Pensieve header: Solving the PR^4 relation degree by degree.

Startup

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio2"];
<< "../Profile/Profile.m";
<< "Engine-Speedy.m";
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.

This version: June 2018. Original version: July 1994.

The Objects

Symmetric Algebra Objects

In[∘]:=
```
sm_{i_,j_→k_} := 𝔼_{{i,j}→{k}}[b_k (β_i + β_j) + t_k (τ_i + τ_j) + a_k (α_i + α_j) + y_k (η_i + η_j) + x_k (ξ_i + ξ_j)];
sΔ_{i_→j_,k_} := 𝔼_{{i}→{j,k}}[β_i (b_j + b_k) + τ_i (t_j + t_k) + α_i (a_j + a_k) + η_i (y_j + y_k) + ξ_i (x_j + x_k)];
sS_{i_} := 𝔼_{{i}→{i}}[-β_i b_i - τ_i t_i - α_i a_i - η_i y_i - ξ_i x_i];
sη_{i_} := 𝔼_{{}→{i}}[0];
sϵ_{i_} := 𝔼_{{i}→{}}[0];
```

In[∘]:=
```
sσ_{i_→j_} := 𝔼_{{i}→{j}}[β_i b_j + τ_i t_j + α_i a_j + η_i y_j + ξ_i x_j];
sΥ_{i_→j_,k_,l_,m_} := 𝔼_{{i}→{j,k,l,m}}[β_i b_k + τ_i t_k + α_i a_l + η_i y_j + ξ_i x_m];
```

The CU Definitions

In[∘]:=

$$\text{c}\Lambda = \left(\eta_i + \frac{e^{-\gamma\,\alpha_i - \epsilon\,\beta_i}\,\eta_j}{1+\gamma\,\epsilon\,\eta_j\,\xi_i}\right) y_k + \left(\beta_i + \beta_j + \frac{\text{Log}[1+\gamma\,\epsilon\,\eta_j\,\xi_i]}{\epsilon}\right) b_k + \left(\alpha_i + \alpha_j + \frac{\text{Log}[1+\gamma\,\epsilon\,\eta_j\,\xi_i]}{\gamma}\right) a_k + \left(\frac{e^{-\gamma\,\alpha_j - \epsilon\,\beta_j}\,\xi_i}{1+\gamma\,\epsilon\,\eta_j\,\xi_i} + \xi_j\right) x_k;$$

```
Define[cm_{i,j→k} = 𝔼_{{i,j}→{k}}[cΛ]]
```

In[∘]:=
```
Define[cσ_{i→j} = sσ_{i,j} /. τ_i → 0, cϵ_i = sϵ_i, cη_i = sη_i, cΔ_{i→j,k} = sΔ_{i→j,k},
  cS_i = sS_i // sΥ_{i→1,2,3,4} // cm_{4,3→i} // cm_{i,2→i} // cm_{i,1→i}];
```

In[∘]:=
```
$k = 1; (*ℏ=γ=1;*)
```

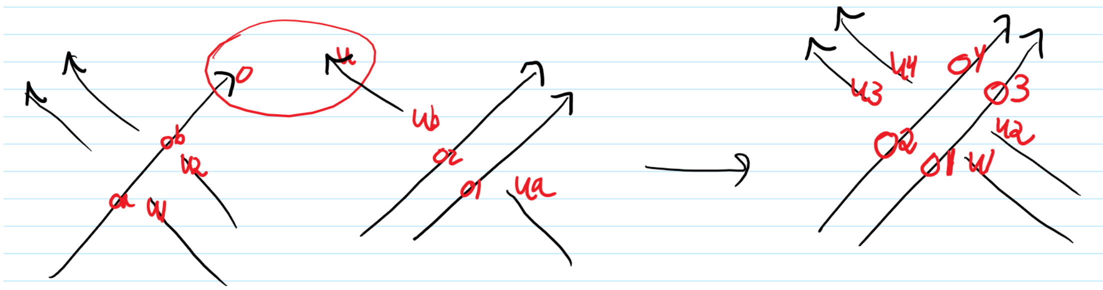
Booting Up QU

In[∘]:=
```
Define[aσ_{i→j} = 𝔼_{{i}→{j}}[a_j α_i + x_j ξ_i], bσ_{i→j} = 𝔼_{{i}→{j}}[b_j β_i + y_j η_i]]
```

In[∘]:=

$$\text{Define}\left[am_{i,j\to k} = \mathbb{E}_{\{i,j\}\to\{k\}}\left[(\alpha_i + \alpha_j)\, a_k + \left(\mathcal{A}_j^{-1}\,\xi_i + \xi_j\right) x_k\right],\ bm_{i,j\to k} = \mathbb{E}_{\{i,j\}\to\{k\}}\left[(\beta_i + \beta_j)\, b_k + \left(\eta_i + e^{-\epsilon\,\beta_i}\,\eta_j\right) y_k\right]\right]$$

The PR^4 Equation



$$\text{rhs} = \left(R_{o1,u1}\, R_{o3,u2}\, R_{o2,u3}\, R_{o4,u4}\right) \,//\, \left(bm_{o1,o3\to o1}\, bm_{o2,o4\to o2}\, am_{u1,u3\to u1}\, am_{u2,u4\to u2}\right)$$

Out[∘]=

$$\mathbb{E}_{\{\}\to\{o1,o2,u1,u2\}}\Big[\hbar\, a_{u1} b_{o1} + \hbar\, a_{u2} b_{o1} + \hbar\, a_{u1} b_{o2} + \hbar\, a_{u2} b_{o2},\ \hbar\, B_{o2} x_{u1} y_{o1} + \hbar\, B_{o2} x_{u2} y_{o1} + \hbar\, x_{u1} y_{o2} + \hbar\, x_{u2} y_{o2},$$
$$1 + \Big(-\hbar^2 a_{u1} B_{o2} x_{u2} y_{o1} - \frac{1}{4}\gamma\hbar^3 B_{o2}^2 x_{u1}^2 y_{o1}^2 - \frac{1}{4}\gamma\hbar^3 B_{o2}^2 x_{u2}^2 y_{o1}^2 -$$
$$\hbar^2 a_{u1} x_{u2} y_{o2} + \gamma\hbar^3 B_{o2} x_{u1} x_{u2} y_{o1} y_{o2} - \frac{1}{4}\gamma\hbar^3 x_{u1}^2 y_{o2}^2 - \frac{1}{4}\gamma\hbar^3 x_{u2}^2 y_{o2}^2\Big)\,\epsilon + O[\epsilon]^2\Big]$$

In[∘]:=

$$\text{lhs} = \left(R_{oa,u1}\, R_{ob,u2} \,//\, bm_{oa,ob\to o}\right)\left(R_{o1,ua}\, R_{o2,ub} \,//\, am_{ua,ub\to u}\right) \,//\, P_{o,u};$$

In[∘]:= lhs ≡ rhs

Out[∘]= True

Solving the PR^4 Equation

In[∘]:=

$$\text{Invert}[R\_,\ k\_]_{i\_,j\_} := \mathbb{E}_{\{i,j\}\to\{\}}\Big[\beta_i\,\alpha_j/\hbar,\ \eta_i\,\xi_j/\hbar,\ 1 + \text{If}\Big[k == 0,\ 0,\ \left(\text{Invert}[R,\ k-1]_{i,j}\right)_k[3] - \left(R_{1,2} \,//\, \left(\left(\text{Invert}[R,\ 0]_{1,j}\right)_k \left(\text{Invert}[R,\ k-1]_{i,2}\right)_k\right)\right)[3]\Big]\Big]$$

In[∘]:=

$$\text{RR}[0]_{i\_,j\_} := \mathbb{E}_{\{\}\to\{i,j\}}[\hbar\, a_j\, b_i,\ \hbar\, x_j\, y_i,\ 1];$$
$$\text{PP}[0]_{i\_,j\_} := \text{Invert}[\text{RR}[0],\ 0]_{i,j};$$

In[∘]:=

$$\text{Eq}[k\_] := \text{Last}\left[\left(\text{RR}[k]_{o1,u1}\,\text{RR}[k]_{o3,u2}\,\text{RR}[k]_{o2,u3}\,\text{RR}[k]_{o4,u4}\right) // \left(bm_{o1,o3\to o1}\, bm_{o2,o4\to o2}\, am_{u1,u3\to u1}\, am_{u2,u4\to u2}\right)\right] -$$
$$\text{Last}\left[\left(\text{RR}[k]_{oa,u1}\,\text{RR}[k]_{ob,u2} // bm_{oa,ob\to o}\right)\left(\text{RR}[k]_{o1,ua}\,\text{RR}[k]_{o2,ub} // am_{ua,ub\to u}\right) // \text{PP}[k]_{o,u}\right]$$

In[∘]:= Block[{\$k = 0}, Eq[0]]

Out[∘]= $O[\epsilon]^1$

In[∘]:= Block[{\$k = 1}, Eq[0]]

Out[∘]= $\gamma\,\hbar^3 B_{o2}\, x_{u1}\, x_{u2}\, y_{o1}\, y_{o2}\,\epsilon + O[\epsilon]^2$

In[∘]:=
```
RR[1]_{i_,j_} := 𝔼_{{}→{i,j}}[ℏ a_j b_i, ℏ x_j y_i,
   1 + ϵ (f_1[b_i] + f_2[b_i] y_i x_j + f_3[b_i] y_i^2 x_j^2 + f_4[b_i] a_j + f_5[b_i] a_j y_i x_j + f_6[b_i] a_j^2) + O[ϵ]^2];
PP[1]_{i_,j_} := Invert[RR[1], 1]_{i,j};
PP[1]_{i,j}
```

Out[∘]=
$$\mathbb{E}_{\{i,j\}\to\{\}}\left[\frac{\alpha_j\,\beta_i}{\hbar},\ \frac{\eta_i\,\xi_j}{\hbar},\ 1+\left(-f_1\left[\frac{\alpha_j}{\hbar}\right]-\frac{\eta_i\,\xi_j\,f_2\left[\frac{\alpha_j}{\hbar}\right]}{\hbar^2}-\frac{\eta_i^2\,\xi_j^2\,f_3\left[\frac{\alpha_j}{\hbar}\right]}{\hbar^4}-\frac{\beta_i\,f_4\left[\frac{\alpha_j}{\hbar}\right]}{\hbar}-\frac{\beta_i\,\eta_i\,\xi_j\,f_5\left[\frac{\alpha_j}{\hbar}\right]}{\hbar^3}-\frac{\beta_i^2\,f_6\left[\frac{\alpha_j}{\hbar}\right]}{\hbar^2}\right)\epsilon+O[\epsilon]^2\right]$$

In[∘]:=
```
Block[{$k = 1}, RR[1]_{i,j} // PP[1]_{i,k}]
```

Out[∘]=
$$\mathbb{E}_{\{k\}\to\{j\}}\left[a_j\,\alpha_k,\ x_j\,\xi_k,\ 1+O[\epsilon]^2\right]$$

In[∘]:=
```
CoefficientRules[Simplify@SeriesCoefficient[Block[{$k = 1}, Eq[1]], 1],
  {a_u1, a_u2, x_u1, x_u2, y_o1, y_o2}] // Column
```

Out[∘]=
$$\{1,1,0,0,0,0\}\to -2 f_6[b_{o1}]-2 f_6[b_{o2}]+2 f_6[b_{o1}+b_{o2}]$$
$$\{1,0,0,1,1,0\}\to -B_{o2} f_5[b_{o1}]+B_{o2} f_5[b_{o1}+b_{o2}]+2\gamma\hbar B_{o2} f_6[b_{o2}]$$
$$\{1,0,0,1,0,1\}\to -f_5[b_{o2}]+f_5[b_{o1}+b_{o2}]$$
$$\{0,1,1,0,1,0\}\to -B_{o2} f_5[b_{o1}]+B_{o2} f_5[b_{o1}+b_{o2}]+2\gamma\hbar B_{o2} f_6[b_{o2}]$$
$$\{0,1,1,0,0,1\}\to -f_5[b_{o2}]+f_5[b_{o1}+b_{o2}]$$
$$\{0,0,1,1,2,0\}\to -2 B_{o2}^2 f_3[b_{o1}]+2 B_{o2}^2 f_3[b_{o1}+b_{o2}]-2\gamma^2\hbar^2 B_{o2}^2 f_6[b_{o2}]$$
$$\{0,0,1,1,1,1\}\to \gamma\hbar^3 B_{o2}+4 B_{o2} f_3[b_{o1}+b_{o2}]+2\gamma\hbar B_{o2} f_5[b_{o2}]$$
$$\{0,0,1,1,0,2\}\to -2 f_3[b_{o2}]+2 f_3[b_{o1}+b_{o2}]$$
$$\{0,0,0,0,0,0\}\to f_1[b_{o1}]+f_1[b_{o2}]-f_1[b_{o1}+b_{o2}]$$

In[∘]:=
```
CoefficientRules[Simplify@SeriesCoefficient[Block[{$k = 1}, Eq[1]] /. f_{1|2|4|5|6}[_] → 0, 1],
  {a_u1, a_u2, x_u1, x_u2, y_o1, y_o2}] // Column
```

Out[∘]=
$$\{0,0,1,1,2,0\}\to -2 B_{o2}^2 f_3[b_{o1}]+2 B_{o2}^2 f_3[b_{o1}+b_{o2}]$$
$$\{0,0,1,1,1,1\}\to \gamma\hbar^3 B_{o2}+4 B_{o2} f_3[b_{o1}+b_{o2}]$$
$$\{0,0,1,1,0,2\}\to -2 f_3[b_{o2}]+2 f_3[b_{o1}+b_{o2}]$$

In[∘]:=
```
CoefficientRules[Simplify@
   SeriesCoefficient[Block[{$k = 1}, Eq[1]] /. {f_{1|4|5|6}[_] → 0, f_3[_] → -γ ℏ^3/4}, 1],
  {a_u1, a_u2, x_u1, x_u2, y_o1, y_o2}] // Column
```

Out[∘]=