lieberum formulas.

Conventions

- a_{ij} = `Ar[i,j]` is an arrow going from i to j.
- Y_{ijk} = **`Y[i,j,k]`** $:= [a_{ik}, a_{jk}] = a_{ik} a_{jk} - a_{jk} a_{ik} = \mathrm{Ad}(a_{ik})(a_{jk}) = -[a_{ij}, a_{jk}]$.
- $x_l Y_{ijk} := [a_{lk}, Y_{ijk}]$.
- IHX: $x_i Y_{jkl} + x_j Y_{kil} + x_k Y_{ijl} = 0$.

Program

```
ReducePrimitives[prims_] := Module[{l, h0, h1}, prims
      //. {
        Y[i_, i_, __] → 0,
        Y[i_, j_, k_, h_] /; i > j ⧴ Y[j, i, k, Expand[-h]],
        c_ * Y[i_, j_, k_, h_] ⧴ Y[i, j, k, Expand[c * h]],
        Y[i_, j_, k_, h1_] + Y[i_, j_, k_, h2_] ⧴ Y[i, j, k, h1 + h2],
        Y[i_, j_, k_, h_] /; ! FreeQ[h, x[l_] /; l < i] ⧴ (
          l = Min[Cases[{h}, x[l_] ⧴ l, Infinity]];
          h0 = Limit[h, x[l] → 0];
          h1 = Expand[(h - h0) / x[l]];
          Y[i, j, k, h0]
           - Y[j, l, k, Expand[h1 x[i]]] - Y[l, i, k, Expand[h1 * x[j]]]
        )
      }
     /. Y[i_, j_, k_, h_] ⧴ Y[i, j, k, Expand[h]]
    /. Y[__, 0] → 0
  ];

S[sigma[i_, j_]] := S[
   Ar[0, j] → Ar[0, j] + Y[0, i, j, -(Exp[-x[i]] - 1) / x[i]],
   Ar[0, i] → Ar[0, i] + Y[0, i, j, (Exp[-x[i]] - 1) / x[i]],
   Ar[j, 0] → Ar[j, 0] + Y[i, j, 0, (Exp[x[i]] - 1) / x[i]]
  ];

S[srules__Rule][prims_] := ReducePrimitives[prims
      /. {
        Ar[i_, j_] ⧴ Distribute[Ar[Ar[i, 0] /. {srules}, Ar[0, j] /. {srules}]],
        Y[i_, j_, k_, h_] ⧴ Distribute[Y[
            Ar[i, 0] , Ar[j, 0], Ar[0, k], h
           ] /. {srules}]
      }
     /. {Ar[i_, 0] ⧴ i, Ar[0, j_] ⧴ j}
    /. {
     Ar[Y[i_, j_, 0, h_], k_] ⧴ Y[i, j, k, h],
     Y[_Y, _Y, __] ⧴ 0,
     Y[i_Integer, Y[j_, k_, 0, h_], l_, h1_] ⧴ Y[j, k, l, x[i] * h * h1],
     Y[Y[j_, k_, 0, h_], i_Integer, l_, h1_] ⧴ Y[j, k, l, -x[i] * h * h1]
    }
   /. {
    Ar[i_, Y[0, j_, k_, h_]] ⧴ Y[i, j, k, h],
    Ar[i_, Y[j_, 0, k_, h_]] ⧴ Y[j, i, k, h],
    Y[i_, j_, Y[0, k_, l_, h_], h1_] ⧴ Y[i, j, l, -x[k] * h * h1],
    Y[i_, j_, Y[k_, 0, l_, h_], h1_] ⧴ Y[i, j, l, x[k] * h * h1]
   }
 ]
```

right action ✓

improve. ✓

split the Ar line from the Y line.

came out uglier

Testing

- **The braid group on two strands is commutative :**

```
S[sigma[1, 2]][Ar[1, 2]]
```

right action ✓

```
Ar[1, 2]
```

- **Locallity in Scale (global over local)**

```
Ar[1, 2] // S[sigma[3, 1]] // S[sigma[3, 2]]
```

```
Ar[1, 2]
```

```
Ar[2, 1] // S[sigma[3, 1]] // S[sigma[3, 2]]
```

```
Ar[2, 1]
```

- **Overcrossings Commute**

```
oc1 = {Ar[1, 4], Ar[2, 4], Ar[3, 4], Ar[4, 1], Ar[4, 2], Ar[4, 3], Ar[1, 2],
    Y[1, 2, 3, 1], Y[2, 3, 1, 1], Y[3, 1, 2, 1]} // S[sigma[1, 2]] // S[sigma[1, 3]]
```

$$\Big\{\text{Ar}[1,4],\ \text{Ar}[2,4]+\text{Y}\Big[1,2,4,-\frac{1}{\text{x}[1]}+\frac{e^{\text{x}[1]}}{\text{x}[1]}\Big],\ \text{Ar}[3,4]+\text{Y}\Big[1,3,4,-\frac{1}{\text{x}[1]}+\frac{e^{\text{x}[1]}}{\text{x}[1]}\Big],$$
$$\text{Ar}[4,1]+\text{Y}\Big[1,4,2,\frac{1}{\text{x}[1]}-\frac{e^{-\text{x}[1]}}{\text{x}[1]}\Big]+\text{Y}\Big[1,4,3,\frac{1}{\text{x}[1]}-\frac{e^{-\text{x}[1]}}{\text{x}[1]}\Big],$$
$$\text{Ar}[4,2]+\text{Y}\Big[1,4,2,-\frac{1}{\text{x}[1]}+\frac{e^{-\text{x}[1]}}{\text{x}[1]}\Big],\ \text{Ar}[4,3]+\text{Y}\Big[1,4,3,-\frac{1}{\text{x}[1]}+\frac{e^{-\text{x}[1]}}{\text{x}[1]}\Big],\ \text{Ar}[1,2],$$
$$\text{Y}[1,2,3,1],\ \text{Y}\Big[1,2,1,\frac{\text{x}[3]}{\text{x}[1]}-\frac{e^{\text{x}[1]}\,\text{x}[3]}{\text{x}[1]}\Big]+\text{Y}\Big[1,2,2,\frac{\text{x}[3]}{\text{x}[1]}-\frac{e^{\text{x}[1]}\,\text{x}[3]}{\text{x}[1]}\Big]+$$
$$\text{Y}\Big[1,2,3,\frac{\text{x}[3]}{\text{x}[1]}-\frac{e^{\text{x}[1]}\,\text{x}[3]}{\text{x}[1]}\Big]+\text{Y}\Big[1,3,1,-\frac{\text{x}[2]}{\text{x}[1]}+\frac{e^{\text{x}[1]}\,\text{x}[2]}{\text{x}[1]}\Big]+$$
$$\text{Y}\Big[1,3,2,-\frac{\text{x}[2]}{\text{x}[1]}+\frac{e^{\text{x}[1]}\,\text{x}[2]}{\text{x}[1]}\Big]+\text{Y}\Big[1,3,3,-\frac{\text{x}[2]}{\text{x}[1]}+\frac{e^{\text{x}[1]}\,\text{x}[2]}{\text{x}[1]}\Big]+\text{Y}[2,3,1,1],\ \text{Y}[1,3,2,-1]\Big\}$$

```
oc2 = {Ar[1, 4], Ar[2, 4], Ar[3, 4], Ar[4, 1], Ar[4, 2], Ar[4, 3], Ar[1, 2],
     Y[1, 2, 3, 1], Y[2, 3, 1, 1], Y[3, 1, 2, 1]} // S[sigma[1, 3]] // S[sigma[1, 2]];
Thread[
 oc1 ==
  oc2]
```

```
True
```

■ Reidemeister 3

```
r31 = {Ar[1, 4], Ar[2, 4], Ar[3, 4], Ar[4, 1], Ar[4, 2], Ar[4, 3]} // S[sigma[1, 2]] //
   S[sigma[1, 3]] // S[sigma[2, 3]]
```

$$
\begin{aligned}
&\Big\{\mathrm{Ar}[1,4],\ \mathrm{Ar}[2,4]+\mathrm{Y}\Big[1,2,4,-\frac{1}{x[1]}+\frac{e^{x[1]}}{x[1]}\Big],\\
&\mathrm{Ar}[3,4]+\mathrm{Y}\Big[1,2,4,-\frac{x[3]}{x[1]\,x[2]}+\frac{e^{x[1]}\,x[3]}{x[1]\,x[2]}+\frac{e^{x[2]}\,x[3]}{x[1]\,x[2]}-\frac{e^{x[1]+x[2]}\,x[3]}{x[1]\,x[2]}\Big]+\\
&\quad \mathrm{Y}\Big[1,3,4,-\frac{e^{x[2]}}{x[1]}+\frac{e^{x[1]+x[2]}}{x[1]}\Big]+\mathrm{Y}\Big[2,3,4,-\frac{1}{x[2]}+\frac{e^{x[2]}}{x[2]}\Big],\\
&\mathrm{Ar}[4,1]+\mathrm{Y}\Big[1,4,2,\frac{1}{x[1]}-\frac{e^{-x[1]}}{x[1]}\Big]+\mathrm{Y}\Big[1,4,3,\frac{1}{x[1]}-\frac{e^{-x[1]}}{x[1]}\Big],\\
&\mathrm{Ar}[4,2]+\mathrm{Y}\Big[1,4,2,-\frac{1}{x[1]}+\frac{e^{-x[1]}}{x[1]}\Big]+\mathrm{Y}\Big[1,4,3,-\frac{1}{x[1]}+\frac{e^{-x[1]}}{x[1]}-\frac{e^{-x[1]-x[2]}}{x[1]}+\frac{e^{-x[2]}}{x[1]}\Big]+\\
&\quad \mathrm{Y}\Big[2,4,3,\frac{1}{x[2]}-\frac{e^{-x[2]}}{x[2]}\Big],\ \mathrm{Ar}[4,3]+\mathrm{Y}\Big[1,4,3,\frac{e^{-x[1]-x[2]}}{x[1]}-\frac{e^{-x[2]}}{x[1]}\Big]+\mathrm{Y}\Big[2,4,3,-\frac{1}{x[2]}+\frac{e^{-x[2]}}{x[2]}\Big]\Big\}
\end{aligned}
$$

```
r32 = {Ar[1, 4], Ar[2, 4], Ar[3, 4], Ar[4, 1], Ar[4, 2], Ar[4, 3]} // S[sigma[2, 3]] //
    S[sigma[1, 3]] // S[sigma[1, 2]];
ReducePrimitives[
 r31 -
  r32]
```

```
{0, 0, 0, 0, 0, 0}
```