Pensieve header: Solving the KV equations ($k=0).

Follows code in Projects/SL2Portfolio/SL2PortfolioProgram.nb.

Startup

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio2"];
Once[<< KnotTheory`];
Once[<< "../Profile/Profile.m";];
<< "Engine-Speedy.m";
<< "Objects.m";
$k = 0;
HL[ℰ_] := Style[ℰ, Background → Green];
```

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

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ToFileName: String or list of strings expected at position 1 in ToFileName[{File, WikiLink, mathematica}].

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, QuantumGroups}].

```
Loading KnotTheory` version of January 20, 2015, 10:42:19.1122.
Read more at http://katlas.org/wiki/KnotTheory.

This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.

This version: June 2018. Original version: July 1994.
```

```
CCF[ℰ_] := PP_CCF@ExpandDenominator@ExpandNumerator@PP_Together@Together[PP_Exp[
        Expand[ℰ] //. e^x_ e^y_ :> e^(x+y) /. e^x_ :> e^CCF[x]]];
CF[ℰ_List] := CF /@ ℰ;
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := PP_CF@Module[
    {vs = Cases[ℰ, (y | a | x | η | β | τ | α | ξ)_, ∞] ⋃ {y, a, x, η, β, τ, α, ξ}},
    Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :> CCF[c] (Times @@ vs^ps)]
   ];
CF[ℰ_𝔼] := CF /@ ℰ; CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

```
R_{i,j}
```

$$\mathbb{E}_{\{\}\to\{i,j\}}[\hbar\, a_j\, b_i,\ \hbar\, x_j\, y_i,\ 1]$$

```
Timing@
 HL[(R_{1,2} R_{4,3} R_{5,6} // dm_{1,4↦1} // dm_{2,5↦2} // dm_{3,6↦3}) ≡ (R_{1,6} R_{2,3} R_{4,5} // dm_{1,4↦1} // dm_{2,5↦2} // dm_{3,6↦3})]
```

```
{0.8125, True}
```

```
Timing@
 HL[(R_{1,2} R_{4,3} R_{5,6} // cm_{1,4↦1} // cm_{2,5↦2} // cm_{3,6↦3}) ≡ (R_{1,6} R_{2,3} R_{4,5} // cm_{1,4↦1} // cm_{2,5↦2} // cm_{3,6↦3})]
```

$$\{0.234375,\ \hbar\, x_2\, y_1 + \gamma\, \hbar^2\, b_2\, x_3\, y_1 + \hbar\, B_2\, x_3\, y_1 + \hbar\, B_1\, x_3\, y_2 == \hbar\, x_2\, y_1 + \hbar\, x_3\, y_1 + \hbar\, B_1\, x_3\, y_2\}$$

```
In[•]:= cR_{i_,j_} := 𝔼_{}→{i,j}[ħ a_j b_i, (1 - B_i)/(γ b_i) x_j y_i, 1];
        cR_{i,j}
```

```
Out[•]= 𝔼_{}→{i,j}[ħ a_j b_i, (1 - B_i) x_j y_i / (γ b_i), 1]
```

```
In[•]:= HL@Simplify[(cR_{1,2} cR_{4,3} cR_{5,6} // cm_{1,4→1} // cm_{2,5→2} // cm_{3,6→3}) ≡
          (cR_{1,6} cR_{2,3} cR_{4,5} // cm_{1,4→1} // cm_{2,5→2} // cm_{3,6→3})]
```

```
Out[•]= True
```

According to IntegrationWithGamma.nb, $\delta = 1$ below.

```
In[•]:= V0_{i_,j_} := 𝔼_{}→{i,j}[δ/2 ħ a_j b_i, 0, 1];
        V0_{i,j}
```

```
Out[•]= 𝔼_{}→{i,j}[1/2 δ ħ a_j b_i, 0, 1]
```

```
In[•]:= R4l = (V0_{-1,-2} (cR_{1,3} // cΔ_{1→1,2})) // cm_{-1,1→1} // cm_{-2,2→2}
```

```
Out[•]= 𝔼_{}→{1,2,3}[1/2 δ ħ a_2 b_1 + a_3 (ħ b_1 + ħ b_2), (1 - B_1 B_2) x_3 y_1 / (γ b_1 + γ b_2) + (B_1^{δ/2} - B_1^{1+δ/2} B_2) x_3 y_2 / (γ b_1 + γ b_2), 1 + O[∈]^1]
```

```
In[•]:= R4r = (cR_{-2,-3} cR_{-1,3} // cm_{-3,3→3}) V0_{1,2} // cm_{-1,1→1} // cm_{-2,2→2}
```

```
Out[•]= 𝔼_{}→{1,2,3}[1/2 δ ħ a_2 b_1 + a_3 (ħ b_1 + ħ b_2), (1 - B_1) x_3 y_1 / (γ b_1) + (B_1 - B_1 B_2) x_3 y_2 / (γ b_2), 1 + O[∈]^1]
```

```
In[•]:= V1_{i_,j_} := 𝔼_{}→{i,j}[ħ/2 a_j b_i, ħ Fij y_i x_j + ħ Fji x_i y_j + ħ Fii x_i y_i + ħ Fjj x_j y_j, ω[B_i, B_j]];
        V1_{i,j}
```

```
Out[•]= 𝔼_{}→{i,j}[1/2 ħ a_j b_i, Fii ħ x_i y_i + Fij ħ x_j y_i + Fji ħ x_i y_j + Fjj ħ x_j y_j, ω[B_i, B_j]]
```

```
In[•]:= R4l = (V1_{-1,-2} (cR_{1,3} // cΔ_{1→1,2})) // cm_{-1,1→1} // cm_{-2,2→2}
```

```
Out[•]= 𝔼_{}→{1,2,3}[1/2 ħ a_2 b_1 + a_3 (ħ b_1 + ħ b_2), Fii ħ x_1 y_1 + Fij ħ x_2 y_1 + 1/(γ b_1 + γ b_2)
        (1 + Fii γ ħ b_1 + Fij γ ħ b_2 - B_1 B_2 - Fii γ ħ b_1 B_1 B_2 - Fij γ ħ b_2 B_1 B_2) x_3 y_1 +
        Fji ħ x_1 y_2 + Fjj ħ x_2 y_2 + 1/(γ b_1 + γ b_2)
        (Fji γ ħ b_1 + Fjj γ ħ b_2 + √B_1 - Fji γ ħ b_1 B_1 B_2 - Fjj γ ħ b_2 B_1 B_2 - B_1^{3/2} B_2) x_3 y_2, ω[B_1, B_2] + O[∈]^1]
```

In[•]:=
```
R4r = (cR_{-2,-3} cR_{-1,3} // cm_{-3,3→3}) V1_{1,2} // cm_{-1,1→1} // cm_{-2,2→2}
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{1,2,3\}}\Big[\frac{1}{2}\hbar\, a_2\, b_1 + a_3\,\big(\hbar\, b_1 + \hbar\, b_2\big),\ \mathrm{Fii}\,\hbar\, x_1\, y_1 + \mathrm{Fij}\,\hbar\, x_2\, y_1 + \frac{\big(1-B_1\big)\,x_3\,y_1}{\gamma\, b_1} + \mathrm{Fji}\,\hbar\, x_1\, y_2 + \mathrm{Fjj}\,\hbar\, x_2\, y_2 + \frac{(B_1 - B_1\,B_2)\,x_3\,y_2}{\gamma\, b_2},\ \omega[B_1, B_2] + O[\epsilon]^1\Big]$$

In[•]:=
```
eqs = Simplify@Coefficient[Numerator@Simplify[R4l[[2]] - R4r[[2]]] / x_3, #] & /@ {y_1, y_2}
```

Out[•]=
$$\Big\{b_2\,\big(b_2\,\big(-1+B_1\big) + \mathrm{Fii}\,\gamma\,\hbar\, b_1^2\,\big(1 - B_1\,B_2\big) + b_1\,\big(-B_1\,\big(-1+B_2\big) + \mathrm{Fij}\,\gamma\,\hbar\, b_2\,\big(1-B_1\,B_2\big)\big)\big),\ b_1\,\Big(b_1\,\big(B_1\,\big(-1+B_2\big) + \mathrm{Fji}\,\gamma\,\hbar\, b_2\,\big(1-B_1\,B_2\big)\big) + b_2\,\Big(-\Big(-\sqrt{B_1}+B_1\Big)\,\Big(1+\sqrt{B_1}\,B_2\Big) + \mathrm{Fjj}\,\gamma\,\hbar\, b_2\,\big(1-B_1\,B_2\big)\Big)\Big)\Big\}$$

In[•]:=
```
Simplify[{Fii, Fjj} /. Solve[(# == 0) & /@ eqs, {Fii, Fjj}]]
```

Out[•]=
$$\Big\{\Big\{\big(-b_1\,B_1\,\big(-1+B_2\big) + b_2\,\big(-1+B_1+\mathrm{Fij}\,\gamma\,\hbar\, b_1\,\big(1-B_1\,B_2\big)\big)\big) \Big/ \big(\gamma\,\hbar\, b_1^2\,\big(-1+B_1\,B_2\big)\big),\ \Big(-b_2\,\Big(-1+\sqrt{B_1}\Big)\,\sqrt{B_1}\,\Big(1+\sqrt{B_1}\,B_2\Big) + b_1\,\big(B_1\,\big(-1+B_2\big) + \mathrm{Fji}\,\gamma\,\hbar\, b_2\,\big(1-B_1\,B_2\big)\big)\Big) \Big/ \big(\gamma\,\hbar\, b_2^2\,\big(-1+B_1\,B_2\big)\big)\Big\}\Big\}$$

In[•]:=
```
V2_{i_,j_} := 𝔼_{}→{i,j}[ħ/2 a_j b_i, ħ Fij y_i x_j + ħ Fji x_i y_j +
  ħ((-b_i B_i (-1 + B_j) + b_2 (-1 + B_i + Fij γ ħ b_i (1 - B_i B_j))))/(γ ħ b_i^2 (-1 + B_i B_j))) x_i y_i +
  ħ((-b_j (-1 + Sqrt[B_i]) Sqrt[B_i] (1 + Sqrt[B_i] B_j) + b_i (B_i (-1 + B_j) + Fji γ ħ b_j (1 - B_i B_j)))/
    (γ ħ b_j^2 (-1 + B_i B_j))) x_j y_j, ω[B_i, B_j]];
V2_{i,j}
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{i,j\}}\Big[\frac{1}{2}\hbar\, a_j\, b_i,\ \big(\big(-b_i\,B_i\,\big(-1+B_j\big) + b_2\,\big(-1+B_i+\mathrm{Fij}\,\gamma\,\hbar\, b_i\,\big(1-B_i\,B_j\big)\big)\big)\,x_i\,y_i\big) \Big/ \big(\gamma\, b_i^2\,\big(-1+B_i\,B_j\big)\big) + \mathrm{Fij}\,\hbar\, x_j\, y_i + \mathrm{Fji}\,\hbar\, x_i\, y_j + \Big(\Big(-b_j\,\Big(-1+\sqrt{B_i}\Big)\,\sqrt{B_i}\,\Big(1+\sqrt{B_i}\,B_j\Big) + b_i\,\big(B_i\,\big(-1+B_j\big) + \mathrm{Fji}\,\gamma\,\hbar\, b_j\,\big(1-B_i\,B_j\big)\big)\Big)\,x_j\,y_j\Big) \Big/ \big(\gamma\, b_j^2\,\big(-1+B_i\,B_j\big)\big),\ \omega[B_i, B_j]\Big]$$

In[•]:=
```
R4l = (V2_{-1,-2} (cR_{1,3} // cΔ_{1→1,2})) // cm_{-1,1→1} // cm_{-2,2→2}
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{1,2,3\}}\Big[\frac{1}{2}\hbar\, a_2\, b_1 + a_3\,\big(\hbar\, b_1 + \hbar\, b_2\big),\ \big(\big(-b_2 + \mathrm{Fij}\,\gamma\,\hbar\, b_1\, b_2 + b_1\, B_1 + b_2\, B_1 - b_1\, B_1\, B_2 - \mathrm{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2\big)\,x_1\,y_1\big) \Big/ \big(-\gamma\, b_1^2 + \gamma\, b_1^2\, B_1\, B_2\big) + \mathrm{Fij}\,\hbar\, x_2\, y_1 + \frac{\big(1-B_1\big)\,x_3\,y_1}{\gamma\, b_1} + \mathrm{Fji}\,\hbar\, x_1\, y_2 + \Big(\Big(\mathrm{Fji}\,\gamma\,\hbar\, b_1\, b_2 + b_2\,\sqrt{B_1} - b_1\, B_1 - b_2\, B_1 + b_1\, B_1\, B_2 + b_2\, B_1\, B_2 - \mathrm{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2 - b_2\, B_1^{3/2}\, B_2\Big)\,x_2\,y_2\Big) \Big/ \big(-\gamma\, b_2^2 + \gamma\, b_2^2\, B_1\, B_2\big) + \frac{(B_1 - B_1\,B_2)\,x_3\,y_2}{\gamma\, b_2},\ \omega[B_1, B_2] + O[\epsilon]^1\Big]$$

In[•]:=
```
R4r = (cR_{-2,-3} cR_{-1,3} // cm_{-3,3→3}) V2_{1,2} // cm_{-1,1→1} // cm_{-2,2→2}
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{1,2,3\}}\Big[\frac{1}{2}\hbar\,a_2\,b_1+a_3\,\big(\hbar\,b_1+\hbar\,b_2\big),\ \Big(\big(-b_2+\text{Fij}\,\gamma\,\hbar\,b_1\,b_2+b_1\,B_1+b_2\,B_1-b_1\,B_1\,B_2-\text{Fij}\,\gamma\,\hbar\,b_1\,b_2\,B_1\,B_2\big)\,x_1\,y_1\Big)\Big/\big(-\gamma\,b_1^2+\gamma\,b_1^2\,B_1\,B_2\big)+\text{Fij}\,\hbar\,x_2\,y_1+\frac{\big(1-B_1\big)\,x_3\,y_1}{\gamma\,b_1}+\text{Fji}\,\hbar\,x_1\,y_2+\Big(\Big(\text{Fji}\,\gamma\,\hbar\,b_1\,b_2+b_2\,\sqrt{B_1}-b_1\,B_1-b_2\,B_1+b_1\,B_1\,B_2+b_2\,B_1\,B_2-\text{Fji}\,\gamma\,\hbar\,b_1\,b_2\,B_1\,B_2-b_2\,B_1^{3/2}\,B_2\Big)\,x_2\,y_2\Big)\Big/\big(-\gamma\,b_2^2+\gamma\,b_2^2\,B_1\,B_2\big)+\frac{(B_1-B_1\,B_2)\,x_3\,y_2}{\gamma\,b_2},\ \omega[B_1,B_2]+O[\in]^1\Big]$$

In[•]:=
```
HL@Simplify[R4l ≡ R4r]
```

Out[•]=
```
True
```

In[•]:=
```
R4l = (V0_{-1,-2} (cR_{3,1} // cΔ_{1→1,2})) // cm_{-1,1→1} // cm_{-2,2→2}
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{1,2,3\}}\Big[\hbar\,a_1\,b_3+\frac{1}{2}\,a_2\,\big(\hbar\,b_1+2\,\hbar\,b_3\big),\ \Big(\big(-b_2\,B_3+\text{Fij}\,\gamma\,\hbar\,b_1\,b_2\,B_3+b_1\,B_1\,B_3+b_2\,B_1\,B_3-b_1\,B_1\,B_2\,B_3-\text{Fij}\,\gamma\,\hbar\,b_1\,b_2\,B_1\,B_2\,B_3\big)\,x_1\,y_1\Big)\Big/\big(-\gamma\,b_1^2+\gamma\,b_1^2\,B_1\,B_2\big)+\text{Fij}\,\hbar\,B_3\,x_2\,y_1+\text{Fji}\,\hbar\,B_3\,x_1\,y_2+\Big(\Big(\text{Fji}\,\gamma\,\hbar\,b_1\,b_2\,B_3+b_2\,\sqrt{B_1}\,B_3-b_1\,B_1\,B_3-b_2\,B_1\,B_3+b_1\,B_1\,B_2\,B_3+b_2\,B_1\,B_2\,B_3-\text{Fji}\,\gamma\,\hbar\,b_1\,b_2\,B_1\,B_2\,B_3-b_2\,B_1^{3/2}\,B_2\,B_3\Big)\,x_2\,y_2\Big)\Big/\big(-\gamma\,b_2^2+\gamma\,b_2^2\,B_1\,B_2\big)+\frac{\big(1-B_3\big)\,x_1\,y_3}{\gamma\,b_3}+\frac{\big(1-B_3\big)\,x_2\,y_3}{\gamma\,b_3},\ \omega[B_1,B_2]+O[\in]^1\Big]$$

In[•]:=
```
R4r = (cR_{-3,-2} cR_{3,-1} // cm_{-3,3→3}) V0_{1,2} // cm_{-1,1→1} // cm_{-2,2→2}
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{1,2,3\}}\Big[\hbar\,a_1\,b_3+\frac{1}{2}\,a_2\,\big(\hbar\,b_1+2\,\hbar\,b_3\big),\ \Big(\big(-b_2\,B_3+\text{Fij}\,\gamma\,\hbar\,b_1\,b_2\,B_3+b_1\,B_1\,B_3+b_2\,B_1\,B_3-b_1\,B_1\,B_2\,B_3-\text{Fij}\,\gamma\,\hbar\,b_1\,b_2\,B_1\,B_2\,B_3\big)\,x_1\,y_1\Big)\Big/\big(-\gamma\,b_1^2+\gamma\,b_1^2\,B_1\,B_2\big)+\text{Fij}\,\hbar\,B_3\,x_2\,y_1+\text{Fji}\,\hbar\,B_3\,x_1\,y_2+\Big(\Big(\text{Fji}\,\gamma\,\hbar\,b_1\,b_2\,B_3+b_2\,\sqrt{B_1}\,B_3-b_1\,B_1\,B_3-b_2\,B_1\,B_3+b_1\,B_1\,B_2\,B_3+b_2\,B_1\,B_2\,B_3-\text{Fji}\,\gamma\,\hbar\,b_1\,b_2\,B_1\,B_2\,B_3-b_2\,B_1^{3/2}\,B_2\,B_3\Big)\,x_2\,y_2\Big)\Big/\big(-\gamma\,b_2^2+\gamma\,b_2^2\,B_1\,B_2\big)+\frac{1}{-\gamma\,b_1\,b_3+\gamma\,b_1\,b_3\,B_1\,B_2}\Big(-b_1-b_2+\text{Fij}\,\gamma\,\hbar\,b_1\,b_2-\text{Fji}\,\gamma\,\hbar\,b_1\,b_2+b_1\,B_1+b_2\,B_1-\text{Fij}\,\gamma\,\hbar\,b_1\,b_2\,B_1\,B_2+\text{Fji}\,\gamma\,\hbar\,b_1\,b_2\,B_1\,B_2+b_1\,B_3+b_2\,B_3-\text{Fij}\,\gamma\,\hbar\,b_1\,b_2\,B_3+\text{Fji}\,\gamma\,\hbar\,b_1\,b_2\,B_3-b_1\,B_1\,B_3-b_2\,B_1\,B_3+\text{Fij}\,\gamma\,\hbar\,b_1\,b_2\,B_1\,B_2\,B_3-\text{Fji}\,\gamma\,\hbar\,b_1\,b_2\,B_1\,B_2\,B_3\Big)\,x_1\,y_3+\frac{1}{-\gamma\,b_2\,b_3+\gamma\,b_2\,b_3\,B_1\,B_2}\Big(-\text{Fij}\,\gamma\,\hbar\,b_1\,b_2+\text{Fji}\,\gamma\,\hbar\,b_1\,b_2-b_1\,B_1-b_2\,B_1+b_1\,B_1\,B_2+b_2\,B_1\,B_2+\text{Fij}\,\gamma\,\hbar\,b_1\,b_2\,B_1\,B_2-\text{Fji}\,\gamma\,\hbar\,b_1\,b_2\,B_1\,B_2+\text{Fij}\,\gamma\,\hbar\,b_1\,b_2\,B_3-\text{Fji}\,\gamma\,\hbar\,b_1\,b_2\,B_3+b_1\,B_1\,B_3+b_2\,B_1\,B_3-b_1\,B_1\,B_2\,B_3-b_2\,B_1\,B_2\,B_3-\text{Fij}\,\gamma\,\hbar\,b_1\,b_2\,B_1\,B_2\,B_3+\text{Fji}\,\gamma\,\hbar\,b_1\,b_2\,B_1\,B_2\,B_3\Big)\,x_2\,y_3,\ \omega[B_1,B_2]+O[\in]^1\Big]$$

In[•]:=
```
CF /@ Simplify[R4l ≡ R4r]
```

Out[•]=
$$\frac{1}{-\gamma\, b_1\, b_3 + \gamma\, b_1\, b_3\, B_1\, B_2}$$
$$\left(b_2 - \text{Fij}\,\gamma\,\hbar\, b_1\, b_2 + \text{Fji}\,\gamma\,\hbar\, b_1\, b_2 - b_1\, B_1 - b_2\, B_1 + b_1\, B_1\, B_2 + \text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2 - \text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2 - b_2\, B_3 + \text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_3 - \text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_3 + b_1\, B_1\, B_3 + b_2\, B_1\, B_3 - b_1\, B_1\, B_2\, B_3 - \text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2\, B_3 + \text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2\, B_3\right) x_1\, y_3 +$$
$$\frac{1}{-\gamma\, b_2\, b_3 + \gamma\, b_2\, b_3\, B_1\, B_2}\left(-b_2 + \text{Fij}\,\gamma\,\hbar\, b_1\, b_2 - \text{Fji}\,\gamma\,\hbar\, b_1\, b_2 + b_1\, B_1 + b_2\, B_1 - b_1\, B_1\, B_2 - \text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2 + \text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2 + b_2\, B_3 - \text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_3 + \text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_3 - b_1\, B_1\, B_3 - b_2\, B_1\, B_3 + b_1\, B_1\, B_2\, B_3 + \text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2\, B_3 - \text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2\, B_3\right) x_2\, y_3 == 0$$

In[•]:=
```
Numerator@CF@First@Simplify[R4l ≡ R4r]
```

Out[•]=
$$\frac{1}{-\gamma\, b_1\, b_3 + \gamma\, b_1\, b_3\, B_1\, B_2}$$
$$\left(b_2 - \text{Fij}\,\gamma\,\hbar\, b_1\, b_2 + \text{Fji}\,\gamma\,\hbar\, b_1\, b_2 - b_1\, B_1 - b_2\, B_1 + b_1\, B_1\, B_2 + \text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2 - \text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2 - b_2\, B_3 + \text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_3 - \text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_3 + b_1\, B_1\, B_3 + b_2\, B_1\, B_3 - b_1\, B_1\, B_2\, B_3 - \text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2\, B_3 + \text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2\, B_3\right) x_1\, y_3 +$$
$$\frac{1}{-\gamma\, b_2\, b_3 + \gamma\, b_2\, b_3\, B_1\, B_2}\left(-b_2 + \text{Fij}\,\gamma\,\hbar\, b_1\, b_2 - \text{Fji}\,\gamma\,\hbar\, b_1\, b_2 + b_1\, B_1 + b_2\, B_1 - b_1\, B_1\, B_2 - \text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2 + \text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2 + b_2\, B_3 - \text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_3 + \text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_3 - b_1\, B_1\, B_3 - b_2\, B_1\, B_3 + b_1\, B_1\, B_2\, B_3 + \text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2\, B_3 - \text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2\, B_3\right) x_2\, y_3$$

In[•]:=
```
eqs =
  (Last[#] == 0) & /@ CoefficientRules[Numerator@CF@First@Simplify[R4l ≡ R4r], {x1, x2, y3}]
```

Out[•]=
$$\left\{-\frac{b_2}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} + \frac{\text{Fij}\,\gamma\,\hbar\, b_1\, b_2}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} - \frac{\text{Fji}\,\gamma\,\hbar\, b_1\, b_2}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} + \frac{b_1\, B_1}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} + \frac{b_2\, B_1}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} - \frac{b_1\, B_1\, B_2}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} - \frac{\text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} + \frac{\text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} + \frac{b_2\, B_3}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} - \frac{\text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_3}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} + \frac{\text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_3}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} - \frac{b_1\, B_1\, B_3}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} - \frac{b_2\, B_1\, B_3}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} + \frac{b_1\, B_1\, B_2\, B_3}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} + \frac{\text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2\, B_3}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} - \frac{\text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2\, B_3}{\gamma\, b_1\, b_3 - \gamma\, b_1\, b_3\, B_1\, B_2} == 0,$$
$$\frac{b_2}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} - \frac{\text{Fij}\,\gamma\,\hbar\, b_1\, b_2}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} + \frac{\text{Fji}\,\gamma\,\hbar\, b_1\, b_2}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} - \frac{b_1\, B_1}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} - \frac{b_2\, B_1}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} + \frac{b_1\, B_1\, B_2}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} + \frac{\text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} - \frac{\text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} - \frac{b_2\, B_3}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} + \frac{\text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_3}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} - \frac{\text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_3}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} + \frac{b_1\, B_1\, B_3}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} + \frac{b_2\, B_1\, B_3}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} - \frac{b_1\, B_1\, B_2\, B_3}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} - \frac{\text{Fij}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2\, B_3}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} + \frac{\text{Fji}\,\gamma\,\hbar\, b_1\, b_2\, B_1\, B_2\, B_3}{\gamma\, b_2\, b_3 - \gamma\, b_2\, b_3\, B_1\, B_2} == 0\right\}$$

In[•]:=
```
Solve[eqs, {Fij, Fji}]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[•]=
$$\left\{\left\{\text{Fji}\to \text{Fij}-\frac{-\text{b}_2+\text{b}_1\,\text{B}_1+\text{b}_2\,\text{B}_1-\text{b}_1\,\text{B}_1\,\text{B}_2}{\gamma\,\hbar\,\text{b}_1\,\text{b}_2\,\left(-1+\text{B}_1\,\text{B}_2\right)}\right\}\right\}$$

In[•]:=
```
Simplify /@ CF@(𝔼{}→{i,j}[1/2 ħ a_j b_i, ((-b_i B_i (-1 + B_j) + b_2 (-1 + B_i + Fij γ ħ b_i (1 - B_i B_j))) x_i y_i)/
    (γ b_i^2 (-1 + B_i B_j)) + Fij ħ x_j y_i + Fji ħ x_i y_j +
   ((-b_j (-1 + Sqrt[B_i]) Sqrt[B_i] (1 + Sqrt[B_i] B_j) + b_i (B_i (-1 + B_j) + Fji γ ħ b_j (1 - B_i B_j))) x_j y_j)/
    (γ b_j^2 (-1 + B_i B_j)), ω[B_i, B_j]] /. Fji → Fij - (-b_j + b_i B_i + b_j B_i - b_i B_i B_j)/(γ ħ b_i b_j (-1 + B_i B_j)))
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{\text{i},\text{j}\}}\left[\frac{1}{2}\,\hbar\,\text{a}_\text{j}\,\text{b}_\text{i},\right.$$
$$-\left(\left(\left(\text{b}_\text{i}\,\text{B}_\text{i}\,\left(-1+\text{B}_\text{j}\right)+\text{b}_2\,\left(1-\text{B}_\text{i}+\text{Fij}\,\gamma\,\hbar\,\text{b}_\text{i}\,\left(-1+\text{B}_\text{i}\,\text{B}_\text{j}\right)\right)\right)\,\text{x}_\text{i}\,\text{y}_\text{i}\right)\Big/\left(\gamma\,\text{b}_\text{i}^2\,\left(-1+\text{B}_\text{i}\,\text{B}_\text{j}\right)\right)\right)+\text{Fij}\,\hbar\,\text{x}_\text{j}\,\text{y}_\text{i}+$$
$$\left(\left(\text{b}_\text{i}\,\text{B}_\text{i}\,\left(-1+\text{B}_\text{j}\right)+\text{b}_\text{j}\,\left(1-\text{B}_\text{i}+\text{Fij}\,\gamma\,\hbar\,\text{b}_\text{i}\,\left(-1+\text{B}_\text{i}\,\text{B}_\text{j}\right)\right)\right)\,\text{x}_\text{i}\,\text{y}_\text{j}\right)\Big/\left(\gamma\,\text{b}_\text{i}\,\text{b}_\text{j}\,\left(-1+\text{B}_\text{i}\,\text{B}_\text{j}\right)\right)-$$
$$\left.\frac{\left(-1+\text{Fij}\,\gamma\,\hbar\,\text{b}_\text{i}+\sqrt{\text{B}_\text{i}}\right)\,\text{x}_\text{j}\,\text{y}_\text{j}}{\gamma\,\text{b}_\text{j}},\,\omega\left[\text{B}_\text{i},\,\text{B}_\text{j}\right]\right]$$

In[•]:=
```
V0_i_,j_ := 𝔼{}→{i,j}[1/2 ħ a_j b_i, 1/(γ b_i^2 b_j (-1 + B_i B_j)) (b_2 b_j (-1 + B_i + Fij γ ħ b_i (1 - B_i B_j)) x_i y_i +
    b_i (b_i ((-1 + Fij γ ħ b_i) x_j + Sqrt[B_i] x_j - B_i^(3/2) B_j x_j + B_i ((-1 + B_j) x_i + (1 - Fij γ ħ b_i) B_j x_j)) y_j +
     b_j (x_i y_j - Fij γ ħ b_i (x_j y_i + x_i y_j) + B_i (Fij γ ħ b_i B_j x_j y_i + x_i
        (-(-1 + B_j) y_i + (-1 + Fij γ ħ b_i B_j) y_j)))), ω[B_i, B_j]] /. Fij → Fij[B_i, B_j]
```

In[•]:=
```
R4l = (V0_-1,-2 (cR_3,1 // cΔ_1→1,2)) // cm_-1,1→1 // cm_-2,2→2
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{1,2,3\}}\left[\hbar\,\text{a}_1\,\text{b}_3+\frac{1}{2}\,\text{a}_2\,\left(\hbar\,\text{b}_1+2\,\hbar\,\text{b}_3\right),\right.$$
$$\left(\left(-\text{b}_2\,\text{B}_3+\gamma\,\hbar\,\text{Fij}\left[\text{B}_1,\,\text{B}_2\right]\,\text{b}_1\,\text{b}_2\,\text{B}_3+\text{b}_1\,\text{B}_1\,\text{B}_3+\text{b}_2\,\text{B}_1\,\text{B}_3-\text{b}_1\,\text{B}_1\,\text{B}_2\,\text{B}_3-\gamma\,\hbar\,\text{Fij}\left[\text{B}_1,\,\text{B}_2\right]\,\text{b}_1\,\text{b}_2\,\text{B}_1\,\text{B}_2\,\text{B}_3\right)\right.$$
$$\left.\text{x}_1\,\text{y}_1\right)\Big/\left(-\gamma\,\text{b}_1^2+\gamma\,\text{b}_1^2\,\text{B}_1\,\text{B}_2\right)+\hbar\,\text{Fij}\left[\text{B}_1,\,\text{B}_2\right]\,\text{B}_3\,\text{x}_2\,\text{y}_1+$$
$$\left(\left(\text{b}_2\,\text{B}_3-\gamma\,\hbar\,\text{Fij}\left[\text{B}_1,\,\text{B}_2\right]\,\text{b}_1\,\text{b}_2\,\text{B}_3-\text{b}_1\,\text{B}_1\,\text{B}_3-\text{b}_2\,\text{B}_1\,\text{B}_3+\text{b}_1\,\text{B}_1\,\text{B}_2\,\text{B}_3+\gamma\,\hbar\,\text{Fij}\left[\text{B}_1,\,\text{B}_2\right]\,\text{b}_1\,\text{b}_2\,\text{B}_1\,\text{B}_2\,\text{B}_3\right)\right.$$
$$\left.\text{x}_1\,\text{y}_2\right)\Big/\left(-\gamma\,\text{b}_1\,\text{b}_2+\gamma\,\text{b}_1\,\text{b}_2\,\text{B}_1\,\text{B}_2\right)+$$
$$\frac{\left(\text{B}_3-\gamma\,\hbar\,\text{Fij}\left[\text{B}_1,\,\text{B}_2\right]\,\text{b}_1\,\text{B}_3-\sqrt{\text{B}_1}\,\text{B}_3\right)\,\text{x}_2\,\text{y}_2}{\gamma\,\text{b}_2}+\frac{\left(1-\text{B}_3\right)\,\text{x}_1\,\text{y}_3}{\gamma\,\text{b}_3}+\frac{\left(1-\text{B}_3\right)\,\text{x}_2\,\text{y}_3}{\gamma\,\text{b}_3},\,\omega\left[\text{B}_1,\,\text{B}_2\right]+\text{O}\left[\epsilon\right]^1\left.\right]$$

In[•]:=
```
R4r = (cR_{-3,-2} cR_{3,-1} // cm_{-3,3→3}) V0_{1,2} // cm_{-1,1→1} // cm_{-2,2→2}
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{1,2,3\}}\Big[\hbar\, a_1 b_3 + \frac{1}{2} a_2 \left(\hbar\, b_1 + 2\hbar\, b_3\right),$$
$$\left(\left(-b_2 B_3 + \gamma\hbar\, \text{Fij}[B_1, B_2]\, b_1 b_2 B_3 + b_1 B_1 B_3 + b_2 B_1 B_3 - b_1 B_1 B_2 B_3 - \gamma\hbar\, \text{Fij}[B_1, B_2]\, b_1 b_2 B_1 B_2 B_3\right) x_1 y_1\right) \Big/ \left(-\gamma b_1^2 + \gamma b_1^2 B_1 B_2\right) + \hbar\, \text{Fij}[B_1, B_2]\, B_3\, x_2 y_1 +$$
$$\left(\left(b_2 B_3 - \gamma\hbar\, \text{Fij}[B_1, B_2]\, b_1 b_2 B_3 - b_1 B_1 B_3 - b_2 B_1 B_3 + b_1 B_1 B_2 B_3 + \gamma\hbar\, \text{Fij}[B_1, B_2]\, b_1 b_2 B_1 B_2 B_3\right) x_1 y_2\right) \Big/ \left(-\gamma b_1 b_2 + \gamma b_1 b_2 B_1 B_2\right) +$$
$$\frac{\left(B_3 - \gamma\hbar\, \text{Fij}[B_1, B_2]\, b_1 B_3 - \sqrt{B_1}\, B_3\right) x_2 y_2}{\gamma b_2} + \frac{\left(1 - B_3\right) x_1 y_3}{\gamma b_3} + \frac{\left(1 - B_3\right) x_2 y_3}{\gamma b_3}, \omega[B_1, B_2] + O[\epsilon]^1\Big]$$

In[•]:=
```
HL[CF /@ Simplify[R4l ≡ R4r]]
```

Out[•]=
```
True
```

In[•]:=
```
V0bar_{i_,j_} := E_{}→{i,j}[-ħ/2 a_j b_i, ħ Gij y_i x_j + ħ Gji x_i y_j + ħ Gii x_i y_i + ħ Gjj x_j y_j, ω̄[B_i, B_j]];
V0bar_{i,j}
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{i,j\}}\Big[-\frac{1}{2}\hbar\, a_j b_i, \text{Gii}\,\hbar\, x_i y_i + \text{Gij}\,\hbar\, x_j y_i + \text{Gji}\,\hbar\, x_i y_j + \text{Gjj}\,\hbar\, x_j y_j, \bar{\omega}[B_i, B_j]\Big]$$

In[•]:=
```
Simplify@(V0bar_{-1,-2} V0_{1,2} // cm_{-1,1→1} // cm_{-2,2→2})
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{1,2\}}\Big[0, \frac{1}{\gamma b_1^2 b_2 \sqrt{B_1}\left(-1 + B_1 B_2\right)}\Big(b_2^2\left(-1 + B_1\right)\sqrt{B_1}\, x_1 y_1 +$$
$$\gamma\hbar\, \text{Fij}[B_1, B_2]\, b_1^3\left(-1 + B_1 B_2\right) x_2\left(-y_2 + \gamma\hbar\, b_2\sqrt{B_1}\left(\left(\text{Gii} - \text{Gij}\right) y_1 + \left(\text{Gji} - \text{Gjj}\right) y_2\right)\right) +$$
$$b_1 b_2 x_1\Big(B_1^{3/2} y_1 - B_1^{3/2} B_2 y_1 + y_2 - B_1 y_2 + \gamma\hbar\, b_2\sqrt{B_1}$$
$$\left(\left(\left(\text{Gii} - \text{Gij}\right)\left(-1 + B_1\right) + \text{Fij}[B_1, B_2]\left(1 - B_1 B_2\right)\right) y_1 + \left(\text{Gji} - \text{Gjj}\right)\left(-1 + B_1\right) y_2\right)\Big) +$$
$$b_1^2\Big(\left(-x_2 + \sqrt{B_1}\, x_2 - B_1^{3/2} B_2 x_2 + B_1\left(\left(-1 + B_2\right) x_1 + B_2 x_2\right)\right) y_2 -$$
$$\gamma^2\hbar^2\, \text{Fij}[B_1, B_2]\, b_2^2\sqrt{B_1}\left(-1 + B_1 B_2\right) x_1\left(\left(\text{Gii} - \text{Gij}\right) y_1 + \left(\text{Gji} - \text{Gjj}\right) y_2\right) +$$
$$\gamma\hbar\, b_2\Big(-\text{Fij}[B_1, B_2]\, x_1 y_2 + \text{Fij}[B_1, B_2]\, B_1 B_2\, x_1 y_2 - \sqrt{B_1}\left(x_1\left(\text{Gii}\, y_1 + \text{Gji}\, y_2\right) + x_2$$
$$\left(\left(\text{Gij} + \text{Fij}[B_1, B_2]\right) y_1 + \text{Gjj}\, y_2\right)\right) + B_1^{3/2}\left(B_2 x_2\left(\left(\text{Gij} + \text{Fij}[B_1, B_2]\right) y_1 + \text{Gjj}\, y_2\right) + x_1$$
$$\left(\left(\text{Gii} - \text{Gij} + \text{Gij}\, B_2\right) y_1 + \left(\text{Gji} - \text{Gjj} + \text{Gjj}\, B_2\right) y_2\right)\Big)\Big)\Big), \omega[B_1, B_2]\,\bar{\omega}[B_1, B_2] + O[\epsilon]^1\Big]$$

In[•]:=
```
eqs = (Last[#] == 0) & /@ CoefficientRules[
    Numerator@Simplify@(V0_{-1,-2} V0_{1,2} // cm_{-1,1→1} // cm_{-2,2→2})[[2]], {x1, x2, y1, y2}]
```

Out[•]=
$$\Big\{-\text{Gii}\,\gamma\,\hbar\,b_1^2\,b_2\sqrt{B_1}-b_2^2\sqrt{B_1}-\text{Gii}\,\gamma\,\hbar\,b_1\,b_2^2\sqrt{B_1}+\text{Gij}\,\gamma\,\hbar\,b_1\,b_2^2\sqrt{B_1}+$$
$$\gamma\,\hbar\,\text{Fij}[B_1,B_2]\,b_1\,b_2^2\sqrt{B_1}+\text{Gii}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^2\,b_2^2\sqrt{B_1}-\text{Gij}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^2\,b_2^2\sqrt{B_1}+$$
$$b_1\,b_2\,B_1^{3/2}+\text{Gii}\,\gamma\,\hbar\,b_1^2\,b_2\,B_1^{3/2}-\text{Gij}\,\gamma\,\hbar\,b_1^2\,b_2\,B_1^{3/2}+b_2^2\,B_1^{3/2}+\text{Gii}\,\gamma\,\hbar\,b_1\,b_2^2\,B_1^{3/2}-$$
$$\text{Gij}\,\gamma\,\hbar\,b_1\,b_2^2\,B_1^{3/2}-b_1\,b_2\,B_1^{3/2}\,B_2+\text{Gij}\,\gamma\,\hbar\,b_1^2\,b_2\,B_1^{3/2}\,B_2-\gamma\,\hbar\,\text{Fij}[B_1,B_2]\,b_1\,b_2^2\,B_1^{3/2}\,B_2-$$
$$\text{Gii}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^2\,b_2^2\,B_1^{3/2}\,B_2+\text{Gij}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^2\,b_2^2\,B_1^{3/2}\,B_2==0,$$
$$b_1\,b_2-\gamma\,\hbar\,\text{Fij}[B_1,B_2]\,b_1^2\,b_2-\text{Gji}\,\gamma\,\hbar\,b_1^2\,b_2\sqrt{B_1}-\text{Gji}\,\gamma\,\hbar\,b_1\,b_2^2\sqrt{B_1}+\text{Gjj}\,\gamma\,\hbar\,b_1\,b_2^2\sqrt{B_1}+$$
$$\text{Gji}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^2\,b_2^2\sqrt{B_1}-\text{Gjj}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^2\,b_2^2\sqrt{B_1}-$$
$$b_1^2\,B_1-b_1\,b_2\,B_1+\text{Gji}\,\gamma\,\hbar\,b_1^2\,b_2\,B_1^{3/2}-\text{Gjj}\,\gamma\,\hbar\,b_1^2\,b_2\,B_1^{3/2}+\text{Gji}\,\gamma\,\hbar\,b_1\,b_2^2\,B_1^{3/2}-$$
$$\text{Gjj}\,\gamma\,\hbar\,b_1\,b_2^2\,B_1^{3/2}+b_1^2\,B_1\,B_2+\gamma\,\hbar\,\text{Fij}[B_1,B_2]\,b_1^2\,b_2\,B_1\,B_2+\text{Gjj}\,\gamma\,\hbar\,b_1^2\,b_2\,B_1^{3/2}\,B_2-$$
$$\text{Gji}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^2\,b_2^2\,B_1^{3/2}\,B_2+\text{Gjj}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^2\,b_2^2\,B_1^{3/2}\,B_2==0,$$
$$-\text{Gij}\,\gamma\,\hbar\,b_1^2\,b_2\sqrt{B_1}-\gamma\,\hbar\,\text{Fij}[B_1,B_2]\,b_1^2\,b_2\sqrt{B_1}-\text{Gii}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^3\,b_2\sqrt{B_1}+$$
$$\text{Gij}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^3\,b_2\sqrt{B_1}+\text{Gij}\,\gamma\,\hbar\,b_1^2\,b_2\,B_1^{3/2}\,B_2+\gamma\,\hbar\,\text{Fij}[B_1,B_2]\,b_1^2\,b_2\,B_1^{3/2}\,B_2+$$
$$\text{Gii}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^3\,b_2\,B_1^{3/2}\,B_2-\text{Gij}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^3\,b_2\,B_1^{3/2}\,B_2==0,$$
$$-b_1^2+\gamma\,\hbar\,\text{Fij}[B_1,B_2]\,b_1^3+b_1^2\sqrt{B_1}-\text{Gjj}\,\gamma\,\hbar\,b_1^2\,b_2\sqrt{B_1}-\text{Gji}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^3\,b_2\sqrt{B_1}+$$
$$\text{Gjj}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^3\,b_2\sqrt{B_1}+b_1^2\,B_1\,B_2-\gamma\,\hbar\,\text{Fij}[B_1,B_2]\,b_1^3\,B_1\,B_2-b_1^2\,B_1^{3/2}\,B_2+$$
$$\text{Gjj}\,\gamma\,\hbar\,b_1^2\,b_2\,B_1^{3/2}\,B_2+\text{Gji}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^3\,b_2\,B_1^{3/2}\,B_2-\text{Gjj}\,\gamma^2\,\hbar^2\,\text{Fij}[B_1,B_2]\,b_1^3\,b_2\,B_1^{3/2}\,B_2==0\Big\}$$

In[•]:=
```
Simplify /@ (V0bar_{i,j} /. (Solve[eqs, {Gii, Gij, Gji, Gjj}][[1]] /. {v_{-1} :> v_i, v_{-2} :> v_j}))
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{i,j\}}\Big[-\frac{1}{2}\,\hbar\,a_j\,b_i,\ \Big(1\Big/\Big(\gamma\,b_i\,b_j\,(b_i+b_j)\sqrt{B_i}\ (1-B_i+\gamma\,\hbar\,\text{Fij}[B_i,B_j]\,b_i\,(-1+B_i\,B_j))\Big)\Big)$$
$$\Big(b_j^2\sqrt{B_i}\ (-1+B_i+\gamma\,\hbar\,\text{Fij}[B_i,B_j]\,b_i\,(1-B_i\,B_j))\,x_i\,y_i+b_i^2\Big(B_i^{3/2}\,((-1+B_j)\,x_i-x_j)+$$
$$(-1+\gamma\,\hbar\,\text{Fij}[B_i,B_j]\,b_i)\,x_j+\sqrt{B_i}\,x_j+B_i\,(1-\gamma\,\hbar\,\text{Fij}[B_i,B_j]\,b_i\,B_j)\,x_j\Big)\,y_j+$$
$$b_i\,b_j\Big((-1+\gamma\,\hbar\,\text{Fij}[B_i,B_j]\,b_i)\,x_j\,y_j+B_i\,(1-\gamma\,\hbar\,\text{Fij}[B_i,B_j]\,b_i\,B_j)\,x_j\,y_j+$$
$$B_i^{3/2}\,(x_i\,(-(-1+B_j)\,y_i+(-1+\gamma\,\hbar\,\text{Fij}[B_i,B_j]\,b_i\,B_j)\,y_j)+$$
$$x_j\,(-y_j+\gamma\,\hbar\,\text{Fij}[B_i,B_j]\,b_i\,B_j\,(y_i+y_j)))+$$
$$\sqrt{B_i}\ ((x_i+x_j)\,y_j-\gamma\,\hbar\,\text{Fij}[B_i,B_j]\,b_i\,(x_i\,y_j+x_j\,(y_i+y_j)))\Big)\Big),\ \bar{\omega}[B_i,B_j]\Big]$$

```
In[•]:= V0bar_{i_,j_} := 𝔼_{}→{i,j}[ -1/2 ħ a_j b_i, ( b_j^2 √B_i (-1 + B_i + Fij γ ħ b_i (1 - B_i B_j)) x_i y_i +
          b_i^2 (B_i^(3/2) ((-1 + B_j) x_i - x_j) + (-1 + Fij γ ħ b_i) x_j + √B_i x_j + B_i (1 - Fij γ ħ b_i B_j) x_j) y_j +
          b_i b_j ((-1 + Fij γ ħ b_i) x_j y_j + B_i (1 - Fij γ ħ b_i B_j) x_j y_j +
            B_i^(3/2) (x_i (-(-1 + B_j) y_i + (-1 + Fij γ ħ b_i B_j) y_j) + x_j (-y_j + Fij γ ħ b_i B_j (y_i + y_j))) +
            √B_i ((x_i + x_j) y_j - Fij γ ħ b_i (x_i y_j + x_j (y_i + y_j))))) /
        (γ b_i b_j (b_i + b_j) √B_i (1 - B_i + Fij γ ħ b_i (-1 + B_i B_j))), ω[B_i, B_j]^-1] /. Fij → Fij[B_i, B_j];
      V0bar_{i,j}

Out[•]= 𝔼_{}→{i,j}[ -1/2 ħ a_j b_i, (1 / (γ b_i b_j (b_i + b_j) √B_i (1 - B_i + γ ħ Fij[B_i, B_j] b_i (-1 + B_i B_j)))
        (b_j^2 √B_i (-1 + B_i + γ ħ Fij[B_i, B_j] b_i (1 - B_i B_j)) x_i y_i +
          b_i^2 (B_i^(3/2) ((-1 + B_j) x_i - x_j) + (-1 + γ ħ Fij[B_i, B_j] b_i) x_j + √B_i x_j +
            B_i (1 - γ ħ Fij[B_i, B_j] b_i B_j) x_j) y_j + b_i b_j ((-1 + γ ħ Fij[B_i, B_j] b_i) x_j y_j +
          B_i (1 - γ ħ Fij[B_i, B_j] b_i B_j) x_j y_j + B_i^(3/2) (x_i ((1 - B_j) y_i + (-1 + γ ħ Fij[B_i, B_j] b_i B_j) y_j) +
            x_j (-y_j + γ ħ Fij[B_i, B_j] b_i B_j (y_i + y_j))) +
          √B_i ((x_i + x_j) y_j - γ ħ Fij[B_i, B_j] b_i (x_i y_j + x_j (y_i + y_j))))), 1/ω[B_i, B_j]]

In[•]:= V0bar_{-1,-2} V0_{1,2} // cm_{-1,1→1} // cm_{-2,2→2}
Out[•]= 𝔼_{}→{1,2}[0, 0, 1 + O[∈]^1]
```

```
In[•]:= Define[
        V0_{i,j} = CF@𝔼_{}→{i,j}[ 1/2 ħ a_j b_i, 1/(γ b_i^2 b_j (-1 + B_i B_j)) (b_2 b_j (-1 + B_i + Fij γ ħ b_i (1 - B_i B_j)) x_i y_i +
          b_i (b_i ((-1 + Fij γ ħ b_i) x_j + √B_i x_j - B_i^(3/2) B_j x_j + B_i ((-1 + B_j) x_i + (1 - Fij γ ħ b_i) B_j x_j)) y_j +
            b_j (x_i y_j - Fij γ ħ b_i (x_j y_i + x_i y_j) + B_i (Fij γ ħ b_i B_j x_j y_i +
              x_i (-(-1 + B_j) y_i + (-1 + Fij γ ħ b_i B_j) y_j))))), ω[B_i, B_j]] /. Fij → 0];
      V0_{i,j}

Out[•]= 𝔼_{}→{i,j}[ 1/2 ħ a_j b_i,
        ((-b_2 + b_2 B_i + b_i B_i - b_i B_i B_j) x_i y_i)/(-γ b_i^2 + γ b_i^2 B_i B_j) + ((b_j - b_i B_i - b_j B_i + b_i B_i B_j) x_i y_j)/(-γ b_i b_j + γ b_i b_j B_i B_j) + ((1 - √B_i) x_j y_j)/(γ b_j), ω[B_i, B_j]]
```

In[•]:=
```
Define[V0ᵢ,ⱼ = CF@𝔼{}→{i,j}[-1/2 ħ aⱼ bᵢ, (bⱼ² √Bᵢ (-1 + Bᵢ + Fij γ ħ bᵢ (1 - Bᵢ Bⱼ)) xᵢ yᵢ +
    bᵢ² (Bᵢ^(3/2) ((-1 + Bⱼ) xᵢ - xⱼ) + (-1 + Fij γ ħ bᵢ) xⱼ + √Bᵢ xⱼ + Bᵢ (1 - Fij γ ħ bᵢ Bⱼ) xⱼ) yⱼ +
    bᵢ bⱼ ((-1 + Fij γ ħ bᵢ) xⱼ yⱼ + Bᵢ (1 - Fij γ ħ bᵢ Bⱼ) xⱼ yⱼ +
      Bᵢ^(3/2) (xᵢ (-(-1 + Bⱼ) yᵢ + (-1 + Fij γ ħ bᵢ Bⱼ) yⱼ) + xⱼ (-yⱼ + Fij γ ħ bᵢ Bⱼ (yᵢ + yⱼ))) +
      √Bᵢ ((xᵢ + xⱼ) yⱼ - Fij γ ħ bᵢ (xᵢ yⱼ + xⱼ (yᵢ + yⱼ))))) /
   (γ bᵢ bⱼ (bᵢ + bⱼ) √Bᵢ (1 - Bᵢ + Fij γ ħ bᵢ (-1 + Bᵢ Bⱼ))), ω[Bᵢ, Bⱼ]^-1] /. Fij → 0];
V0ᵢ,ⱼ
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{i,j\}}\left[-\frac{1}{2}\hbar\, a_j\, b_i,\ \frac{\left(-b_j+b_i B_i+b_j B_i-b_i B_i B_j\right)x_i y_i}{\gamma b_i^2+\gamma b_i b_j-\gamma b_i^2 B_i-\gamma b_i b_j B_i}+\frac{\left(b_j-b_i B_i-b_j B_i+b_i B_i B_j\right)x_i y_j}{\gamma b_i b_j+\gamma b_j^2-\gamma b_i b_j B_i-\gamma b_j^2 B_i}+\right.$$
$$\left.\left(\left(-b_i-b_j+b_i\sqrt{B_i}+b_j\sqrt{B_i}+b_i B_i+b_j B_i-b_i B_i^{3/2}-b_j B_i^{3/2}\right)x_j y_j\right)\Big/\left(\gamma b_i b_j\sqrt{B_i}+\gamma b_j^2\sqrt{B_i}-\gamma b_i b_j B_i^{3/2}-\gamma b_j^2 B_i^{3/2}\right),\ \frac{1}{\omega[B_i,B_j]}\right]$$

In[•]:=
```
Define[Θ0ᵢ,ⱼ,ₖ = Module[{s1, s2, s3},
    (((V0ᵢ,ₖ // cΔᵢ→ᵢ,ⱼ) V0s1,s2 // cmᵢ,s1→ᵢ // cmⱼ,s2→ⱼ) V0s2,s3 // cmⱼ,s2→ⱼ // cmₖ,s3→ₖ)
     (V0s1,s2 // cΔs2→s2,s3) // cmᵢ,s1→ᵢ // cmⱼ,s2→ⱼ // cmₖ,s3→ₖ
  ]];
Θ0ᵢ,ⱼ,ₖ
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{i,j,k\}}\left[0,\ \left(\left(-b_2 b_i b_j-b_2 b_j^2+b_i b_j^2+b_j^3-b_2 b_i b_k-2 b_2 b_j b_k+2 b_i b_j b_k+b_j^2 b_k-b_2 b_k^2+b_i b_k^2+b_2 b_i b_k B_i+\right.\right.\right.$$
$$b_j^2 b_k B_i-b_i b_k^2 B_i+b_j b_k^2 B_i+\langle\cdots 146\cdots\rangle+b_2 b_j^2 B_i^2 B_j^2 B_k^2+b_2 b_i b_k B_i^2 B_j^2 B_k^2+b_i^2 b_k B_i^2 B_j^2 B_k^2+b_j b_k B_i^2 B_j^2 B_k^2+b_i b_j b_k B_i^2 B_j^2 B_k^2+b_2 b_i^2 B_i^2 B_j^3 B_k^2+b_2 b_i b_j B_i^2 B_j^3 B_k^2-b_i b_j^2 B_i^2 B_j^3 B_k^2-b_j^3 B_i^2 B_j^3 B_k^2-$$
$$\left.\left.\left.b_i^2 b_k B_i^2 B_j^3 B_k^2-3 b_i b_j b_k B_i^2 B_j^3 B_k^2-2 b_j^2 b_k B_i^2 B_j^3 B_k^2-b_i b_k^2 B_i^2 B_j^3 B_k^2-b_j b_k^2 B_i^2 B_j^3 B_k^2\right)x_i y_i\right)\Big/$$
$$\left(-\gamma b_i^3 b_j-2\gamma b_i^2 b_j^2-\gamma b_i b_j^3-\gamma b_i^3 b_k-3\gamma b_i^2 b_j b_k-2\gamma b_i b_j^2 b_k-\gamma b_i^2 b_k^2-\gamma b_i b_j b_k^2+\right.$$
$$\gamma b_i^3 b_j B_i B_j+2\gamma b_i^2 b_j^2 B_i B_j+\gamma b_i b_j^3 B_i B_j+\gamma b_i^3 b_k B_i B_j+\langle\cdots 68\cdots\rangle+$$
$$\gamma b_i^3 b_j B_i^2 B_j^3 B_k^2+2\gamma b_i^2 b_j^2 B_i^2 B_j^3 B_k^2+\gamma b_i b_j^3 B_i^2 B_j^3 B_k^2+\gamma b_i^3 b_k B_i^2 B_j^3 B_k^2+$$
$$\left.3\gamma b_i^2 b_j b_k B_i^2 B_j^3 B_k^2+2\gamma b_i b_j^2 b_k B_i^2 B_j^3 B_k^2+\gamma b_i^2 b_k^2 B_i^2 B_j^3 B_k^2+\gamma b_i b_j b_k^2 B_i^2 B_j^3 B_k^2\right)+$$
$$\frac{\langle\cdots 1\cdots\rangle}{\langle\cdots 1\cdots\rangle}+\langle\cdots 5\cdots\rangle+\frac{\langle\cdots 1\cdots\rangle}{\langle\cdots 1\cdots\rangle}+\frac{\left(\langle\cdots 1\cdots\rangle\right)x_k y_k}{\gamma b_i b_j b_k+\gamma b_j^2 b_k+\langle\cdots 27\cdots\rangle+\gamma b_k^3 B_i B_j^2 B_k},$$
$$\left.\frac{\omega[B_i,B_j B_k]\,\omega[B_j,B_k]}{\omega[B_i,B_j]\,\omega[B_i B_j,B_k]}+O[\epsilon]^1\right]$$

large output | show less | show more | show all | set size limit...

In[•]:=
```
lhs = (Φ0_{1,2,3} (Φ0_{-1,-2,4} // cΔ_{-2→-2,-3}) // cm_{1,-1→1} // cm_{2,-2→2} // cm_{3,-3→3}) Φ0_{-2,-3,-4} // cm_{2,-2→2} //
    cm_{3,-3→3} // cm_{4,-4→4}
```

Out[•]=

$$\mathbb{E}_{\{\}\to\{1,2,3,4\}}\Big[0,\ \Big(b_1^2 b_2 b_3 x_1 y_3 + b_1 b_2^2 b_3 x_1 y_3 + b_1 b_2 b_3^2 x_1 y_3 - b_1^2 b_2 b_3 \sqrt{B_2}\ x_1 y_3 - 2 b_1 b_2^2 b_3 \sqrt{B_2}\ x_1 y_3 - b_2^3 b_3 \sqrt{B_2}\ x_1 y_3 - b_1 b_2 b_3^2 \sqrt{B_2}\ x_1 y_3 - b_2^2 b_3^2 \sqrt{B_2}\ x_1 y_3 + b_1 b_2^2 b_3 \sqrt{B_1}\ \sqrt{B_2}\ x_1 y_3 + b_2^3 b_3 \sqrt{B_1}\ \sqrt{B_2}\ x_1 y_3 + b_2^2 b_3^2 \sqrt{B_1}\ \sqrt{B_2}\ x_1 y_3 - b_1^3 b_3 x_2 y_3 - b_1^2 b_2 b_3 x_2 y_3 - b_1^2 b_3^2 x_2 y_3 + b_1^3 b_3 \sqrt{B_2}\ x_2 y_3 + 2 b_1^2 b_2 b_3 \sqrt{B_2}\ x_2 y_3 + b_1 b_2^2 b_3 \sqrt{B_2}\ x_2 y_3 + b_1^2 b_3^2 \sqrt{B_2}\ x_2 y_3 + b_1 b_2 b_3^2 \sqrt{B_2}\ x_2 y_3 - b_1^2 b_2 b_3 \sqrt{B_1}\ \sqrt{B_2}\ x_2 y_3 - b_1 b_2^2 b_3 \sqrt{B_1}\ \sqrt{B_2}\ x_2 y_3 - b_1 b_2 b_3^2 \sqrt{B_1}\ \sqrt{B_2}\ x_2 y_3 + b_1^2 b_2 b_3 x_1 y_4 + b_1 b_2^2 b_3 x_1 y_4 - b_1^2 b_2 b_3 \sqrt{B_2}\ \sqrt{B_3}\ x_1 y_4 - 2 b_1 b_2^2 b_3 \sqrt{B_2}\ \sqrt{B_3}\ x_1 y_4 - b_2^3 b_3 \sqrt{B_2}\ \sqrt{B_3}\ x_1 y_4 - b_1 b_2 b_3^2 \sqrt{B_2}\ \sqrt{B_3}\ x_1 y_4 - b_2^2 b_3^2 \sqrt{B_2}\ \sqrt{B_3}\ x_1 y_4 + b_1 b_2^2 b_3 \sqrt{B_1}\ \sqrt{B_2}\ \sqrt{B_3}\ x_1 y_4 + b_2^3 b_3 \sqrt{B_1}\ \sqrt{B_2}\ \sqrt{B_3}\ x_1 y_4 + b_1 b_2 b_3^2 \sqrt{B_1}\ \sqrt{B_2}\ \sqrt{B_3}\ x_1 y_4 + b_2^2 b_3^2 \sqrt{B_1}\ \sqrt{B_2}\ \sqrt{B_3}\ x_1 y_4 + b_1^2 b_2 b_3 x_2 y_4 + b_1 b_2^2 b_3 x_2 y_4 - b_1^3 b_3 \sqrt{B_3}\ x_2 y_4 - 2 b_1^2 b_2 b_3 \sqrt{B_3}\ x_2 y_4 - b_1 b_2^2 b_3 \sqrt{B_3}\ x_2 y_4 - b_1^2 b_3^2 \sqrt{B_3}\ x_2 y_4 - b_1 b_2 b_3^2 \sqrt{B_3}\ x_2 y_4 + b_1^3 b_3 \sqrt{B_2}\ \sqrt{B_3}\ x_2 y_4 + 2 b_1^2 b_2 b_3 \sqrt{B_2}\ \sqrt{B_3}\ x_2 y_4 + b_1 b_2^2 b_3 \sqrt{B_2}\ \sqrt{B_3}\ x_2 y_4 + b_1^2 b_3^2 \sqrt{B_2}\ \sqrt{B_3}\ x_2 y_4 + b_1 b_2 b_3^2 \sqrt{B_2}\ \sqrt{B_3}\ x_2 y_4 - b_1^2 b_2 b_3 \sqrt{B_1}\ \sqrt{B_2}\ \sqrt{B_3}\ x_2 y_4 - b_1 b_2^2 b_3 \sqrt{B_1}\ \sqrt{B_2}\ \sqrt{B_3}\ x_2 y_4 - b_1 b_2 b_3^2 \sqrt{B_1}\ \sqrt{B_2}\ \sqrt{B_3}\ x_2 y_4 - b_1^2 b_2 b_3^2 x_3 y_4 - 2 b_1^2 b_2^2 x_3 y_4 - b_1 b_2^3 x_3 y_4 + b_1^3 b_2 \sqrt{B_3}\ x_3 y_4 + 2 b_1^2 b_2^2 \sqrt{B_3}\ x_3 y_4 + b_1 b_2^3 \sqrt{B_3}\ x_3 y_4 + b_1^2 b_2 b_3 \sqrt{B_3}\ x_3 y_4 + b_1 b_2^2 b_3 \sqrt{B_3}\ x_3 y_4 - b_1^2 b_2 b_3 \sqrt{B_1}\ \sqrt{B_2}\ \sqrt{B_3}\ x_3 y_4 - b_1 b_2^2 b_3 \sqrt{B_1}\ \sqrt{B_2}\ \sqrt{B_3}\ x_3 y_4\Big) \Big/ \Big(\gamma b_1^3 b_2 b_3 + 2\gamma b_1^2 b_2^2 b_3 + \gamma b_1 b_2^3 b_3 + \gamma b_1^2 b_2 b_3^2 + \gamma b_1 b_2^2 b_3^2\Big),\ \frac{\omega[B_1, B_2 B_3 B_4]\ \omega[B_2, B_3 B_4]\ \omega[B_3, B_4]}{\omega[B_1, B_2]\ \omega[B_1 B_2, B_3]\ \omega[B_1 B_2 B_3, B_4]} + O[\epsilon]^1\Big]$$

In[•]:= **rhs = $\left(\text{Φ0}_{1,3,4} \;//\; \text{cΔ}_{1\to 1,2}\right)\left(\text{Φ0}_{-1,-2,-3} \;//\; \text{cΔ}_{-3\to -3,-4}\right) \;//\; \text{cm}_{1,-1\to 1} \;//\; \text{cm}_{2,-2\to 2} \;//\; \text{cm}_{3,-3\to 3} \;//\; \text{cm}_{4,-4\to 4}$**

Out[•]=
$$\mathbb{E}_{\{\}\to\{1,2,3,4\}}\Big[0,$$
$$\Big(b_1^2 b_2 b_3 x_1 y_3 + b_1 b_2^2 b_3 x_1 y_3 + b_1 b_2 b_3^2 x_1 y_3 - b_1^2 b_2 b_3 \sqrt{B_2}\, x_1 y_3 - 2 b_1 b_2^2 b_3 \sqrt{B_2}\, x_1 y_3 - b_2^3 b_3 \sqrt{B_2}\, x_1 y_3 - b_1 b_2 b_3^2 \sqrt{B_2}\, x_1 y_3 - b_2^2 b_3^2 \sqrt{B_2}\, x_1 y_3 + b_1 b_2^2 b_3 \sqrt{B_1}\, \sqrt{B_2}\, x_1 y_3 + b_2^3 b_3 \sqrt{B_1}\, \sqrt{B_2}\, x_1 y_3 + b_2^2 b_3^2 \sqrt{B_1}\, \sqrt{B_2}\, x_1 y_3 - b_1^3 b_3 x_2 y_3 - b_1^2 b_2 b_3 x_2 y_3 - b_1^2 b_3^2 x_2 y_3 + b_1^3 b_3 \sqrt{B_2}\, x_2 y_3 + 2 b_1^2 b_2 b_3 \sqrt{B_2}\, x_2 y_3 + b_1 b_2^2 b_3 \sqrt{B_2}\, x_2 y_3 + b_1^2 b_3^2 \sqrt{B_2}\, x_2 y_3 + b_1 b_2 b_3^2 \sqrt{B_2}\, x_2 y_3 - b_1^2 b_2 b_3 \sqrt{B_1}\, \sqrt{B_2}\, x_2 y_3 - b_1 b_2^2 b_3 \sqrt{B_1}\, \sqrt{B_2}\, x_2 y_3 - b_1 b_2 b_3^2 \sqrt{B_1}\, \sqrt{B_2}\, x_2 y_3 + b_1^2 b_2 b_3 x_1 y_4 + b_1 b_2^2 b_3 x_1 y_4 + b_1 b_2 b_3^2 x_1 y_4 - b_1 b_2 b_3^2 \sqrt{B_2}\, x_1 y_4 - b_2^2 b_3^2 \sqrt{B_2}\, x_1 y_4 + b_2^2 b_3^2 \sqrt{B_1}\, \sqrt{B_2}\, x_1 y_4 - b_1^2 b_2 b_3 \sqrt{B_2}\, \sqrt{B_3}\, x_1 y_4 - 2 b_1 b_2^2 b_3 \sqrt{B_2}\, \sqrt{B_3}\, x_1 y_4 - b_2^3 b_3 \sqrt{B_2}\, \sqrt{B_3}\, x_1 y_4 - b_1 b_2 b_3^2 \sqrt{B_2}\, \sqrt{B_3}\, x_1 y_4 - b_2^2 b_3^2 \sqrt{B_2}\, \sqrt{B_3}\, x_1 y_4 + b_1 b_2^2 b_3 \sqrt{B_1}\, \sqrt{B_2}\, \sqrt{B_3}\, x_1 y_4 + b_2^3 b_3 \sqrt{B_1}\, \sqrt{B_2}\, \sqrt{B_3}\, x_1 y_4 + b_2^2 b_3^2 \sqrt{B_1}\, \sqrt{B_2}\, \sqrt{B_3}\, x_1 y_4 + b_1 b_2 b_3^2 \sqrt{B_1}\, B_2 \sqrt{B_3}\, x_1 y_4 + b_2^2 b_3^2 \sqrt{B_1}\, B_2 \sqrt{B_3}\, x_1 y_4 - b_2^2 b_3^2 B_1 B_2 \sqrt{B_3}\, x_1 y_4 + b_1^2 b_2 b_3 x_2 y_4 + b_1 b_2^2 b_3 x_2 y_4 - b_1^2 b_3^2 x_2 y_4 + b_1^2 b_3^2 \sqrt{B_2}\, x_2 y_4 + b_1 b_2 b_3^2 \sqrt{B_2}\, x_2 y_4 - b_1 b_2 b_3^2 \sqrt{B_1}\, \sqrt{B_2}\, x_2 y_4 - b_1^3 b_3 \sqrt{B_3}\, x_2 y_4 - 2 b_1^2 b_2 b_3 \sqrt{B_3}\, x_2 y_4 - b_1 b_2^2 b_3 \sqrt{B_3}\, x_2 y_4 - b_1^2 b_3^2 \sqrt{B_3}\, x_2 y_4 - b_1 b_2 b_3^2 \sqrt{B_3}\, x_2 y_4 + b_1^3 b_3 \sqrt{B_2}\, \sqrt{B_3}\, x_2 y_4 + 2 b_1^2 b_2 b_3 \sqrt{B_2}\, \sqrt{B_3}\, x_2 y_4 + b_1 b_2^2 b_3 \sqrt{B_2}\, \sqrt{B_3}\, x_2 y_4 + b_1^2 b_3^2 \sqrt{B_2}\, \sqrt{B_3}\, x_2 y_4 + b_1 b_2 b_3^2 \sqrt{B_2}\, \sqrt{B_3}\, x_2 y_4 - b_1^2 b_2 b_3 \sqrt{B_1}\, \sqrt{B_2}\, \sqrt{B_3}\, x_2 y_4 - b_1 b_2^2 b_3 \sqrt{B_1}\, \sqrt{B_2}\, \sqrt{B_3}\, x_2 y_4 + b_1^2 b_3^2 \sqrt{B_1}\, \sqrt{B_2}\, \sqrt{B_3}\, x_2 y_4 - b_1^2 b_3^2 \sqrt{B_1}\, B_2 \sqrt{B_3}\, x_2 y_4 - b_1 b_2 b_3^2 \sqrt{B_1}\, B_2 \sqrt{B_3}\, x_2 y_4 + b_1 b_2 b_3^2 B_1 B_2 \sqrt{B_3}\, x_2 y_4 - b_1^3 b_2 x_3 y_4 - 2 b_1^2 b_2^2 x_3 y_4 - b_1 b_2^3 x_3 y_4 + b_1^3 b_2 \sqrt{B_3}\, x_3 y_4 + 2 b_1^2 b_2^2 \sqrt{B_3}\, x_3 y_4 + b_1 b_2^3 \sqrt{B_3}\, x_3 y_4 + b_1^2 b_2 b_3 \sqrt{B_3}\, x_3 y_4 + b_1 b_2^2 b_3 \sqrt{B_3}\, x_3 y_4 - b_1^2 b_2 b_3 \sqrt{B_1}\, \sqrt{B_2}\, \sqrt{B_3}\, x_3 y_4 - b_1 b_2^2 b_3 \sqrt{B_1}\, \sqrt{B_2}\, \sqrt{B_3}\, x_3 y_4\Big) \Big/$$
$$\left(\gamma b_1^3 b_2 b_3 + 2\gamma b_1^2 b_2^2 b_3 + \gamma b_1 b_2^3 b_3 + \gamma b_1^2 b_2 b_3^2 + \gamma b_1 b_2^2 b_3^2\right),$$
$$\frac{\omega[B_1, B_2 B_3 B_4]\,\omega[B_2, B_3 B_4]\,\omega[B_3, B_4]}{\omega[B_1, B_2]\,\omega[B_1 B_2, B_3]\,\omega[B_1 B_2 B_3, B_4]} + O[\epsilon]^1\Big]$$

In[•]:= **Simplify[lhs ≡ rhs]**

Out[•]=
$$\Big(b_3\left(b_1\left(-1+\sqrt{B_2}\right) - b_2\left(-1+\sqrt{B_1}\right)\sqrt{B_2}\right)\left(-1+\sqrt{B_1}\,\sqrt{B_2}\,\sqrt{B_3}\right)\left(-b_2 x_1 + b_1 x_2\right) y_4\Big) \Big/$$
$$\left(\gamma b_1 b_2 \left(b_1+b_2\right)\left(b_1+b_2+b_3\right)\right) == 0$$

In[•]:= **$\text{cΔ}_{1\to 1,2}$**

Out[•]=
$$\mathbb{E}_{\{1\}\to\{1,2\}}\left[\left(a_1+a_2\right)\alpha_1 + \left(b_1+b_2\right)\beta_1,\ \left(y_1+y_2\right)\eta_1 + \left(x_1+x_2\right)\xi_1,\ 1\right]$$