Pensieve header: Exp relative to am, bm, cm, dm.

Follows code in Projects/SL2Portfolio/SL2PortfolioProgram.nb.

Startup

In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio2"];
(*Once[<< KnotTheory`];*)
Once[<< "../Profile/Profile.m";];
<< "Engine-Speedy.m";
<< "Objects.m";
$k = 3;
HL[ℰ_] := Style[ℰ, Background → Green];
```

```
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.

This version: April 2020. Original version: July 1994.
```

» Warning: On Sep 4 2019 I swapped the operations ϵ and η. Some incompatibilities may arise in older notebooks.

Exponentials as needed.

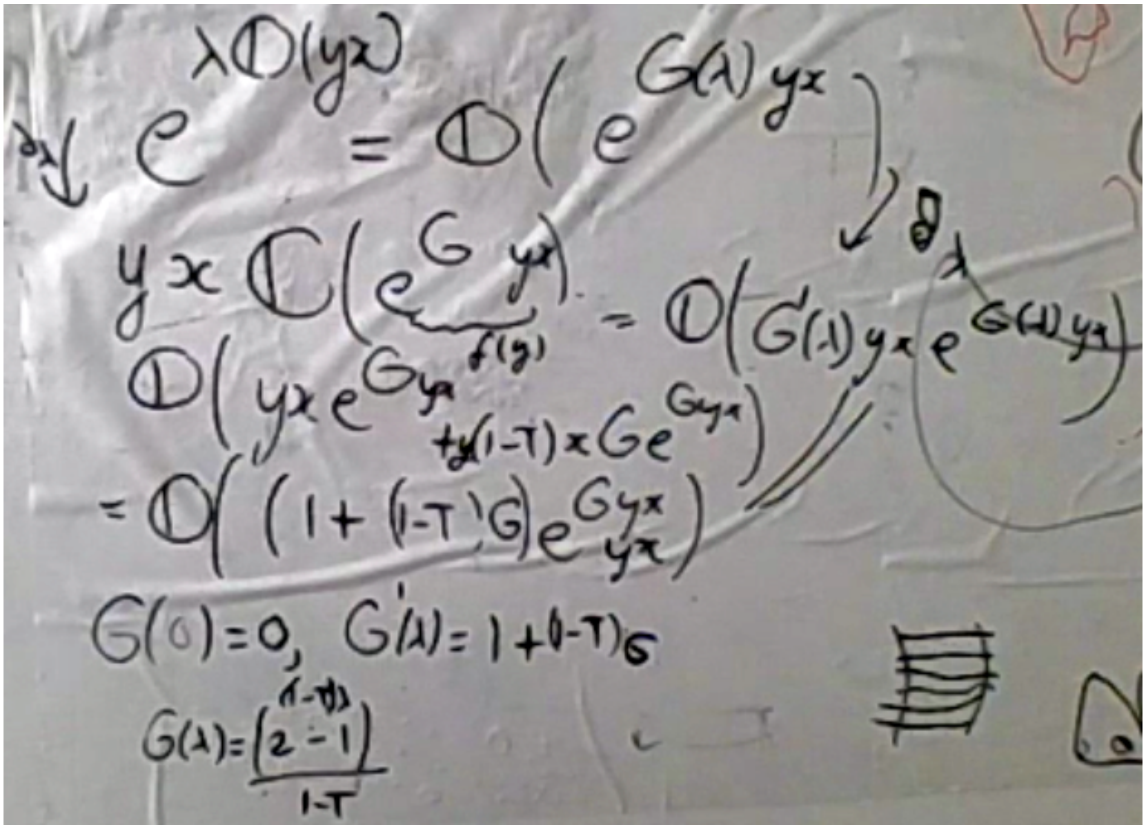
Task. Define $\mathrm{Exp}_{m,i,k}[P]$ to compute $e^{\mathbb{O}(P)}$ to ϵ^k in the using the $\mathrm{mm}_{i,i\to i}$ multiplication, where P is an ϵ-dependent near-docile element, giving the answer in $\mathbb{E}$-form.

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\lambda\,\mathbb{O}(P)} = \mathbb{O}\left(e^{\lambda P_0}\,F(\lambda)\right)$, then $F(\lambda=0)=1$ and we have:

$\mathbb{O}\left(e^{\lambda P_0}(P_0\,F(\lambda)+\partial_\lambda F)\right) = \mathbb{O}\left(\partial_\lambda e^{\lambda P_0}\,F(\lambda)\right) = \partial_\lambda\,\mathbb{O}\left(e^{\lambda P_0}\,F(\lambda)\right) = \partial_\lambda e^{\lambda\,\mathbb{O}(P)} = e^{\lambda\,\mathbb{O}(P)}\,\mathbb{O}(P) = \mathbb{O}\left(e^{\lambda P_0}\,F(\lambda)\right)\mathbb{O}(P).$

This is a linear ODE for F. Setting inductively $F_k = F_{k-1} + \epsilon^k\,\varphi$ we find that $F_0 = 1$ and solve for φ.

In[•]:=
```
(* Bug: The first line is valid only if 𝕆(𝕖^P₀) == 𝕖^𝕆(P₀). *)
Exp_{mm_,i_,0}[P_] := Module[{LQ = Normal@P /. ϵ → 0},
   𝔼[LQ /. (x | y)_i → 0, LQ /. (b | a | t)_i → 0, 1] ];
```



```
mm = dm; i = 1; P0 = b1 a1 + y1 x1;
```

```
G =
```

$k=0, Example 1

In[∘]:= `mm = dm; P0 = yi xi;`

In[∘]:= `𝔼G = 𝔼{}→{0}[LG[λ] b0 a0, QG[λ] y0 x0, 1 + O[ϵ]]`

Out[∘]= $\mathbb{E}_{\{\}\to\{0\}}\left[\text{LG}[\lambda]\, a_0 b_0,\ \text{QG}[\lambda]\, x_0 y_0,\ 1+O[\epsilon]^1\right]$

In[∘]:= `(𝔼{}→{i}[0, 0, P0 + O[ϵ]] 𝔼{}→{0}[LG[λ] b0 a0, QG[λ] y0 x0, 1 + O[ϵ]]) // mmi,0→0`

Out[∘]= $\mathbb{E}_{\{\}\to\{0\}}\left[\text{LG}[\lambda]\, a_0 b_0,\ \text{QG}[\lambda]\, x_0 y_0,\ \frac{\left(\text{QG}[\lambda] - \text{QG}[\lambda]\, B_0 + \hbar\, B_0^{\frac{\text{LG}[\lambda]}{\hbar}}\right) x_0 y_0}{\hbar} + O[\epsilon]^1\right]$

In[∘]:= `𝔼{}→{i}[0, 0, P0 + O[ϵ]] 𝔼{}→{0}[0, QG[λ] y0 x0, 1 + O[ϵ]] // mmi,0→0`

Out[∘]= $\mathbb{E}_{\{\}\to\{0\}}\left[0,\ \text{QG}[\lambda]\, x_0 y_0,\ \frac{(\hbar + \text{QG}[\lambda] - \text{QG}[\lambda]\, B_0)\, x_0 y_0}{\hbar} + O[\epsilon]^1\right]$

In[∘]:= `𝔼{}→{0}[LG[λ] b0 a0, QG[λ] y0 x0, ∂λ (LG[λ] b0 a0 + QG[λ] y0 x0) + O[ϵ]]`

Out[∘]= $\mathbb{E}_{\{\}\to\{0\}}\left[\text{LG}[\lambda]\, a_0 b_0,\ \text{QG}[\lambda]\, x_0 y_0,\ (a_0 b_0\, \text{LG}'[\lambda] + x_0 y_0\, \text{QG}'[\lambda]) + O[\epsilon]^1\right]$

In[∘]:=
```
sol = DSolve[QG′[λ] == (ℏ + QG[λ] - QG[λ] B0)/ℏ ∧ QG[0] == 0, QG[λ], λ]
```

Out[∘]=
$$\left\{\left\{QG[\lambda] \to \frac{e^{\frac{\lambda}{\hbar}-\frac{\lambda B_0}{\hbar}}\left(-1+e^{\frac{\lambda(-1+B_0)}{\hbar}}\right)\hbar}{-1+B_0}\right\}\right\}$$

In[∘]:=
```
Simplify[QG[λ] /. sol]
```

Out[∘]=
$$\left\{\frac{\hbar - e^{\frac{\lambda-\lambda B_0}{\hbar}}\hbar}{-1+B_0}\right\}$$

$k>0

In[∘]:=
```
Exp_{mm_,i_,k_}[P_] := Block[{$k = k},
  Module[{P0, λ, φ, F, j, rhs, err = 0, pows, at0, atλ},
   P0 = Normal@P /. ϵ → 0;
   F = Normal@Last@Exp_{mm,i,k-1}[λ P];
   (*Unary*) While[
    If[err =!= 0,
     pows = Echo[First /@ CoefficientRules[err, {y_i, b_i, a_i, x_i}]];
     F += Sum[ϵ^k φ_js[λ] Times @@ {y_i, b_i, a_i, x_i}^js, {js, pows}];
     rhs = Normal@Last@mm_{i,j→i}[
        𝔼_{{}→{i}}[λ P0 /. (x | y)_i → 0, λ P0 /. (b | a | t)_i → 0, F]_k sσ_{i→j}@𝔼_{{}→{i}}[0, 0, P]_k];
     err = CF[(∂_λ F) + P0 F - rhs];
     at0 = Table[φ_js[0] == 0, {js, pows}];
     atλ = (# == 0) & /@ (pows /. CoefficientRules[err, {y_i, b_i, a_i, x_i}]);
     F = F /. DSolve[And @@ (at0 ⋃ atλ), Table[φ_js[λ], {js, pows}], λ][[1]]
    ];
    rhs = Normal@Last@mm_{i,j→i}[
       𝔼_{{}→{i}}[λ P0 /. (x | y)_i → 0, λ P0 /. (b | a | t)_i → 0, F]_k sσ_{i→j}@𝔼_{{}→{i}}[0, 0, P]_k];
    err = CF[(∂_λ F) + P0 F - rhs];
    err =!= 0
   ];
   𝔼_{{}→{i}}[P0 /. (x | y)_i → 0, P0 /. (b | a | t)_i → 0, F + O[ϵ]^(k+1) /. λ → 1]]
 ]
```

In[∘]:=
```
Exp_{dm,1,2}[ξ (x1 + ϵ y1)]
```

» {{1, 0, 0, 0}, {0, 0, 0, 0}}

» {{2, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}, {0, 0, 0, 0}}

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ \xi x_1,\ 1+\left(-\frac{\xi^2(-1+B_1)}{2\hbar}+\xi y_1\right)\epsilon+\left(\frac{\xi^4(-1+B_1)^2}{8\hbar^2}+\frac{1}{2}\xi^2 a_1 B_1-\frac{1}{6}\gamma\xi^3(-1+3B_1)x_1-\frac{\xi^3(-1+B_1)y_1}{2\hbar}+\frac{1}{2}\gamma\xi^2\hbar x_1 y_1+\frac{1}{2}\xi^2 y_1^2\right)\epsilon^2+O[\epsilon]^3\right]$

In[∘]:= `dS₁[𝔼{}→{1}[0, 0, #]] & /@ {y₁, x₁}`

Out[∘]= $\left\{\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -\frac{y_1}{B_1}+\frac{\gamma\hbar y_1\epsilon}{B_1}-\frac{(\gamma^2\hbar^2 y_1)\epsilon^2}{2B_1}+\frac{\gamma^3\hbar^3 y_1\epsilon^3}{6B_1}+O[\epsilon]^4\right],\ \mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -x_1-\hbar a_1 x_1\epsilon-\frac{1}{2}(\hbar^2 a_1^2 x_1)\epsilon^2-\frac{1}{6}(\hbar^3 a_1^3 x_1)\epsilon^3+O[\epsilon]^4\right]\right\}$

In[∘]:= `Timing@{lhs = 𝔼{1}→{1}[0, ξ₁ x₁, 1] // dS₁,`
`rhs = Exp_{dm,1,$k}[ξ₁ Last@dS₁[𝔼{}→{1}[0, 0, x₁]]] /. {{} → {1}}; HL[lhs ≡ rhs]}`

» {{0, 0, 1, 1}, {0, 0, 0, 2}}

» {{0, 0, 2, 2}, {0, 0, 2, 1}, {0, 0, 1, 3}, {0, 0, 1, 2}, {0, 0, 0, 4}, {0, 0, 0, 3}, {0, 0, 0, 2}}

» {{0, 0, 3, 3}, {0, 0, 3, 2}, {0, 0, 3, 1}, {0, 0, 2, 4}, {0, 0, 2, 3}, {0, 0, 2, 2}, {0, 0, 1, 5}, {0, 0, 1, 4}, {0, 0, 1, 3}, {0, 0, 1, 2}, {0, 0, 0, 6}, {0, 0, 0, 5}, {0, 0, 0, 4}, {0, 0, 0, 3}, {0, 0, 0, 2}}

Out[∘]= $\left\{5.67188,\ \left\{\mathbb{E}_{\{1\}\to\{1\}}\left[0,\ -x_1\xi_1,\ 1+\left(-\hbar a_1 x_1\xi_1-\frac{1}{2}\gamma\hbar x_1^2\xi_1^2\right)\epsilon+\left(-\frac{1}{2}\hbar^2 a_1^2 x_1\xi_1+\frac{1}{4}\gamma^2\hbar^2 x_1^2\xi_1^2-\gamma\hbar^2 a_1 x_1^2\xi_1^2+\frac{1}{2}\hbar^2 a_1^2 x_1^2\xi_1^2-\frac{1}{2}\gamma^2\hbar^2 x_1^3\xi_1^3+\frac{1}{2}\gamma\hbar^2 a_1 x_1^3\xi_1^3+\frac{1}{8}\gamma^2\hbar^2 x_1^4\xi_1^4\right)\epsilon^2+\left(-\frac{1}{6}\hbar^3 a_1^3 x_1\xi_1-\frac{1}{12}\gamma^3\hbar^3 x_1^2\xi_1^2+\frac{1}{2}\gamma^2\hbar^3 a_1 x_1^2\xi_1^2-\gamma\hbar^3 a_1^2 x_1^2\xi_1^2+\frac{1}{2}\hbar^3 a_1^3 x_1^2\xi_1^2+\frac{2}{3}\gamma^3\hbar^3 x_1^3\xi_1^3-\frac{7}{4}\gamma^2\hbar^3 a_1 x_1^3\xi_1^3+\frac{5}{4}\gamma\hbar^3 a_1^2 x_1^3\xi_1^3-\frac{1}{6}\hbar^3 a_1^3 x_1^3\xi_1^3-\frac{19}{24}\gamma^3\hbar^3 x_1^4\xi_1^4+\gamma^2\hbar^3 a_1 x_1^4\xi_1^4-\frac{1}{4}\gamma\hbar^3 a_1^2 x_1^4\xi_1^4+\frac{1}{4}\gamma^3\hbar^3 x_1^5\xi_1^5-\frac{1}{8}\gamma^2\hbar^3 a_1 x_1^5\xi_1^5-\frac{1}{48}\gamma^3\hbar^3 x_1^6\xi_1^6\right)\epsilon^3+O[\epsilon]^4\right],\ \text{True}\right\}\right\}$

In[∘]:= `Timing@{lhs = 𝔼{1}→{1}[0, η₁ y₁, 1] // dS₁,`
`rhs = Exp_{dm,1,$k}[η Last@dS₁[𝔼{}→{1}[0, 0, y₁]]] /. {η → η₁, {} → {1}}, HL[lhs ≡ rhs]}`

» {{2, 0, 0, 0}, {1, 0, 0, 0}}

» {{4, 0, 0, 0}, {3, 0, 0, 0}, {2, 0, 0, 0}, {1, 0, 0, 0}}

» {{6, 0, 0, 0}, {5, 0, 0, 0}, {4, 0, 0, 0}, {3, 0, 0, 0}, {2, 0, 0, 0}, {1, 0, 0, 0}}

Out[∘]=
$$\Big\{5.15625, \Big\{\mathbb{E}_{\{1\}\to\{1\}}\Big[0, -\frac{y_1\,\eta_1}{B_1}, 1+\left(\frac{\gamma\,\hbar\,y_1\,\eta_1}{B_1}-\frac{\gamma\,\hbar\,y_1^2\,\eta_1^2}{2\,B_1^2}\right)\epsilon+\left(-\frac{\gamma^2\,\hbar^2\,y_1\,\eta_1}{2\,B_1}+\frac{7\,\gamma^2\,\hbar^2\,y_1^2\,\eta_1^2}{4\,B_1^2}-\frac{\gamma^2\,\hbar^2\,y_1^3\,\eta_1^3}{B_1^3}+\frac{\gamma^2\,\hbar^2\,y_1^4\,\eta_1^4}{8\,B_1^4}\right)\epsilon^2+\left(\frac{\gamma^3\,\hbar^3\,y_1\,\eta_1}{6\,B_1}-\frac{25\,\gamma^3\,\hbar^3\,y_1^2\,\eta_1^2}{12\,B_1^2}+\frac{23\,\gamma^3\,\hbar^3\,y_1^3\,\eta_1^3}{6\,B_1^3}-\frac{49\,\gamma^3\,\hbar^3\,y_1^4\,\eta_1^4}{24\,B_1^4}+\frac{3\,\gamma^3\,\hbar^3\,y_1^5\,\eta_1^5}{8\,B_1^5}-\frac{\gamma^3\,\hbar^3\,y_1^6\,\eta_1^6}{48\,B_1^6}\right)\epsilon^3+O[\epsilon]^4\Big],$$
$$\mathbb{E}_{\{1\}\to\{1\}}\Big[0, -\frac{y_1\,\eta_1}{B_1}, 1-\frac{\left(\gamma\,\hbar\,\left(-2\,B_1\,y_1\,\eta_1+y_1^2\,\eta_1^2\right)\right)\epsilon}{2\,B_1^2}+\frac{\gamma^2\,\hbar^2\,\left(-4\,B_1^3\,y_1\,\eta_1+14\,B_1^2\,y_1^2\,\eta_1^2-8\,B_1\,y_1^3\,\eta_1^3+y_1^4\,\eta_1^4\right)\epsilon^2}{8\,B_1^4}-\frac{\left(\gamma^3\,\hbar^3\,\left(-8\,B_1^5\,y_1\,\eta_1+100\,B_1^4\,y_1^2\,\eta_1^2-184\,B_1^3\,y_1^3\,\eta_1^3+98\,B_1^2\,y_1^4\,\eta_1^4-18\,B_1\,y_1^5\,\eta_1^5+y_1^6\,\eta_1^6\right)\right)\epsilon^3}{48\,B_1^6}+O[\epsilon]^4\Big], \text{True}\Big\}\Big\}$$

In[∘]:=
**Timing@{lhs = $\mathbb{E}_{\{1\}\to\{1\}}$[0, ξ_1 x_1, 1] // cS_1,
rhs = $Exp_{cm,1,\$k}$[$\xi_1$ Last@cS_1[$\mathbb{E}_{\{\}\to\{1\}}$[0, 0, x_1]]] /. {} → {1}; HL[lhs ≡ rhs]}**

Out[∘]= $\{5.78125, \{\mathbb{E}_{\{1\}\to\{1\}}[0, -x_1\,\xi_1, 1+O[\epsilon]^4], \text{True}\}\}$

In[∘]:=
**Timing@{lhs = $\mathbb{E}_{\{1\}\to\{1\}}$[0, η_1 y_1, 1] // cS_1,
rhs = $Exp_{cm,1,\$k}$[$\eta_1$ Last@cS_1[$\mathbb{E}_{\{\}\to\{1\}}$[0, 0, y_1]]] /. {} → {1}; HL[lhs ≡ rhs]}**

Out[∘]= $\{0.484375, \{\mathbb{E}_{\{1\}\to\{1\}}[0, -y_1\,\eta_1, 1+O[\epsilon]^4], \text{True}\}\}$