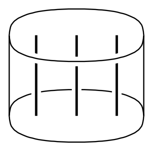
Emergent Knots in a Pole Dancing Studio — SMRI Visit Proposal for 2025

It's a bit tough to write a research proposal for research that will take place more than a year from now. In the optimistic scenario, we will finish everything that's on our desks right now, and be ready to sail on into the horizon. In the pessimistic scenario, we will still be struggling with the details of a paper we started 10 years ago. In the most realistic scenario, we will be pushing our understanding of **emergent knots** (and other knotted objects: links, braids, tangles, etc.) in a **pole dancing studio**.



What's a **pole dancing studio**? It's a big room (namely, a disk cross an interval) with a few stationary poles (namely, straight vertical lines going from floor to ceiling) removed. A picture is on the right, and in dryer language, it is simply a punctured disk cross an interval. Let's call the pole dancing studio with n poles PDS_n.

What's an **emergent knot**? It's an object living in a quotient of knot theory in which almost all knottedness is killed, and only a tiny bit remains. Precisely, for the strands of the knotted object itself (not for the poles), we mod out by the relation $\doublepoint\doublepoint = 0$, where $\doublepoint := \overcrossing - \undercrossing$ (namely, a knotted object with two double points is set to 0, where a double point is a mnemonic for the difference between an over crossing and an under crossing). Had we been moding out by the stronger relation $\doublepoint = 0$, we would be declaring that an over crossing is equal to an under crossing, and that kills all knottedness. We are not that cruel, and we let a whiff of knottedness survive. Let us call the space of emergent knots $\mathcal{K}_1$.

Collapsed knots, namely knots modulo $\doublepoint = 0$, are simply homotopy classes of curves in PDS_n. If we project them to the floor of the studio we lose nothing and we see curves in a punctured disk, namely, elements of the free group FG_n (at this level of detail we are ignoring base points). The free group has a commutative algebra analog, the free associative algebra FA_n, and there is the so-called Magnus expansion $Z_0\colon FG_n \to FA_n$ which plays a significant role in combinatorial group theory.

It turns out that the procedure that takes FG to FA ("associated graded") can be imitated in the case of emergent knots, and then it takes $\mathcal{K}_1$ to $\mathcal{A}_1$, "emergent chord diagrams", which we will not define here, except to say that it is a whiff more than the free associative algebra FA. And it makes sense to ask the question, is there an expansion $Z_1\colon \mathcal{K}_1 \to \mathcal{A}_1$? Is there such an expansion which is homomorphic, meaning, which preserves several additional structures that $\mathcal{K}_1$ and $\mathcal{A}_1$ inherit from knots?

The answer is YES. Furthermore, it turns out that Z_1 allows us to construct an expansion for the Goldman-Turaev Lie bialgebra, a certain algebraic structure constructed from curves in the punctured disk. In itself, this was shown by Alekseev, Kawazumi, Kuno, and Naef to be equivalent to a solution of the Kashiwara-Vergne problem which in itself implies an equivalence statement between convolutions of invariant functions on Lie groups and on Lie algebras. We don't know a direct relation between emergent knots and this convolutions statement. We will certainly try to find this relationship during my proposed visit to SMRI!

By an earlier work of Dancso and myself, that same Kashiwara-Vergne problem is also related to "w-knots", a class of 2-dimensional knotted objects living in 4-dimensional space. Thus emergent knots ought to be related to w-knots, but we don't know a direct relation. Please add that to our SMRI to do list.

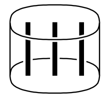
It is well known that expansions for ("full", not merely emergent) knots are closely related to Drinfel'd associators and to the Grothendieck-Teichmüller group GT. There are analogs of these notions for emergent knots, and seeing that in emergent knots almost all complexity is killed off, emergent associators are simpler to compute than full associators and the study of GT also becomes simpler. Yet recent computations by Kuno and myself show that up to degree 10 or so, the simpler emergent theory is equal to the harder full theory. We don't know why this is so. Perhaps we will figure it out in 2025!

Thus I hope that SMRI will extend its hospitality to me and to my research with Dr. Zsuzsanna Dancso, and will fund my visit to Sydney between May 24 2025 and July 6 2025. During that time I will be delighted to speak about this subject in Sydney. I expect that I will spend most of that period in Sydney, though it is possible that I will visit Melbourne and/or Brisbane for 2-3 days each.

A handout from a talk I gave on the subject in Switzerland in 2022 is on the following two pages. It has many more details, and a few relevant references. A video of that talk is linked at http://drorbn.net/ld22.

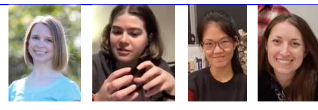
Tangles in a Pole Dance Studio: A Reading of Massuyeau, Alekseev, and Naef

Preliminary Definitions. Fix $p \in \mathbb{N}$ and $\mathbb{F} = \mathbb{Q}/\mathbb{C}$. Let $D_p := D^2\backslash(p \text{ pts})$, and let the Pole Dance Studio be $PDS_p := D_p \times I$.



PDS_3

Abstract. I will report on joint work with Zsuzsanna Dancso, Tamara Hogan, Jessica Liu, and Nancy Scherich. Little of what we do is original, and much of it is simply a reading of Massuyeau [Ma] and Alekseev and Naef [AN1].



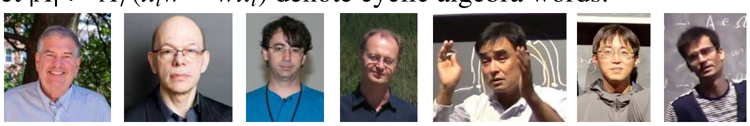
Dancso Hogan Liu Scherich

We study the pole-strand and strand-strand double filtration on the space of tangles in a pole dance studio (a punctured disk cross an interval), the corresponding homomorphic expansions, and a strand-only HOMFLY-PT relation. When the strands are transparent or nearly transparent to each other we recover and perhaps simplify substantial parts of the work of the aforementioned authors on expansions for the Goldman-Turaev Lie bi-algebra.

Jessica, Nancy, Tamara, Zsuzsi, & Dror in PDS_4

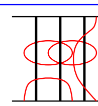
$\Longrightarrow$ Expansions $W\colon FG\langle X_i\rangle \to FA\langle x_i\rangle$:
Magnus: $X_i \mapsto 1 + x_i$, $X_i^{-1} \mapsto 1 - x_i + x_i^2 - \dots$
Exponential: $X_i^{\pm 1} \mapsto e^{\pm x_i}$

Definitions. Let $\pi := FG\langle X_1, \dots, X_p\rangle$ be the free group (of deformation classes of based curves in D_p), $\bar\pi$ be the framed free group (deformation classes of based immersed curves), $|\pi|$ and $|\bar\pi|$ denote $\mathbb{F}$-linear combinations of cyclic words ($|x_iw| = |wx_i|$, unbased curves), $A := FA\langle x_1, \dots, x_p\rangle$ be the free associative algebra, and let $|A| := A/(x_iw = wx_i)$ denote cyclic algebra words.



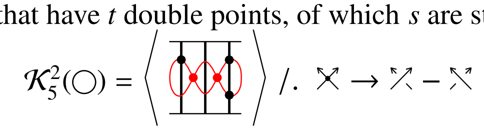
Theorem 1 (Goldman, Turaev, Massuyeau, Alekseev, Kawazumi, Kuno, Naef). $|\bar\pi|$ and $|A|$ are Lie bialgebras, and there is a "homomorphic expansion" $W\colon |\bar\pi| \to |A|$: a morphism of Lie bialgebras with $W(|X_i|) = 1 + |x_i| + \dots$.

Further Definitions. • $\mathcal{K} = \mathcal{K}_0 = \mathcal{K}_0^0 = \mathcal{K}(S) := \mathbb{F}\langle\text{framed tangles in } PDS_p\rangle$.



• $\mathcal{K}_t^s$:=(the image via $\times \to \nearrow\!\!\!\searrow - \searrow\!\!\!\nearrow$ of tangles in PDS_p that have t double points, of which s are strand-strand).

E.g.,

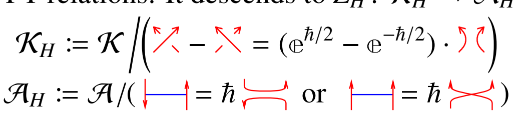


• $\mathcal{K}^{/s} := \mathcal{K}/\mathcal{K}^s$. Most important, $\mathcal{K}^{/1}(\bigcirc) = |\bar\pi|$, and there is $P\colon \mathcal{K}(\bigcirc) \to |\bar\pi|$.

• $\mathcal{A} := \prod \mathcal{K}_t/\mathcal{K}_{t+1}$, $\mathcal{A}^s := \prod \mathcal{K}_t^s/\mathcal{K}_{t+1}^s \subset \mathcal{A}$, $\mathcal{A}^{/s} := \mathcal{A}/\mathcal{A}^s$.

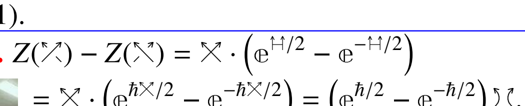
Fact 1. The Kontsevich Integral is an "expansion" $Z\colon \mathcal{K} \to \mathcal{A}$, compatible with several noteworthy structures.

Fact 2 (Le-Murakami, [LM1]). Z satisfies the strand-strand HOMFLY-PT relations: It descends to $Z_H\colon \mathcal{K}_H \to \mathcal{A}_H$, where



and $\deg \hbar = (1, 1)$.

Proof of Fact 2.

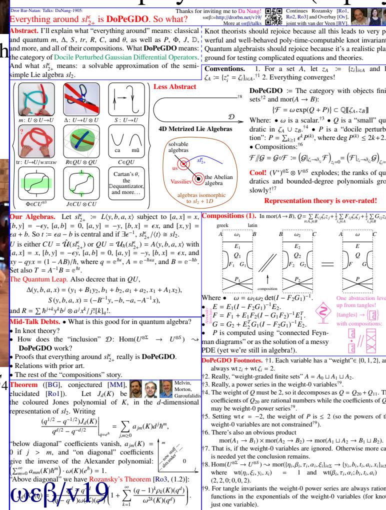


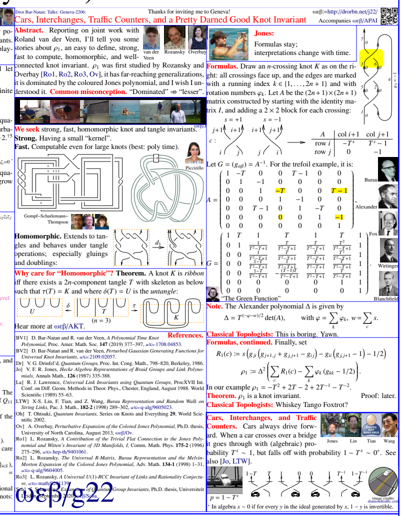
$\square$

Le, Murakami

Other Passions. With Roland van der Veen, I use "solvable approximation" and "Perturbed Gaussian Differential Operators" to unveil simple, strong, fast to compute, and topologically meaningful knot invariants near the Alexander polynomial. ($\subset$ polymath!)

van der Veen





Key 1. $W\colon |\bar\pi| \to |A|$ is $Z_H^{/1}\colon \mathcal{K}_H^{/1}(\bigcirc) \to \mathcal{A}_H^{/1}(\bigcirc)$.

Key 2 (Schematic). Suppose $\lambda_0, \lambda_1\colon |\bar\pi| \to \mathcal{K}(\bigcirc)$ are two ways of lifting plane curves into knots in PDS_p (namely, $P \circ \lambda_i = I$). Then for $\gamma \in |\bar\pi|$, **Lemma 1.** "Division by $\hbar$" is well-defined.

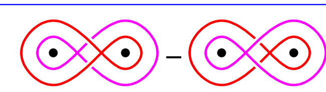
$$\eta(\gamma) := (\lambda_0(\gamma) - \lambda_1(\gamma))/\hbar \in \mathcal{K}_H^{/1}(\bigcirc\bigcirc) = |\bar\pi| \otimes |\bar\pi|$$

and we get an operation η on plane curves. If Kontsevich likes λ_0 and λ_1 (namely if there are λ_i^a with $Z^{/2}(\lambda_i(\gamma)) = \lambda_i^a(W(\gamma))$), then η will have a compatible algebraic companion η^a:

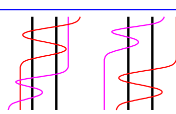
$$\eta^a(\alpha) := (\lambda_0^a(\alpha) - \lambda_1^a(\alpha))/\hbar \in \mathcal{A}_H^{/1}(\bigcirc\bigcirc) = |A| \otimes |A|.$$

For indeed, in $\mathcal{A}_H^{/2}$ we have $\hbar W(\eta(\gamma)) = \hbar Z(\eta(\gamma)) = Z(\lambda_0(\gamma)) - Z(\lambda_1(\gamma)) = \lambda_0^a(W(\gamma)) - \lambda_1^a(W(\gamma)) = \hbar\eta^a(W(\gamma))$.

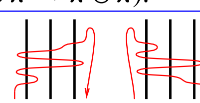
Example 1. With $\gamma_1, \gamma_2 \in |\pi|$ (or $|\bar\pi|$) set $\lambda_0(\gamma_1, \gamma_2) = \tilde\gamma_1 \cdot \tilde\gamma_2$ and $\lambda_1(\gamma_1, \gamma_2) = \tilde\gamma_2 \cdot \tilde\gamma_1$ where $\tilde\gamma_i$ are arbitrary lifts of γ_i. Then η_1 is the Goldman bracket! Note that here λ_0 and λ_1 are not well-defined, yet η_1 is.



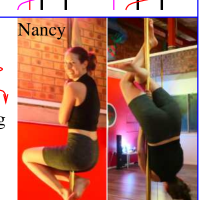
Example 2. With $\gamma_1, \gamma_2 \in \pi$ (or $\bar\pi$) and with λ_0, λ_1 as on the right, we get the "double bracket" $\eta_2\colon \pi \otimes \pi \to \pi \otimes \pi$ (or $\bar\pi \otimes \bar\pi \to \bar\pi \otimes \bar\pi$).



Example 3. With $\gamma \in \bar\pi$ and $\lambda_0(\gamma)$ its ascending realization as a bottom tangle and $\lambda_1(\gamma)$ its descending realization as a bottom tangle, we get $\eta_3\colon \bar\pi \to \bar\pi \otimes |\bar\pi|$. Closing the first component and anti-symmetrizing, this is the Turaev cobracket.

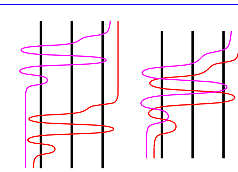


ascending descending



ascending descending

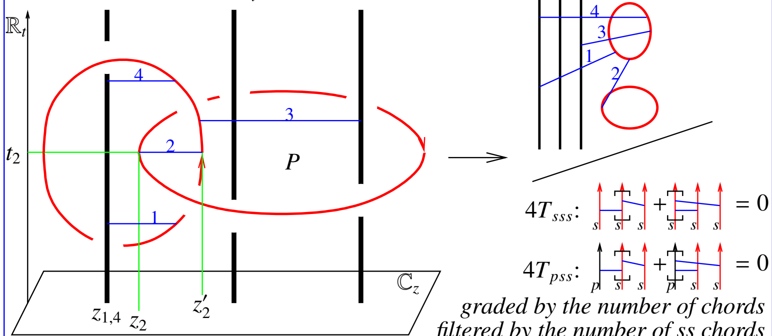
Example 4 [Ma]. With $\gamma \in \bar\pi$ and $\lambda_0(\gamma)$ its ascending outer double and $\lambda_1(\gamma)$ its ascending inner double we get $\eta_4\colon \bar\pi \to \bar\pi \otimes \bar\pi$. After some massaging, it too becomes the Turaev cobracket.



The rest is essentially **Exercises:** **1.** Lemma 1? **2.** $\mathcal{A}$? **3.** Fact 2? **4.** $\mathcal{A}^{/1}$? Especially, $\mathcal{A}^{/1}(\bigcirc) \cong |A|$! **5.** Explain why Kontsevich likes our λ's. **6.** Figure out η_i^a, $i = 1, \dots, 4$.

Kontsevich in a Pole Dance Studio. (w/o poles? See [Ko, BN])

$$Z=\left(\sum_{m=0}^{\infty}\frac{1}{(2\pi i)^m}\int_{\substack{t_1<\dots<t_m\\ P=\{(z_i,z_i')\}}}(-1)^{\#P_\downarrow}D_P\bigwedge_{i=1}^{m}\frac{dz_i-dz_i'}{z_i-z_i'}\right)^{\sim}\in\mathcal{A}$$



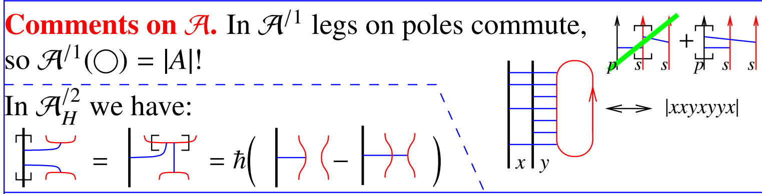
graded by the number of chords
filtered by the number of ss chords

Comments on the Kontsevich Integral.

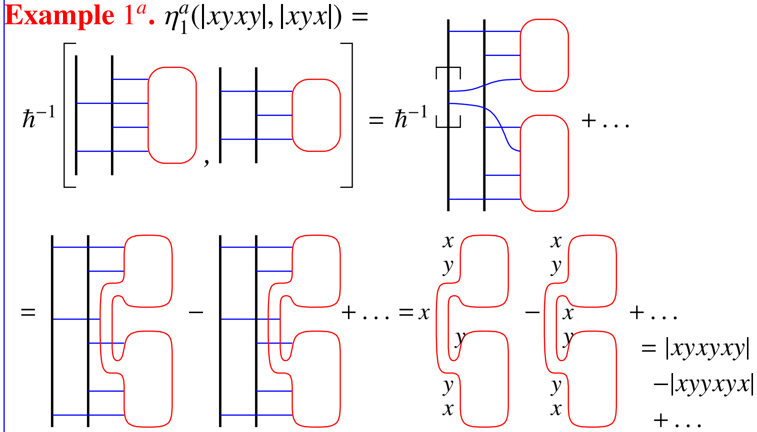
1. In the tangle case, the endpoints are fixed at top and bottom.
2. The $(\cdots)^{\sim}$ means "a correction is needed near the caps and the cups" (for the framed version, see [LM2, Da]).
3. There are never pp chords, and no $4T_{pps}$ and $4T_{ppp}$ relations.
4. Z is an "expansion".
5. Z respects the ss filtration and so descends to $Z^{/s}\colon \mathcal{K}^{/s}\to\mathcal{A}^{/s}$.

Comments on $\mathcal{A}$. In $\mathcal{A}^{/1}$ legs on poles commute, so $\mathcal{A}^{/1}(\bigcirc)=|A|$!

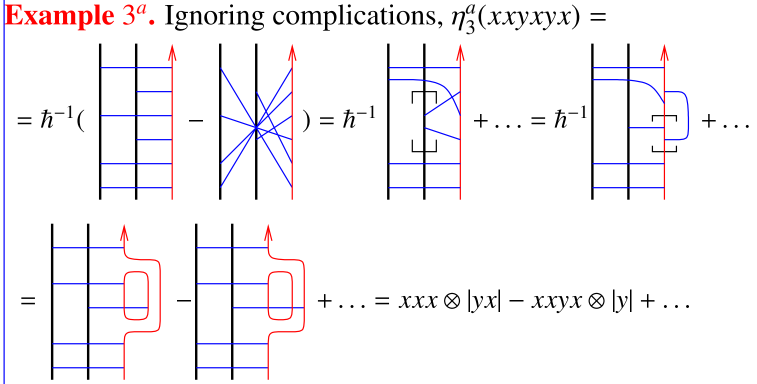
In $\mathcal{A}^{/2}_H$ we have:



Example 1^a. $\eta_1^a(|xyxy|,|xyx|)=$



Example 3^a. Ignoring complications, $\eta_3^a(xxyxyx)=$



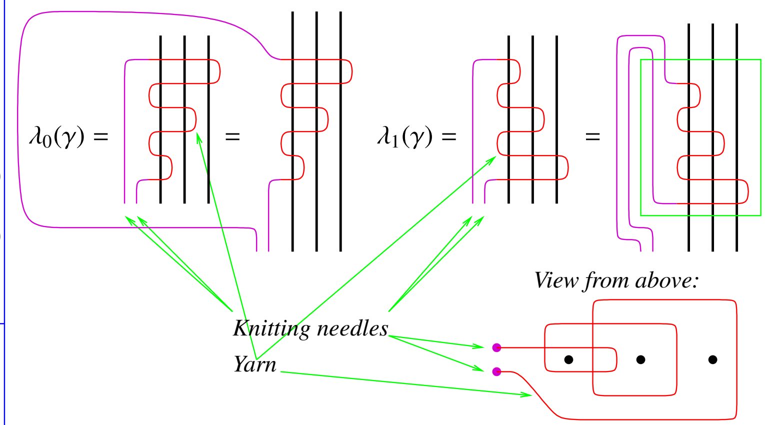
Proof of Lemma 1. We partially prove Theorem 2 instead:

Theorem 2. $\operatorname{gr}^\bullet\mathcal{K}_H\cong\mathbb{F}[\![\hbar]\!]\otimes(\mathcal{K}^{/1})_0$.

Proof mod $\hbar^2$. The map $\leftarrow$ is obvious. To go $\to$, map $\mathcal{K}_H\to\mathbb{F}[\![\hbar]\!]\otimes\mathcal{K}^{/1}$ using $\nearrow\!\!\!\!\nwarrow\mapsto\times+\frac{\hbar}{2})($ and $\nwarrow\!\!\!\!\nearrow\mapsto\times-\frac{\hbar}{2})($ and apply the functor $\operatorname{gr}^\bullet$.

Unignoring the Complications. We need λ_0 and λ_1 such that:

1. $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by flipping all self-intersections from ascending to descending.
2. Up to conjugation, $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by a global flip.
3. $Z(\lambda_i(\gamma))$ is computable from $W(\gamma)$ and $Z^{/1}(\lambda_i(\gamma))=W(\gamma)$.



Homework

1. Is there more than Examples 1–4?
2. Derive the bialgebra axioms from this perspective.
3. What more do we get if we don't mod out by HOMFLY-PT?
4. What more do we get if we allow more than one strand-strand interaction?
5. In this language, recover Kashiwara-Vergne [AKKN1, AKKN2].
6. How is all this related to w-knots?
7. Do the same with associators. Use that to derive formulas for solutions of Kashiwara-Vergne.
8. What's the relationship with the Habiro-Massuyeau invariants of links in handlebodies [HM] (different filtration!).
9. Pole dance on other surfaces!
10. Explore the action of the mapping class group.

Kashiwara Vergne

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