Pensieve header: Full testing of all functions.

Startup

In[ ]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio2"];
<< KnotTheory`;
<< "../Profile/Profile.m";
<< "Engine-With-w.m";
<< "Objects-With-w.m";
<< "KT.m";
BeginProfile[];
PopupWindow[Button["Show Profile Monitor"],
 Dynamic[PrintProfile[], UpdateInterval → 3, TrackedSymbols → {}]]
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
This version: April 2020. Original version: July 1994.
» Warning: On Sep 4 2019 I swapped the operations
  ∈ and η. Some incompatibilities may arise in older notebooks.
```

Out[ ]= Show Profile Monitor

In[ ]:=
```
$k = 2; (*ħ=γ=1;*)
```

Utilities

In[ ]:=
```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Testing

In[ ]:=
```
Block[{$k = 1}, {
   am → am_{i,j→k}, bm → bm_{i,j→k}, dm → dm_{i,j→k}, R → R_{i,j}, R̄ → R̄_{i,j}, P → P_{i,j},
   aS → aS_i, aS̄ → aS̄_i, bS → bS_i, bS̄ → bS̄_i, dS → dS_i, aΔ → aΔ_{i→j,k}, bΔ → bΔ_{i→j,k},
   dΔ → dΔ_{i→j,k}, C → C_i, C̄ → C̄_i, Kink → Kink_i, K̄ink → K̄ink_i, b2t → b2t_i, t2b → t2b_i
  }] //
 Column
```

Out[•]=

$$\mathrm{am} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\,\alpha_i + a_k\,\alpha_j,\ \frac{x_k\,\xi_i}{\mathcal{A}_j} + x_k\,\xi_j,\ 1\right]$$

$$\mathrm{bm} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[b_k\,\beta_i + b_k\,\beta_j,\ y_k\,\eta_i + y_k\,\eta_j,\ 1 - y_k\,\beta_i\,\eta_j\,\epsilon + O[\epsilon]^2\right]$$

$$\mathrm{dm} \to \mathbb{E}_{\{i,j\}\to\{k\}}\Big[a_k\,\alpha_i + a_k\,\alpha_j + b_k\,\beta_i + b_k\,\beta_j,\ y_k\,\eta_i + \frac{y_k\,\eta_j}{\mathcal{A}_i} + \frac{x_k\,\xi_i}{\mathcal{A}_j} + \frac{(1-B_k)\,\eta_j\,\xi_i}{\hbar} + x_k\,\xi_j,$$
$$1 + \left(-\frac{y_k\,\beta_i\,\eta_j}{\mathcal{A}_i} - \frac{x_k\,\beta_j\,\xi_i}{\mathcal{A}_j} + a_k\,B_k\,\eta_j\,\xi_i + \frac{\gamma\,\hbar\,x_k\,y_k\,\eta_j\,\xi_i}{\mathcal{A}_i\,\mathcal{A}_j} + \frac{(\gamma-3\,\gamma\,B_k)\,y_k\,\eta_j^2\,\xi_i}{2\,\mathcal{A}_i} + \frac{(\gamma-3\,\gamma\,B_k)\,x_k\,\eta_j\,\xi_i^2}{2\,\mathcal{A}_j} + \frac{\left(\gamma-4\,\gamma\,B_k+3\,\gamma\,B_k^2\right)\,\eta_j^2\,\xi_i^2}{4\,\hbar}\right)\epsilon +$$
$$O[\epsilon]^2\Big]$$

$$\mathrm{R} \to \mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\,a_j\,b_i,\ \hbar\,x_j\,y_i,\ 1 - \frac{1}{4}\left(\gamma\,\hbar^3\,x_j^2\,y_i^2\right)\epsilon + O[\epsilon]^2\right]$$

$$\overline{\mathrm{R}} \to \mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\,a_j\,b_i,\ -\frac{\hbar\,x_j\,y_i}{B_i},\ 1 + \left(-\frac{\hbar^2\,a_j\,x_j\,y_i}{B_i} - \frac{3\,\gamma\,\hbar^3\,x_j^2\,y_i^2}{4\,B_i^2}\right)\epsilon + O[\epsilon]^2\right]$$

$$\mathrm{P} \to \mathbb{E}_{\{i,j\}\to\{\}}\left[\frac{\alpha_j\,\beta_i}{\hbar},\ \frac{\eta_i\,\xi_j}{\hbar},\ 1 + \frac{\gamma\,\eta_i^2\,\xi_j^2\,\epsilon}{4\,\hbar} + O[\epsilon]^2\right]$$

$$\mathrm{aS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i,\ -x_i\,\mathcal{A}_i\,\xi_i,\ 1 + \left(-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i - \frac{1}{2}\,\gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right)\epsilon + O[\epsilon]^2\right]$$

$$\overline{\mathrm{aS}} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i,\ -x_i\,\mathcal{A}_i\,\xi_i,\ 1 + \left(\gamma\,\hbar\,x_i\,\mathcal{A}_i\,\xi_i - \hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i - \frac{1}{2}\,\gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right)\epsilon + O[\epsilon]^2\right]$$

$$\mathrm{bS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i,\ -\frac{y_i\,\eta_i}{B_i},\ 1 + \left(-\frac{y_i\,\beta_i\,\eta_i}{B_i} - \frac{\gamma\,\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right)\epsilon + O[\epsilon]^2\right]$$

$$\overline{\mathrm{bS}} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i,\ -\frac{y_i\,\eta_i}{B_i},\ 1 + \left(\frac{\gamma\,\hbar\,y_i\,\eta_i}{B_i} - \frac{y_i\,\beta_i\,\eta_i}{B_i} - \frac{\gamma\,\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right)\epsilon + O[\epsilon]^2\right]$$

$$\mathrm{dS} \to \mathbb{E}_{\{i\}\to\{i\}}\Big[-a_i\,\alpha_i - b_i\,\beta_i,\ -\frac{y_i\,\mathcal{A}_i\,\eta_i}{B_i} - x_i\,\mathcal{A}_i\,\xi_i + \frac{(\mathcal{A}_i - B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{\hbar\,B_i},$$
$$1 + \Big(\frac{\gamma\,\hbar\,y_i\,\mathcal{A}_i\,\eta_i}{B_i} - \frac{y_i\,\mathcal{A}_i\,\beta_i\,\eta_i}{B_i} - \frac{\gamma\,\hbar\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2}{2\,B_i^2} - \hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i - x_i\,\mathcal{A}_i\,\beta_i\,\xi_i + \frac{a_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{B_i} -$$
$$\frac{\gamma\,\hbar\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i}{B_i} + \frac{(-\gamma\,\mathcal{A}_i + \gamma\,B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{B_i} + \frac{(\mathcal{A}_i - B_i\,\mathcal{A}_i)\,\beta_i\,\eta_i\,\xi_i}{\hbar\,B_i} + \frac{y_i\left(3\,\gamma\,\mathcal{A}_i^2 - \gamma\,B_i\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i}{2\,B_i^2} -$$
$$\frac{1}{2}\,\gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2 + \frac{x_i\left(3\,\gamma\,\mathcal{A}_i^2 - \gamma\,B_i\,\mathcal{A}_i^2\right)\eta_i\,\xi_i^2}{2\,B_i} + \frac{\left(-3\,\gamma\,\mathcal{A}_i^2 + 4\,\gamma\,B_i\,\mathcal{A}_i^2 - \gamma\,B_i^2\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i^2}{4\,\hbar\,B_i^2}\Big)\epsilon + O[\epsilon]^2\Big]$$

$$\mathrm{a}\triangle \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i + a_k\,\alpha_i,\ x_j\,\xi_i + x_k\,\xi_i,\ 1 + \left(-\hbar\,a_j\,x_k\,\xi_i + \frac{1}{2}\,\gamma\,\hbar\,x_j\,x_k\,\xi_i^2\right)\epsilon + O[\epsilon]^2\right]$$

$$\mathrm{b}\triangle \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[b_j\,\beta_i + b_k\,\beta_i,\ B_k\,y_j\,\eta_i + y_k\,\eta_i,\ 1 + \frac{1}{2}\,\gamma\,\hbar\,B_k\,y_j\,y_k\,\eta_i^2\,\epsilon + O[\epsilon]^2\right]$$

$$\mathrm{d}\triangle \to \mathbb{E}_{\{i\}\to\{j,k\}}\Big[a_j\,\alpha_i + a_k\,\alpha_i + b_j\,\beta_i + b_k\,\beta_i,\ y_j\,\eta_i + B_j\,y_k\,\eta_i + x_j\,\xi_i + x_k\,\xi_i,$$
$$1 + \left(\frac{1}{2}\,\gamma\,\hbar\,B_j\,y_j\,y_k\,\eta_i^2 - \hbar\,a_j\,x_k\,\xi_i + \frac{1}{2}\,\gamma\,\hbar\,x_j\,x_k\,\xi_i^2\right)\epsilon + O[\epsilon]^2\Big]$$

$$\mathrm{C} \to \mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \sqrt{B_i} - \frac{1}{2}\left(\hbar\,a_i\,\sqrt{B_i}\right)\epsilon + O[\epsilon]^2\right]$$

$$\overline{\mathrm{C}} \to \mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \frac{1}{\sqrt{B_i}} + \frac{\hbar\,a_i\,\epsilon}{2\,\sqrt{B_i}} + O[\epsilon]^2\right]$$

$$\mathrm{Kink} \to \mathbb{E}_{\{\}\to\{i\}}\left[\hbar\,a_i\,b_i,\ \hbar\,x_i\,y_i,\ \frac{1}{\sqrt{B_i}} + \left(\frac{\hbar\,a_i}{2\,\sqrt{B_i}} - \frac{\gamma\,\hbar^3\,x_i^2\,y_i^2}{4\,\sqrt{B_i}}\right)\epsilon + O[\epsilon]^2\right]$$

$$\overline{\mathrm{Kink}} \to \mathbb{E}_{\{\}\to\{i\}}\left[-\hbar\,a_i\,b_i,\ -\frac{\hbar\,x_i\,y_i}{B_i},\ \sqrt{B_i} + \left(-\frac{1}{2}\,\hbar\,a_i\,\sqrt{B_i} - \frac{\hbar^2\,a_i\,x_i\,y_i}{\sqrt{B_i}} - \frac{3\,\gamma\,\hbar^3\,x_i^2\,y_i^2}{4\,B_i^{3/2}}\right)\epsilon + O[\epsilon]^2\right]$$

$$\mathrm{b2t} \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i + \frac{t_i\,\beta_i}{\gamma},\ y_i\,\eta_i + x_i\,\xi_i,\ 1 + \frac{a_i\,\beta_i\,\epsilon}{\gamma} + O[\epsilon]^2\right]$$

$$\mathrm{t2b} \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i + \gamma\,b_i\,\tau_i,\ y_i\,\eta_i + x_i\,\xi_i,\ 1 - a_i\,\tau_i\,\epsilon + O[\epsilon]^2\right]$$

Check that on the generators this agrees with our conventions in the handout:

```
In[∘]:= Timing@
  {{"[a,x]" → ((𝔼{}→{1,2}[0, 0, a2 x1] // am1,2→1)[3] - (𝔼{}→{1,2}[0, 0, a1 x2] // am1,2→1)[3]),
     "[b,y]" → ((𝔼{}→{1,2}[0, 0, y2 b1] // bm1,2→1)[3] - (𝔼{}→{1,2}[0, 0, y1 b2] // bm1,2→1)[3])} /.
    z_1 → z,
   {"Δ[y]" → Last[𝔼{}→{1}[0, 0, y1] ~B1~ bΔ1→1,2],
    "Δ[b]" → Last[𝔼{}→{1}[0, 0, b1] ~B1~ bΔ1→1,2],
    "Δ[a]" → Last[𝔼{}→{1}[0, 0, a1] ~B1~ aΔ1→1,2],
    "Δ[x]" → Last[𝔼{}→{1}[0, 0, x1] ~B1~ aΔ1→1,2]},
   {
     "S(a)" → ((𝔼{}→{1}[0, 0, a1] ~B1~ aS1)[3]),
     "S(x)" → ((𝔼{}→{1}[0, 0, x1] ~B1~ aS1)[3]),
     "S(b)" → ((𝔼{}→{1}[0, 0, b1] ~B1~ bS1)[3]),
     "S(y)" → ((𝔼{}→{1}[0, 0, y1] ~B1~ bS1)[3])
   } /. z_1 → z}
```

Out[∘]= $\{1.48438, \{\{[a,x] \to -x\gamma,\ [b,y] \to -y\epsilon + O[\epsilon]^3\},\ \{\Delta[y] \to (B_2 y_1 + y_2) + O[\epsilon]^3,\ \Delta[b] \to (b_1 + b_2) + O[\epsilon]^3,$
$\Delta[a] \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta[x] \to (x_1 + x_2) - \hbar a_1 x_2 \epsilon + \frac{1}{2}\hbar^2 a_1^2 x_2 \epsilon^2 + O[\epsilon]^3\},\ \{S(a) \to -a + O[\epsilon]^3,$
$S(x) \to -x - a x \hbar \epsilon - \frac{1}{2}(a^2 x \hbar^2)\epsilon^2 + O[\epsilon]^3,\ S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + O[\epsilon]^3\}\}\}$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

```
In[∘]:= Timing@Block[{$k = 3},
   HL /@ {(am1,2→1 // am1,3→1) ≡ (am2,3→2 // am1,2→1), (bm1,2→1 // bm1,3→1) ≡ (bm2,3→2 // bm1,2→1)}
  ]
```

Out[∘]= {0.25, {True, True}}

R and *P* are inverses:

```
In[∘]:= Timing@Block[{$k = 3}, {Ri,j, Pi,k, HL[(Ri,j // Pi,k) ≡ aσk→j]}]
```

Out[∘]= $\{0.5, \{\mathbb{E}_{\{\}\to\{i,j\}}[\hbar a_j b_i, \hbar x_j y_i, 1 - \frac{1}{4}(\gamma\hbar^3 x_j^2 y_i^2)\epsilon + (\frac{1}{9}\gamma^2\hbar^5 x_j^3 y_i^3 + \frac{1}{32}\gamma^2\hbar^6 x_j^4 y_i^4)\epsilon^2 +$
$(\frac{1}{48}\gamma^3\hbar^5 x_j^2 y_i^2 - \frac{1}{16}\gamma^3\hbar^7 x_j^4 y_i^4 - \frac{1}{36}\gamma^3\hbar^8 x_j^5 y_i^5 - \frac{1}{384}\gamma^3\hbar^9 x_j^6 y_i^6)\epsilon^3 + O[\epsilon]^4],$
$\mathbb{E}_{\{i,k\}\to\{\}}[\frac{\alpha_k\beta_i}{\hbar}, \frac{\eta_i\xi_k}{\hbar}, 1 + \frac{\gamma\eta_i^2\xi_k^2\epsilon}{4\hbar} + \frac{(36\gamma^2\hbar^2\eta_i^2\xi_k^2 + 40\gamma^2\hbar\eta_i^3\xi_k^3 + 9\gamma^2\eta_i^4\xi_k^4)\epsilon^2}{288\hbar^2} +$
$(\frac{1}{24}\gamma^3\hbar\eta_i^2\xi_k^2 + \frac{1}{6}\gamma^3\eta_i^3\xi_k^3 + \frac{13\gamma^3\eta_i^4\xi_k^4}{96\hbar} + \frac{5\gamma^3\eta_i^5\xi_k^5}{144\hbar^2} + \frac{\gamma^3\eta_i^6\xi_k^6}{384\hbar^3})\epsilon^3 + O[\epsilon]^4], \text{True}\}\}$

as and $\overline{\text{aS}}$ are inverses, bs and $\overline{\text{bS}}$ are inverses:

```
In[∘]:= Timing[HL /@ {(aS̄1 // aS1) ≡ aσ1→1, (bS̄1 // bS1) ≡ bσ1→1}]
```

Out[∘]= {0.390625, {True, True}}

(co)-associativity on both sides

In[∘]:=
```
Timing[
 HL /@ {(aΔ_{1→1,2} // aΔ_{2→2,3}) ≡ (aΔ_{1→1,3} // aΔ_{1→1,2}), (bΔ_{1→1,2} // bΔ_{2→2,3}) ≡ (bΔ_{1→1,3} // bΔ_{1→1,2}),
   (am_{1,2→1} // am_{1,3→1}) ≡ (am_{2,3→2} // am_{1,2→1}), (bm_{1,2→1} // bm_{1,3→1}) ≡ (bm_{2,3→2} // bm_{1,2→1})}]
```

Out[∘]= {0.375, {True, True, True, True}}

Δ is an algebra morphism

In[∘]:=
```
Timing[HL /@ {(am_{1,2→1} // aΔ_{1→1,2}) ≡ ((aΔ_{1→1,3} aΔ_{2→2,4}) // (am_{3,4→2} am_{1,2→1})),
   (bm_{1,2→1} // bΔ_{1→1,2}) ≡ ((bΔ_{1→1,3} bΔ_{2→2,4}) // (bm_{3,4→2} bm_{1,2→1}))}]
```

Out[∘]= {0.40625, {True, True}}

An explicit formula for aS_i

In[∘]:=
```
Timing@Block[{$k = 4}, HL[aS_i ≡ (𝔼_{{i}→{i,j}}[-α_i a_j, -ξ_i x_i,
      Sum[Expand[(ⅇ^{ξ_i x_i} (-ℏ γ ϵ)^k)/(2^k k!) Nest[Expand[x_i^2 ∂_{{x_i,2}} #] &, ⅇ^{-ξ_i ⅇ^{ℏ ϵ a_i} x_i}, k]], {k, 0, $k}]]_{$k} //
    am_{i,j→i})]]
```

Out[∘]= {3.5, True}

S is convolution inverse of id

In[∘]:=
```
Timing[HL[# ≡ sϵ_1 sη_1] & /@ {
   (aΔ_{1→1,2} ~ B_1 ~ aS_1) ~ B_{1,2} ~ am_{1,2→1}, (aΔ_{1→1,2} ~ B_2 ~ aS_2) ~ B_{1,2} ~ am_{1,2→1},
   (bΔ_{1→1,2} ~ B_1 ~ bS_1) ~ B_{1,2} ~ bm_{1,2→1}, (bΔ_{1→1,2} ~ B_2 ~ bS_2) ~ B_{1,2} ~ bm_{1,2→1}}]
```

Out[∘]= {0.484375, {True, True, True, True}}

But not with the opposite product:

In[∘]:=
```
Timing[Short[# ≡ sϵ_1 sη_1] & /@ {
   (aΔ_{1→1,2} ~ B_1 ~ aS_1) ~ B_{1,2} ~ am_{2,1→1}, (aΔ_{1→1,2} ~ B_2 ~ aS_2) ~ B_{1,2} ~ am_{2,1→1},
   (bΔ_{1→1,2} ~ B_1 ~ bS_1) ~ B_{1,2} ~ bm_{2,1→1}, (bΔ_{1→1,2} ~ B_2 ~ bS_2) ~ B_{1,2} ~ bm_{2,1→1}}]
```

Out[∘]= $\Big\{0.59375, \Big\{\frac{1}{2}\left(-2\gamma\epsilon\hbar x_1 \mathcal{A}_1 \xi_1 + \gamma^2\epsilon^2\hbar^2 x_1 \mathcal{A}_1 \xi_1 - 2 \ll 6 \gg \ll 1 \gg + 2\gamma^2\epsilon^2\hbar^2 x_1^2 \mathcal{A}_1^2 \xi_1^2\right) == 0,$

$\frac{1}{2}\left(-2\gamma\epsilon\hbar x_1 \xi_1 - \gamma^2\epsilon^2\hbar^2 x_1 \xi_1 + 2\gamma^2\epsilon^2\hbar^2 x_1^2 \xi_1^2\right) == 0,$

$\frac{1}{2}\left(-2\gamma\epsilon\hbar y_1 \eta_1 - \gamma^2\epsilon^2\hbar^2 y_1 \eta_1 + 2\gamma^2\epsilon^2\hbar^2 y_1^2 \eta_1^2\right) == 0,$

$\frac{-2\gamma\epsilon\hbar B_1 y_1 \eta_1 + \ll 1 \gg \ll 1 \gg \ll 1 \gg + 2\gamma^2 \ll 3 \gg \eta_1^2}{2 B_1^2} == 0\Big\}\Big\}$

S is an algebra anti-(co)morphism

```
In[∘]:= Timing[HL /@ {am1,2→1 ~ B1 ~ aS1 ≡ (aS1 aS2) ~ B1,2 ~ am2,1→1, bm1,2→1 ~ B1 ~ bS1 ≡ (bS1 bS2) ~ B1,2 ~ bm2,1→1,
    aS1 ~ B1 ~ aΔ1→1,2 ≡ aΔ1→2,1 ~ B1,2 ~ (aS1 aS2), bS1 ~ B1 ~ bΔ1→1,2 ≡ bΔ1→2,1 ~ B1,2 ~ (bS1 bS2)}]
Out[∘]= {0.734375, {True, True, True, True}}
```

Pairing axioms

```
In[∘]:= Timing[HL /@ {(bm1,2→1 𝔼{3}→{3}[α3 a3, ξ3 x3, 1]) ~ B1,3 ~ P1,3 ≡
      (𝔼{1}→{1}[β1 b1, η1 y1, 1] 𝔼{2}→{2}[β2 b2, η2 y2, 1] aΔ3→4,5) ~ B1,4 ~ P1,4 ~ B2,5 ~ P2,5,
    (bΔ1→1,2 𝔼{3}→{3}[α3 a3, ξ3 x3, 1] 𝔼{4}→{4}[α4 a4, ξ4 x4, 1]) ~ B1,3 ~ P1,3 ~ B2,4 ~ P2,4 ≡
      (𝔼{1}→{1}[β1 b1, η1 y1, 1] am3,4→3) ~ B1,3 ~ P1,3 }]
Out[∘]= {0.328125, {True, True}}
```

```
In[∘]:= Timing[HL /@ {((bS1 aσ2→2) // P1,2) ≡ ((bσ1→1 aS2) // P1,2),
    (bS̄1 aσ2→2) ~ B1,2 ~ P1,2 ≡ (bσ1→1 aS̄2) ~ B1,2 ~ P1,2}]
Out[∘]= {0.3125, {True, True}}
```

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

In[•]:=
```
Timing@{{
    "[a,y]" →
     ((𝔼_{}→{1,2}[0, 0, y₂ a₁] ~B₁,₂~dm₁,₂→₁)[3] - (𝔼_{}→{1,2}[0, 0, y₁ a₂] ~B₁,₂~dm₁,₂→₁)[3]),
    "[b,x]"  → ((𝔼_{}→{1,2}[0, 0, x₂ b₁] ~B₁,₂~dm₁,₂→₁)[3] -
      (𝔼_{}→{1,2}[0, 0, x₁ b₂] ~B₁,₂~dm₁,₂→₁)[3]),
    "xy-qyx" → ((𝔼_{}→{1,2}[0, 0, x₁ y₂] ~B₁,₂~dm₁,₂→₁)[3] -
      (1 + ϵ) (𝔼_{}→{1,2}[0, 0, y₁ x₂] ~B₁,₂~dm₁,₂→₁)[3])
   } /. {z_1 → z} // Expand // Factor,
  {
   "Δ(a)" → ((𝔼_{}→{1}[0, 0, a₁] ~B₁~dΔ₁→₁,₂)[3]),
   "Δ(x)" → ((𝔼_{}→{1}[0, 0, x₁] ~B₁~dΔ₁→₁,₂)[3]),
   "Δ(b)" → ((𝔼_{}→{1}[0, 0, b₁] ~B₁~dΔ₁→₁,₂)[3]),
   "Δ(y)" → ((𝔼_{}→{1}[0, 0, y₁] ~B₁~dΔ₁→₁,₂)[3])
  } // Simplify,
  {
    "S(a)" → ((𝔼_{}→{1}[0, 0, a₁] ~B₁~dS₁)[3]),
    "S(x)" → ((𝔼_{}→{1}[0, 0, x₁] ~B₁~dS₁)[3]),
    "S(b)" → ((𝔼_{}→{1}[0, 0, b₁] ~B₁~dS₁)[3]),
    "S(y)" → ((𝔼_{}→{1}[0, 0, y₁] ~B₁~dS₁)[3])
   } /. {z_1 → z} // Simplify
 }
```

Out[•]=
$$\Big\{4.125,\ \Big\{\Big\{[a,y] \to -y\,\gamma + O[\epsilon]^3,\ [b,x] \to x\,\epsilon + O[\epsilon]^3,$$
$$\text{xy-qyx} \to \frac{1-B}{\hbar} + (a\,B - x\,y + x\,y\,\gamma\,\hbar)\,\epsilon + \left(-\frac{1}{2}a^2\,B\,\hbar + \frac{1}{2}x\,y\,\gamma^2\,\hbar^2\right)\epsilon^2 + O[\epsilon]^3\Big\},$$
$$\Big\{\Delta(a) \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta(x) \to (x_1 + x_2) - \hbar\,a_1\,x_2\,\epsilon + \frac{1}{2}\hbar^2\,a_1^2\,x_2\,\epsilon^2 + O[\epsilon]^3,$$
$$\Delta(b) \to (b_1 + b_2) + O[\epsilon]^3,\ \Delta(y) \to (y_1 + B_1\,y_2) + O[\epsilon]^3\Big\},$$
$$\Big\{S(a) \to -a + O[\epsilon]^3,\ S(x) \to -x - a\,x\,\hbar\,\epsilon - \frac{1}{2}\left(a^2\,x\,\hbar^2\right)\epsilon^2 + O[\epsilon]^3,$$
$$S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + \frac{y\,\gamma\,\hbar\,\epsilon}{B} - \frac{\left(y\,\gamma^2\,\hbar^2\right)\epsilon^2}{2\,B} + O[\epsilon]^3\Big\}\Big\}\Big\}$$

In[•]:=
```
{HL[(𝔼_{1,2}→{1,2}[α₁ a₁ + α₂ a₂, ξ₁ x₁ + ξ₂ x₂, 1] // dm₁,₂→₁) ≡ am₁,₂→₁],
 HL[(𝔼_{1,2}→{1,2}[β₁ b₁ + β₂ b₂, η₁ y₁ + η₂ y₂, 1] // dm₁,₂→₁) ≡ bm₁,₂→₁]}
```

Out[•]= {True, True}

(co)-associativity

In[•]:=
```
Timing[
 HL /@ {(dΔ₁→₁,₂ // dΔ₂→₂,₃) ≡ (dΔ₁→₁,₃ // dΔ₁→₁,₂), (dm₁,₂→₁ // dm₁,₃→₁) ≡ (dm₂,₃→₂ // dm₁,₂->₁)}]
```

Out[•]= {2.0625, {True, True}}

Δ is an algebra morphism

In[•]:=
```
Timing@HL[(dm₁,₂→₁ // dΔ₁→₁,₂) ≡ ((dΔ₁→₁,₃ dΔ₂→₂,₄) // (dm₃,₄→₂ dm₁,₂→₁))]
```

Out[•]= {2.07813, True}

dS and $\overline{\text{dS}}$ are inverses:

```
In[∘]:= Timing@HL[(dS̄₁ // dS₁) ≡ dσ₁→₁]
Out[∘]= {1.60938, True}
```

S_2 inverts R, but not S_1:

```
In[∘]:= Timing@{(R₁,₂ // dS₁) ≡ R̄₁,₂, HL[(R₁,₂ // dS₂) ≡ R̄₁,₂]}
```

$$\text{Out}= \left\{0.421875, \left\{\frac{1}{4 B_1^3}\left(4\gamma\epsilon\hbar^2 B_1^2 x_2 y_1 - 2\gamma^2\epsilon^2\hbar^3 B_1^2 x_2 y_1 + 4\gamma\epsilon^2\hbar^3 a_2 B_1^2 x_2 y_1 + 8\gamma^2\epsilon^2\hbar^4 B_1 x_2^2 y_1^2 - 4\gamma\epsilon^2\hbar^4 a_2 B_1 x_2^2 y_1^2 - 3\gamma^2\epsilon^2\hbar^5 x_2^3 y_1^3\right) == 0, \text{True}\right\}\right\}$$

dS is convolution inverse of id

```
In[∘]:= Timing[HL[# ≡ dϵ₁ dη₁] & /@ {(dΔ₁→₁,₂ // dS₁) // dm₁,₂→₁, (dΔ₁→₁,₂ // dS₂) // dm₁,₂→₁}]
Out[∘]= {3.625, {True, True}}
```

dS is a (co)-algebra anti-morphism

```
In[∘]:= Timing[HL /@
          Expand /@ {(dm₁,₂→₁ // dS₁) ≡ ((dS₁ dS₂) // dm₂,₁→₁), (dS₁ // dΔ₁→₁,₂) ≡ (dΔ₁→₂,₁ // (dS₁ dS₂))}]
Out[∘]= {7.26563, {True, True}}
```

Quasi-triangular axiom 1:

```
In[∘]:= Timing[
         HL /@ {(R₁,₃ // dΔ₁→₁,₂) ≡ ((R₁,₄ R₂,₃) // dm₃,₄→₃), (R₁,₂ // dΔ₂→₂,₃) ≡ ((R₁,₂ R₄,₃) // dm₁,₄→₁)}]
Out[∘]= {0.453125, {True, True}}
```

Quasi-triangular axiom 2:

```
In[∘]:= Timing@HL[((dΔ₁->₁,₂ R₃,₄) // (dm₁,₃→₁ dm₂,₄→₂)) ≡ ((R₁,₂ dΔ₁→₃,₄) // (dm₁,₄→₁ dm₂,₃→₂))]
Out[∘]= {1.8125, True}
```

The Drinfel'd element inverse property, $(u_1\,\overline{u}_2)$ // $\text{dm}_{1,2\to 1} \equiv \text{d}\epsilon_i$:

```
In[∘]:= Timing@HL[((R₁,₂ ~ B₁ ~ dS₁ ~ B₁,₂ ~ dm₂,₁→i) (R₁,₂ ~ B₂ ~ dS₂ ~ B₂ ~ dS₂ ~ B₁,₂ ~ dm₂,₁→j)) ~ Bi,j ~ dmi,j→i ≡ dηi]
Out[∘]= {1.51563, True}
```

The ribbon element v satisfies $v^2 = S(u)\,u$. The spinner C=uv^{-1}. It is convenient to compute $z = S(u)\,u^{-1}$ which is something easy.

```
In[∘]:= Timing@Block[{$k = 2},
         (((R₁,₂ ~ B₁ ~ dS₁ ~ B₁,₂ ~ dm₂,₁→i) ~ Bi ~ dSi) (R₁,₂ ~ B₂ ~ dS₂ ~ B₂ ~ dS₂ ~ B₁,₂ ~ dm₂,₁→j)) ~ Bi,j ~ dmi,j→i]
```

$$\text{Out}= \left\{2.0625, \mathbb{E}_{\{\}\to\{i\}}\left[0, 0, \frac{1}{B_i} + \frac{\hbar a_i \epsilon}{B_i} + \frac{\hbar^2 a_i^2 \epsilon^2}{2 B_i} + O[\epsilon]^3\right]\right\}$$

In[∘]:=
```
Timing@Block[{$k = 2}, HL /@ {((C_i C̄_j) // dm_{i,j→i}) ≡ dη_i, ((C̄_i C̄_j) // dm_{i,j→i}) ≡
    (((R_{1,2} // dS_1 // dm_{2,1→i}) // dS_i) (R_{1,2} // dS_2 // dS_2 // dm_{2,1→j}) // dm_{i,j→i})}]
```
Out[∘]= {2.3125, {True, True}}

In[∘]:=
```
Timing@Block[{$k = 2}, HL /@ {(C_i C̄_j) ~B_{i,j}~dm_{i,j→i} ≡ 𝔼_{{}→{i}}[0, 0, 1], (C̄_i C̄_j) ~B_{i,j}~dm_{i,j→i} ≡
    (((R_{1,2}~B_1~dS_1~B_{1,2}~dm_{2,1→i}) ~B_i~dS_i) (R_{1,2}~B_2~dS_2~B_2~dS_2~B_{1,2}~dm_{2,1→j})) ~B_{i,j}~dm_{i,j→i}}]
```
Out[∘]= {2.29688, {True, True}}

Reidemeister 2:

In[∘]:=
```
Timing[HL[# ≡ dη_1 dη_2] & /@ {(R̄_{1,2} R_{3,4}) // (dm_{1,3→1} dm_{2,4→2}), (R_{1,2} R̄_{3,4}) // (dm_{1,3→1} dm_{2,4→2})}]
```
Out[∘]= {1.32813, {True, True}}

Cyclic Reidemeister 2:

In[∘]:=
```
Timing@HL[((R_{1,4} R̄_{5,2} C̄_3) // dm_{2,4→2} // dm_{1,3→1} // dm_{1,5→1}) ≡ C̄_1 dη_2]
```
Out[∘]= {0.90625, True}

Reidemeister 3:

In[∘]:=
```
Timing@HL[(R_{1,2} R_{6,3} R_{4,5} // dm_{1,6→1} dm_{2,4→2} dm_{3,5→3} ) ≡ (R_{2,3} R_{1,4} R_{5,6} // dm_{1,5→1} dm_{2,6→2} dm_{3,4→3})]
```
Out[∘]= {2.92188, True}

Relations between the four kinks:

In[∘]:=
```
Timing[HL /@ {Kink_i ≡ ((R_{3,1} C_2) // dm_{1,2→1} // dm_{1,3→i}),
   K̄ink_j ≡ ((R̄_{3,1} C̄_2) // dm_{1,2→1} // dm_{1,3→j}), ((Kink_i K̄ink_j) // dm_{i,j→1}) ≡ dη_1}]
```
Out[∘]= {2.67188, {True, True, True}}

The Trefoil

In[∘]:=
```
Timing@Block[{$k = 1},
  Z31 = R_{1,5} R_{6,2} R_{3,7} C̄_4 K̄ink_8 K̄ink_9 K̄ink_10;
  Do[Z31 = Z31 // dm_{1,r→1}, {r, 2, 10}];
  {Simplify /@ Z31, Simplify /@ (Z31~B_1~b2t_1 /. T_1 → T)}]
```

Out[∘]=

$$\left\{2.20313, \left\{\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, \frac{B_1}{1 - B_1 + B_1^2} - \frac{\hbar B_1 \left(-a_1 \left(-1 + B_1 - B_1^3 + B_1^4\right) + \gamma \left(B_1 - 2 B_1^2 - 2 B_1^4 + 2 \hbar x_1 y_1 + B_1^3 (3 + 2 \hbar x_1 y_1)\right)\right) \epsilon}{\left(1 - B_1 + B_1^2\right)^3} + O[\epsilon]^2\right], \mathbb{E}_{\{\}\to\{1\}}\left[0, 0, \frac{T}{1 - T + T^2} + \frac{T \hbar \left(T \left(-1 + 2 T - 3 T^2 + 2 T^3\right) \gamma + 2 \left(-1 + T - T^3 + T^4\right) a_1 - 2 \left(1 + T^3\right) \gamma \hbar x_1 y_1\right) \epsilon}{\left(1 - T + T^2\right)^3} + O[\epsilon]^2\right]\right\}\right\}$$

b2t, t2b, knot tensors.

In[∘]:= $\text{HL}[(\text{b2t}_i \mathbin{//} \text{t2b}_i) \equiv d\sigma_{i\to i}]$

Out[∘]= True

In[∘]:= $\text{t2b}_i \mathbin{//} \text{b2t}_i$

Out[∘]= $\mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i + t_i\,\tau_i,\ y_i\,\eta_i + x_i\,\xi_i,\ 1 + O[\epsilon]^3\right]$

Reidemeister 2:

In[∘]:= $\text{Timing}\left[\text{HL}[\#\equiv d\eta_1\,d\eta_2]\ \&\ /@\left\{\left(\overline{\text{kR}}_{1,2}\,\text{kR}_{3,4}\right) \mathbin{//} (\text{km}_{1,3\to 1}\,\text{km}_{2,4\to 2}),\ \left(\text{kR}_{1,2}\,\overline{\text{kR}}_{3,4}\right) \mathbin{//} (\text{km}_{1,3\to 1}\,\text{km}_{2,4\to 2})\right\}\right]$

Out[∘]= {2.21875, {True, True}}

Cyclic Reidemeister 2:

In[∘]:= $\text{Timing}@\text{HL}\left[\left(\left(\text{kR}_{1,4}\,\overline{\text{kR}}_{5,2}\,\overline{\text{kC}}_3\right) \mathbin{//} \text{km}_{2,4\to 2} \mathbin{//} \text{km}_{1,3\to 1} \mathbin{//} \text{km}_{1,5\to 1}\right) \equiv \overline{\text{kC}}_1\,d\eta_2\right]$

Out[∘]= {1.01563, True}

Reidemeister 3:

In[∘]:= $\text{Timing}@\text{HL}[(\text{kR}_{1,2}\,\text{kR}_{4,3}\,\text{kR}_{5,6} \mathbin{//} \text{km}_{1,4\to 1} \mathbin{//} \text{km}_{2,5\to 2} \mathbin{//} \text{km}_{3,6\to 3}) \equiv$
$(\text{kR}_{1,6}\,\text{kR}_{2,3}\,\text{kR}_{4,5} \mathbin{//} \text{km}_{1,4\to 1} \mathbin{//} \text{km}_{2,5\to 2} \mathbin{//} \text{km}_{3,6\to 3})]$

Out[∘]= {1.5625, True}

Relations between the four kinks:

In[∘]:= $\text{Timing}\left[\text{HL}\ /@\left\{\text{kKink}_i \equiv ((\text{kR}_{3,1}\,\text{kC}_2) \mathbin{//} \text{km}_{1,2\to 1} \mathbin{//} \text{km}_{1,3\to i}),\right.\right.$
$\left.\left.\overline{\text{kKink}}_j \equiv \left(\left(\overline{\text{kR}}_{3,1}\,\overline{\text{kC}}_2\right) \mathbin{//} \text{km}_{1,2\to 1} \mathbin{//} \text{km}_{1,3\to j}\right),\ \left(\left(\text{kKink}_i\,\overline{\text{kKink}}_j\right) \mathbin{//} \text{km}_{i,j\to 1}\right) \equiv d\eta_1\right\}\right]$

Out[∘]= {1.51563, {True, True, True}}

The Trefoil

In[∘]:= $\text{Timing}@\text{Block}\Big[\{\$k = 1\},$
$\quad \text{Z31} = \text{kR}_{1,5}\,\text{kR}_{6,2}\,\text{kR}_{3,7}\,\overline{\text{kC}}_4\,\overline{\text{kKink}}_8\,\overline{\text{kKink}}_9\,\overline{\text{kKink}}_{10};$
$\quad \text{Do}[\text{Z31} = \text{Z31} \mathbin{//} \text{km}_{1,r\to 1},\ \{r, 2, 10\}];$
$\quad \text{Simplify}\ /@\ \text{Z31}\Big]$

Out[∘]= $\left\{1.48438,\ \mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \frac{T}{1-T+T^2} + \frac{T\,\hbar\left(T\left(-1+2T-3T^2+2T^3\right)\gamma + 2\left(-1+T-T^3+T^4\right)a_1 - 2\left(1+T^3\right)\gamma\,\hbar\,x_1\,y_1\right)\epsilon}{\left(1-T+T^2\right)^3} + O[\epsilon]^2\right]\right\}$

In[∘]:=
```
Timing@Block[{$k = 1},
  Z31 = kR1,5 kR6,2 kR3,7 kC̄4 kKink̄8 kKink̄9 kKink̄10;
  Do[Z31 = Z31 ~ B1,r ~ km1,r→1, {r, 2, 10}];
  Simplify /@ Z31]
```

Out[∘]=
$$\left\{1.23438,\ \mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \frac{T}{1-T+T^2}+\frac{T\hbar\left(T\left(-1+2T-3T^2+2T^3\right)\gamma+2\left(-1+T-T^3+T^4\right)a_1-2\left(1+T^3\right)\gamma\hbar x_1 y_1\right)\epsilon}{\left(1-T+T^2\right)^3}+O[\epsilon]^2\right]\right\}$$

CU

Associativity of CU:

In[∘]:=
```
Timing@Block[{$k = 3}, HL[(cm1,2→1 // cm1,3→1) ≡ (cm2,3→2 // cm1,2→1)]]
```

Out[∘]= {6.0625, True}

Associativity, co-associativity, and Δ is an algebra morphism:

In[∘]:=
```
Timing@Block[{$k = 3}, HL /@ {(cm1,2→1 // cm1,3→1) ≡ (cm2,3→2 // cm1,2→1),
    (cΔ1→1,2 // cΔ2→2,3) ≡ (cΔ1→1,3 // cΔ1→1,2),
    (cm1,2→1 // cΔ1→1,2) ≡ ((cΔ1→1,3 cΔ2→2,4) // (cm3,4→2 cm1,2→1))}]
```

Out[∘]= {7.57813, {True, True, True}}

S is convolution inverse of id:

In[∘]:=
```
Timing@Block[{$k = 3}, HL[# ≡ cϵ1 cη1] & /@ {
    (cΔ1→1,2 ~ B1 ~ cS1) ~ B1,2 ~ cm1,2→1, (cΔ1→1,2 ~ B2 ~ cS2) ~ B1,2 ~ cm1,2→1}]
```

Out[∘]= {3.375, {True, True}}

S is an algebra anti-(co)morphism

In[∘]:=
```
Timing@Block[{$k = 3},
  HL /@ {cm1,2→1 ~ B1 ~ cS1 ≡ (cS1 cS2) ~ B1,2 ~ cm2,1→1, cS1 ~ B1 ~ cΔ1→1,2 ≡ cΔ1→2,1 ~ B1,2 ~ (cS1 cS2)}]
```

Out[∘]= {11.4375, {True, True}}

Classical is the $\hbar \to 0$ limit of quantum:

In[∘]:=
```
ClassicalLimit[f_] := Normal@Series[Normal[f] /. U2l, {ℏ, 0, 0}] //. l2U;
(*ClassicalLimit[f_]:=Limit[Normal[f]/.U2l,ℏ→0]//.l2U;*)
Timing[HL /@ Simplify /@
   {cm1,2→3 ≡ ClassicalLimit /@ dm1,2→3,
    (cΔ1→2,3 /. τ1 → 0) ≡ ClassicalLimit /@ dΔ1→2,3, cS1 ≡ ClassicalLimit /@ dS1}]
```

Out[∘]= {3.64063, {True, True, True}}

In[∘]:=
```
EndProfile[];
```

In[∘]:=
```
PrintProfile[]
```

```
Out[•]= ProfileRoot is root. Profiled time: 90.793
   (   210)   1.047/ 74.511 above B
   (    59)   2.659/ 16.109 above Boot
   (   270)   0.079/  0.173 above CF
 CCF: called 67319 times, time in 21.671/36.418
   ( 67319)  21.671/ 36.418 under CF
   ( 67319)  10.948/ 14.747 above Together
 Zip: called 3168 times, time in 20.285/114.069
   (   294)   2.314/ 16.128 under LZip
   (   294)   1.725/  8.084 under QZip
   (  2580)  16.246/ 89.857 under Zip
   (  3168)   3.927/  3.927 above Collect
   (  2580)  16.246/ 89.857 above Zip
 CF: called 13068 times, time in 17.881/54.299
   (  1433)   1.003/  1.844 under EEQ
   (   179)   0.205/  0.344 under Boot
   (  1416)   7.565/ 26.166 under LZip
   (   270)   0.079/  0.173 under ProfileRoot
   (  9770)   9.029/ 25.772 under QZip
   ( 67319)  21.671/ 36.418 above CCF
 Together: called 67319 times, time in 10.948/14.747
   ( 67319)  10.948/ 14.747 under CCF
   ( 67319)   3.799/  3.799 above Exp
 LZip: called 294 times, time in 4.084/49.226
   (   294)   4.084/ 49.226 under B
   (  1433)   1.004/  2.848 above EEQ
   (  1416)   7.565/ 26.166 above CF
   (   294)   2.314/ 16.128 above Zip
 Collect: called 3168 times, time in 3.927/3.927
   (  3168)   3.927/  3.927 under Zip
 Boot: called 83 times, time in 3.894/23.5
   (    24)   1.235/  7.391 under Boot
   (    59)   2.659/ 16.109 under ProfileRoot
   (    84)   0.250/ 11.871 above B
   (    24)   1.235/  7.391 above Boot
   (   179)   0.205/  0.344 above CF
 Exp: called 67319 times, time in 3.799/3.799
   ( 67319)   3.799/  3.799 under Together
 QZip: called 294 times, time in 2.003/35.859
   (   294)   2.003/ 35.859 under B
   (  9770)   9.029/ 25.772 above CF
   (   294)   1.725/  8.084 above Zip
 B: called 294 times, time in 1.297/86.382
   (    84)   0.250/ 11.871 under Boot
   (   210)   1.047/ 74.511 under ProfileRoot
   (   294)   4.084/ 49.226 above LZip
   (   294)   2.003/ 35.859 above QZip
 EEQ: called 1433 times, time in 1.004/2.848
   (  1433)   1.004/  2.848 under LZip
   (  1433)   1.003/  1.844 above CF
```