Pensieve header: The CU definitions.

# Startup

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio2"];
Once[<< KnotTheory`];
Once[<< "../Profile/Profile.m";];
<< "Engine-Speedy.m";
<< "Objects.m";
$k = 2;
HL[ℰ_] := Style[ℰ, Background → Green];
```

```
Loading KnotTheory` version of January 20, 2015, 10:42:19.1122.
Read more at http://katlas.org/wiki/KnotTheory.
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
This version: June 2018. Original version: July 1994.
```

In[∘]:= $\rho y = \begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix}$; $\rho b = \begin{pmatrix} 0 & 0 \\ 0 & -\epsilon \end{pmatrix}$; $\rho a = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$; $\rho x = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$;

```
HL /@ {ρa.ρx - ρx.ρa == γ ρx, ρa.ρy - ρy.ρa == -γ ρy,
   ρb.ρy - ρy.ρb == -ϵ ρy, ρb.ρx - ρx.ρb == ϵ ρx, ρx.ρy - ρy.ρx == γ ρb + ϵ ρa}
```

Out[∘]= {True, True, True, True, True}

In[∘]:=
```
eq = With[{ME = MatrixExp}, ME[ξ0 ρx].ME[η0 ρy] == ME[η1 ρy].ME[β1 ρb].ME[α1 ρa].ME[ξ1 ρx]]
```

Out[∘]= $\{\{1+\gamma\epsilon\,\eta0\,\xi0,\ \gamma\,\xi0\},\{\epsilon\,\eta0,\ 1\}\} == \{\{e^{\alpha1\,\gamma},\ e^{\alpha1\,\gamma}\gamma\,\xi1\},\{e^{\alpha1\,\gamma}\epsilon\,\eta1,\ e^{-\beta1\,\epsilon}+e^{\alpha1\,\gamma}\gamma\,\epsilon\,\eta1\,\xi1\}\}$

In[∘]:= `Det /@ eq`

Out[∘]= $1 == e^{\alpha1\,\gamma-\beta1\,\epsilon}$

In[∘]:=
```
{so} = Solve[Thread[Flatten /@ eq /. α1 → β1 ϵ / γ], {η1, β1, ξ1}] /. C@1 → 0
```

Out[∘]= $\left\{\left\{\eta1\to\frac{\eta0}{1+\gamma\,\epsilon\,\eta0\,\xi0},\ \xi1\to\frac{\xi0}{1+\gamma\,\epsilon\,\eta0\,\xi0},\ \beta1\to\frac{\mathrm{Log}[1+\gamma\,\epsilon\,\eta0\,\xi0]}{\epsilon}\right\}\right\}$

In[∘]:=
```
eq = With[{ME = MatrixExp},
  ME[η1 ρy].ME[β1 ρb].ME[α1 ρa].ME[ξ1 ρx].ME[η2 ρy].ME[β2 ρb].ME[α2 ρa].ME[ξ2 ρx] ==
   ME[η ρy].ME[β ρb].ME[α ρa].ME[ξ ρx]
 ]
```

Out[∘]= $\{\{e^{\alpha2\,\gamma}(e^{\alpha1\,\gamma}+e^{\alpha1\,\gamma}\gamma\,\epsilon\,\eta2\,\xi1),\ e^{\alpha1\,\gamma-\beta2\,\epsilon}\gamma\,\xi1+e^{\alpha2\,\gamma}\gamma\,(e^{\alpha1\,\gamma}+e^{\alpha1\,\gamma}\gamma\,\epsilon\,\eta2\,\xi1)\,\xi2\},$
$\{e^{\alpha2\,\gamma}(e^{\alpha1\,\gamma}\epsilon\,\eta1+\epsilon\,\eta2\,(e^{-\beta1\,\epsilon}+e^{\alpha1\,\gamma}\gamma\,\epsilon\,\eta1\,\xi1)),$
$e^{-\beta2\,\epsilon}(e^{-\beta1\,\epsilon}+e^{\alpha1\,\gamma}\gamma\,\epsilon\,\eta1\,\xi1)+e^{\alpha2\,\gamma}\gamma\,(e^{\alpha1\,\gamma}\epsilon\,\eta1+\epsilon\,\eta2\,(e^{-\beta1\,\epsilon}+e^{\alpha1\,\gamma}\gamma\,\epsilon\,\eta1\,\xi1))\,\xi2\}\} ==$
$\{\{e^{\alpha\,\gamma},\ e^{\alpha\,\gamma}\gamma\,\xi\},\{e^{\alpha\,\gamma}\epsilon\,\eta,\ e^{-\beta\,\epsilon}+e^{\alpha\,\gamma}\gamma\,\epsilon\,\eta\,\xi\}\}$

In[∘]:= `Det /@ eq`

Out[∘]= $e^{\alpha1\,\gamma+\alpha2\,\gamma-\beta1\,\epsilon-\beta2\,\epsilon} == e^{\alpha\,\gamma-\beta\,\epsilon}$

In[∘]:=
```
{s0} = Solve[PowerExpand /@ Log /@ Det /@ eq, α]
```

Out[∘]=
$$\left\{\left\{\alpha\to\frac{\alpha1\,\gamma+\alpha2\,\gamma+\beta\,\epsilon-\beta1\,\epsilon-\beta2\,\epsilon}{\gamma}\right\}\right\}$$

In[∘]:=
```
{s1} = Solve[Thread[Flatten /@ eq /. s0], {η, β, ξ}] /. C@1 → 0
```

··· Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is
$\alpha1\,\gamma+\alpha2\,\gamma+\mathrm{Log}[e^{(\beta-\beta1-\beta2)\,\epsilon}]-\mathrm{Log}[e^{\alpha2\,\gamma}\,(e^{\alpha1\,\gamma}+e^{\mathrm{Times}[\ll2\gg]}\,\gamma\,\epsilon\,\eta2\,\xi1)]==0.$

··· Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[∘]=
$$\Bigg\{\Bigg\{\eta\to\Bigg(e^{-\alpha1\,\gamma-\beta1\,\epsilon-\epsilon\left(-\beta1-\beta2+\frac{-\alpha1\,\gamma-\alpha2\,\gamma+\beta1\,\epsilon+\beta2\,\epsilon+\mathrm{Log}\left[e^{\alpha2\,\gamma}\,\left(e^{\alpha1\,\gamma}+e^{\alpha1\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right)\right]}{\epsilon}\right)}$$
$$\Bigg(-1+e^{\epsilon\left(-\beta1-\beta2+\frac{-\alpha1\,\gamma-\alpha2\,\gamma+\beta1\,\epsilon+\beta2\,\epsilon+\mathrm{Log}\left[e^{\alpha2\,\gamma}\,\left(e^{\alpha1\,\gamma}+e^{\alpha1\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right)\right]}{\epsilon}\right)}+e^{\alpha1\,\gamma+\beta1\,\epsilon+\epsilon\left(-\beta1-\beta2+\frac{-\alpha1\,\gamma-\alpha2\,\gamma+\beta1\,\epsilon+\beta2\,\epsilon+\mathrm{Log}\left[e^{\alpha2\,\gamma}\,\left(e^{\alpha1\,\gamma}+e^{\alpha1\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right)\right]}{\epsilon}\right)}$$
$$\gamma\,\epsilon\,\eta1\,\xi1+e^{\alpha1\,\gamma+\alpha2\,\gamma+\beta1\,\epsilon+\beta2\,\epsilon+\epsilon\left(-\beta1-\beta2+\frac{-\alpha1\,\gamma-\alpha2\,\gamma+\beta1\,\epsilon+\beta2\,\epsilon+\mathrm{Log}\left[e^{\alpha2\,\gamma}\,\left(e^{\alpha1\,\gamma}+e^{\alpha1\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right)\right]}{\epsilon}\right)}\,\gamma\,\epsilon\,\eta1\,\xi2+$$
$$e^{\alpha2\,\gamma+\beta2\,\epsilon+\epsilon\left(-\beta1-\beta2+\frac{-\alpha1\,\gamma-\alpha2\,\gamma+\beta1\,\epsilon+\beta2\,\epsilon+\mathrm{Log}\left[e^{\alpha2\,\gamma}\,\left(e^{\alpha1\,\gamma}+e^{\alpha1\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right)\right]}{\epsilon}\right)}\,\gamma\,\epsilon\,\eta2\,\xi2+$$
$$e^{\alpha1\,\gamma+\alpha2\,\gamma+\beta1\,\epsilon+\beta2\,\epsilon+\epsilon\left(-\beta1-\beta2+\frac{-\alpha1\,\gamma-\alpha2\,\gamma+\beta1\,\epsilon+\beta2\,\epsilon+\mathrm{Log}\left[e^{\alpha2\,\gamma}\,\left(e^{\alpha1\,\gamma}+e^{\alpha1\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right)\right]}{\epsilon}\right)}\,\gamma^2\,\epsilon^2\,\eta1\,\eta2\,\xi1\,\xi2\Bigg)\Bigg)\Bigg/$$
$$\left(\gamma\,\epsilon\,\left(\xi1+e^{\alpha2\,\gamma+\beta2\,\epsilon}\,\xi2+e^{\alpha2\,\gamma+\beta2\,\epsilon}\,\gamma\,\epsilon\,\eta2\,\xi1\,\xi2\right)\right),$$
$$\beta\to\frac{1}{\epsilon}\left(-\alpha1\,\gamma-\alpha2\,\gamma+\beta1\,\epsilon+\beta2\,\epsilon+\mathrm{Log}\left[e^{\alpha2\,\gamma}\,\left(e^{\alpha1\,\gamma}+e^{\alpha1\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right)\right]\right),$$
$$\xi\to$$
$$\left(e^{\alpha1\,\gamma-\alpha2\,\gamma-\beta2\,\epsilon}\,\left(\xi1+e^{\alpha2\,\gamma+\beta2\,\epsilon}\,\xi2+e^{\alpha2\,\gamma+\beta2\,\epsilon}\,\gamma\,\epsilon\,\eta2\,\xi1\,\xi2\right)\right)\Big/\left(e^{\alpha1\,\gamma}+e^{\alpha1\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right)\Bigg\}\Bigg\}$$

In[∘]:=
```
{so} = Solve[Thread[Flatten /@ eq /. {}], {η, β, α, ξ}] /. C@1 → 0
```

··· Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is
$\mathrm{Log}[e^{\alpha\,\gamma}]-\mathrm{Log}[e^{\alpha2\,\gamma}\,(e^{\alpha1\,\gamma}+e^{\mathrm{Times}[\ll2\gg]}\,\gamma\,\epsilon\,\eta2\,\xi1)]==0.$

··· Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is
$-\mathrm{Log}[e^{\alpha\,\gamma}]+\mathrm{Log}[e^{\alpha1\,\gamma+\alpha2\,\gamma}+e^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1]==0.$

··· Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[∘]=
$$\left\{\left\{\eta\to\left(e^{\alpha2\,\gamma-\beta1\,\epsilon}\,\left(e^{\alpha1\,\gamma+\beta1\,\epsilon}\,\eta1+\eta2+e^{\alpha1\,\gamma+\beta1\,\epsilon}\,\gamma\,\epsilon\,\eta1\,\eta2\,\xi1\right)\right)\Big/\left(e^{\alpha1\,\gamma+\alpha2\,\gamma}+e^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right),\right.\right.$$
$$\beta\to\frac{1}{\epsilon}\,\mathrm{Log}\Big[\left(e^{\beta1\,\epsilon+\beta2\,\epsilon}\,\left(e^{\alpha1\,\gamma+\alpha2\,\gamma}+e^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right)\right)\Big/\Big(e^{\alpha1\,\gamma+\alpha2\,\gamma}-e^{2\,\alpha1\,\gamma+\alpha2\,\gamma+\beta1\,\epsilon}\,\gamma\,\epsilon\,\eta1\,\xi1-$$
$$e^{2\,\alpha1\,\gamma+\alpha2\,\gamma+\beta1\,\epsilon}\,\gamma^2\,\epsilon^2\,\eta1\,\eta2\,\xi1^2+e^{\alpha1\,\gamma+\beta1\,\epsilon}\,\gamma\,\epsilon\,\eta1\,\xi1\,\left(e^{\alpha1\,\gamma+\alpha2\,\gamma}+e^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right)-$$
$$e^{2\,\alpha1\,\gamma+2\,\alpha2\,\gamma+\beta1\,\epsilon+\beta2\,\epsilon}\,\gamma\,\epsilon\,\eta1\,\xi2-e^{\alpha1\,\gamma+2\,\alpha2\,\gamma+\beta2\,\epsilon}\,\gamma\,\epsilon\,\eta2\,\xi2-2\,e^{2\,\alpha1\,\gamma+2\,\alpha2\,\gamma+\beta1\,\epsilon+\beta2\,\epsilon}\,\gamma^2\,\epsilon^2\,\eta1\,\eta2\,\xi1\,\xi2-$$
$$e^{\alpha1\,\gamma+2\,\alpha2\,\gamma+\beta2\,\epsilon}\,\gamma^2\,\epsilon^2\,\eta2^2\,\xi1\,\xi2-e^{2\,\alpha1\,\gamma+2\,\alpha2\,\gamma+\beta1\,\epsilon+\beta2\,\epsilon}\,\gamma^3\,\epsilon^3\,\eta1\,\eta2^2\,\xi1^2\,\xi2+e^{\alpha1\,\gamma+\alpha2\,\gamma+\beta1\,\epsilon+\beta2\,\epsilon}\,\gamma\,\epsilon$$
$$\eta1\,\left(e^{\alpha1\,\gamma+\alpha2\,\gamma}+e^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right)\,\xi2+e^{\alpha2\,\gamma+\beta2\,\epsilon}\,\gamma\,\epsilon\,\eta2\,\left(e^{\alpha1\,\gamma+\alpha2\,\gamma}+e^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right)\,\xi2+$$
$$e^{\alpha1\,\gamma+\alpha2\,\gamma+\beta1\,\epsilon+\beta2\,\epsilon}\,\gamma^2\,\epsilon^2\,\eta1\,\eta2\,\xi1\,\left(e^{\alpha1\,\gamma+\alpha2\,\gamma}+e^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right)\,\xi2\Big)\Big],$$
$$\alpha\to\frac{\mathrm{Log}\left[e^{\alpha1\,\gamma+\alpha2\,\gamma}+e^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right]}{\gamma},\ \xi\to\left(e^{\alpha1\,\gamma-\beta2\,\epsilon}\,\left(\xi1+e^{\alpha2\,\gamma+\beta2\,\epsilon}\,\xi2+e^{\alpha2\,\gamma+\beta2\,\epsilon}\,\gamma\,\epsilon\,\eta2\,\xi1\,\xi2\right)\right)\Big/$$
$$\left.\left.\left(e^{\alpha1\,\gamma+\alpha2\,\gamma}+e^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta2\,\xi1\right)\right\}\right\}$$

In[∘]:= `Λ = PowerExpand@Simplify[η y + α a + β b + ξ x /. so]`

Out[∘]= $$y\,\eta 1+\frac{e^{-\alpha 1\,\gamma-\beta 1\,\epsilon}\,y\,\eta 2}{1+\gamma\,\epsilon\,\eta 2\,\xi 1}+\frac{e^{-\alpha 2\,\gamma-\beta 2\,\epsilon}\,x\,\xi 1}{1+\gamma\,\epsilon\,\eta 2\,\xi 1}+x\,\xi 2+\frac{a\,\left((\alpha 1+\alpha 2)\,\gamma+\mathrm{Log}[1+\gamma\,\epsilon\,\eta 2\,\xi 1]\right)}{\gamma}+\frac{b\,\left((\beta 1+\beta 2)\,\epsilon+\mathrm{Log}[1+\gamma\,\epsilon\,\eta 2\,\xi 1]\right)}{\epsilon}$$

In[∘]:= `Q = Limit[Λ, ϵ → 0]`

Out[∘]= $$a\,(\alpha 1+\alpha 2)+y\,\eta 1+e^{-\alpha 1\,\gamma}\,y\,\eta 2+e^{-\alpha 2\,\gamma}\,x\,\xi 1+b\,(\beta 1+\beta 2+\gamma\,\eta 2\,\xi 1)+x\,\xi 2$$

In[∘]:= `Simplify[Λ - Q]`

Out[∘]= $$-e^{-\alpha 1\,\gamma}\,y\,\eta 2-e^{-\alpha 2\,\gamma}\,x\,\xi 1-b\,\gamma\,\eta 2\,\xi 1+\frac{e^{-\alpha 1\,\gamma-\beta 1\,\epsilon}\,y\,\eta 2}{1+\gamma\,\epsilon\,\eta 2\,\xi 1}+\frac{e^{-\alpha 2\,\gamma-\beta 2\,\epsilon}\,x\,\xi 1}{1+\gamma\,\epsilon\,\eta 2\,\xi 1}+\left(\frac{a}{\gamma}+\frac{b}{\epsilon}\right)\mathrm{Log}[1+\gamma\,\epsilon\,\eta 2\,\xi 1]$$

In[∘]:= `Series[e^(Λ-Q), {ϵ, 0, 2}]`

Out[∘]= $$1+\left(a\,\eta 2\,\xi 1-\frac{1}{2}\,b\,\gamma^2\,\eta 2^2\,\xi 1^2-e^{-\alpha 1\,\gamma}\,y\,\eta 2\,(\beta 1+\gamma\,\eta 2\,\xi 1)-e^{-\alpha 2\,\gamma}\,x\,\xi 1\,(\beta 2+\gamma\,\eta 2\,\xi 1)\right)\epsilon+\frac{1}{2}\left(\left(a\,\eta 2\,\xi 1-\frac{1}{2}\,b\,\gamma^2\,\eta 2^2\,\xi 1^2-e^{-\alpha 1\,\gamma}\,y\,\eta 2\,(\beta 1+\gamma\,\eta 2\,\xi 1)-e^{-\alpha 2\,\gamma}\,x\,\xi 1\,(\beta 2+\gamma\,\eta 2\,\xi 1)\right)^2+2\left(-\frac{1}{2}\,a\,\gamma\,\eta 2^2\,\xi 1^2+\frac{1}{3}\,b\,\gamma^3\,\eta 2^3\,\xi 1^3+\frac{1}{2}\,e^{-\alpha 1\,\gamma}\,y\,\eta 2\,(\beta 1^2+2\,\beta 1\,\gamma\,\eta 2\,\xi 1+2\,\gamma^2\,\eta 2^2\,\xi 1^2)+\frac{1}{2}\,e^{-\alpha 2\,\gamma}\,x\,\xi 1\,(\beta 2^2+2\,\beta 2\,\gamma\,\eta 2\,\xi 1+2\,\gamma^2\,\eta 2^2\,\xi 1^2)\right)\right)\epsilon^2+O[\epsilon]^3$$

In[∘]:= `QQ = Q /. {η1 → η_i, β1 → β_i, α1 → α_i, ξ1 → ξ_i, η2 → η_j, β2 → β_j, α2 → α_j, ξ2 → ξ_j, y → y_k, b → b_k, a → a_k, x → x_k}`

Out[∘]= $$a_k\,(\alpha_i+\alpha_j)+y_k\,\eta_i+e^{-\gamma\,\alpha_i}\,y_k\,\eta_j+e^{-\gamma\,\alpha_j}\,x_k\,\xi_i+b_k\,(\beta_i+\beta_j+\gamma\,\eta_j\,\xi_i)+x_k\,\xi_j$$

In[∘]:= `QQ /. (η | ξ)_ → 0`

Out[∘]= $$a_k\,(\alpha_i+\alpha_j)+b_k\,(\beta_i+\beta_j)$$

In[∘]:= `FullSimplify[Simplify[QQ - (QQ /. (η | ξ)_ → 0)] //. l2U]`

Out[∘]= $$y_k\left(\eta_i+\frac{\eta_j}{\mathcal{A}_i}\right)+\gamma\,b_k\,\eta_j\,\xi_i+x_k\left(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\right)$$

In[∘]:= `α2𝒜 = {e^(d_.+c_. α_i_) :> 𝒜_i^(c/γ) e^d, e^(α c_.+d_.) :> 𝒜^(c/γ) e^d, e^ℰ_ :> e^Expand[ℰ //. α2𝒜]}`

Out[∘]= $$\left\{e^{d\_.+c\_.\,\alpha_{i\_}}:\!\to \mathcal{A}_i^{c/\gamma}\,e^d,\ e^{\alpha\,c\_.+d\_.}:\!\to \mathcal{A}^{c/\gamma}\,e^d,\ e^{\mathcal{E}\_}:\!\to e^{\mathrm{Expand}[\mathcal{E}//.\alpha 2\mathcal{A}]}\right\}$$

In[∘]:= `FullSimplify[Exp@FullSimplify[Λ - Q] /. {η1 → η_i, β1 → β_i, α1 → α_i, ξ1 → ξ_i, η2 → η_j, β2 → β_j, α2 → α_j, ξ2 → ξ_j, y → y_k, b → b_k, a → a_k, x → x_k} //. α2𝒜]`

Out[∘]= $$e^{y_k\,\eta_j\left(-\frac{1}{\mathcal{A}_i}+\frac{e^{-\epsilon\,\beta_i}}{\mathcal{A}_i+\gamma\,\epsilon\,\mathcal{A}_i\,\eta_j\,\xi_i}\right)+\xi_i\left(-\gamma\,b_k\,\eta_j+x_k\left(-\frac{1}{\mathcal{A}_j}+\frac{e^{-\epsilon\,\beta_j}}{\mathcal{A}_j+\gamma\,\epsilon\,\mathcal{A}_j\,\eta_j\,\xi_i}\right)\right)}\,(1+\gamma\,\epsilon\,\eta_j\,\xi_i)^{\frac{a_k}{\gamma}+\frac{b_k}{\epsilon}}$$

In[∘]:=

$$\text{Define}\Big[\text{cm}_{i,j\to k}=\text{CF}@\mathbb{E}_{\{i,j\}\to\{k\}}\Big[a_k\,(\alpha_i+\alpha_j)+b_k\,(\beta_i+\beta_j),\ y_k\left(\eta_i+\frac{\eta_j}{\mathcal{A}_i}\right)+\gamma\,b_k\,\eta_j\,\xi_i+x_k\left(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\right),$$
$$e^{y_k\,\eta_j\left(\frac{e^{-\epsilon\beta_i}}{\mathcal{A}_i+\gamma\,\epsilon\,\mathcal{A}_i\,\eta_j\,\xi_i}-\frac{1}{\mathcal{A}_i}\right)+\xi_i\left(x_k\left(\frac{e^{-\epsilon\beta_j}}{\mathcal{A}_j+\gamma\,\epsilon\,\mathcal{A}_j\,\eta_j\,\xi_i}-\frac{1}{\mathcal{A}_j}\right)-\gamma\,b_k\,\eta_j\right)}\left(1+\gamma\,\epsilon\,\eta_j\,\xi_i\right)^{\frac{a_k}{\gamma}+\frac{b_k}{\epsilon}}$$
$$\Big]_{\$k}\Big]$$

In[∘]:= $\text{cm}_{1,2\to3}$

Out[∘]=

$$\mathbb{E}_{\{1,2\}\to\{3\}}\Big[a_3\,\alpha_1+a_3\,\alpha_2+b_3\,\beta_1+b_3\,\beta_2,\ y_3\,\eta_1+\frac{y_3\,\eta_2}{\mathcal{A}_1}+\frac{x_3\,\xi_1}{\mathcal{A}_2}+\gamma\,b_3\,\eta_2\,\xi_1+x_3\,\xi_2,$$
$$1+\left(-\frac{y_3\,\beta_1\,\eta_2}{\mathcal{A}_1}-\frac{x_3\,\beta_2\,\xi_1}{\mathcal{A}_2}+a_3\,\eta_2\,\xi_1-\frac{\gamma\,y_3\,\eta_2^2\,\xi_1}{\mathcal{A}_1}-\frac{\gamma\,x_3\,\eta_2\,\xi_1^2}{\mathcal{A}_2}-\frac{1}{2}\gamma^2\,b_3\,\eta_2^2\,\xi_1^2\right)\epsilon+$$
$$\Bigg(\frac{y_3\,\beta_1^2\,\eta_2}{2\,\mathcal{A}_1}+\frac{y_3^2\,\beta_1^2\,\eta_2^2}{2\,\mathcal{A}_1^2}+\frac{x_3\,\beta_2^2\,\xi_1}{2\,\mathcal{A}_2}+\frac{x_3\,y_3\,\beta_1\,\beta_2\,\eta_2\,\xi_1}{\mathcal{A}_1\,\mathcal{A}_2}+\frac{\gamma\,y_3\,\beta_1\,\eta_2^2\,\xi_1}{\mathcal{A}_1}-\frac{a_3\,y_3\,\beta_1\,\eta_2^2\,\xi_1}{\mathcal{A}_1}+\frac{\gamma\,y_3^2\,\beta_1\,\eta_2^3\,\xi_1}{\mathcal{A}_1^2}+$$
$$\frac{x_3^2\,\beta_2^2\,\xi_1^2}{2\,\mathcal{A}_2^2}+\frac{\gamma\,x_3\,\beta_2\,\eta_2\,\xi_1^2}{\mathcal{A}_2}-\frac{a_3\,x_3\,\beta_2\,\eta_2\,\xi_1^2}{\mathcal{A}_2}-\frac{1}{2}\gamma\,a_3\,\eta_2^2\,\xi_1^2+\frac{1}{2}a_3^2\,\eta_2^2\,\xi_1^2+\frac{\gamma\,x_3\,y_3\,\beta_1\,\eta_2^2\,\xi_1^2}{\mathcal{A}_1\,\mathcal{A}_2}+$$
$$\frac{\gamma\,x_3\,y_3\,\beta_2\,\eta_2^2\,\xi_1^2}{\mathcal{A}_1\,\mathcal{A}_2}+\frac{\gamma^2\,y_3\,\eta_2^3\,\xi_1^2}{\mathcal{A}_1}-\frac{\gamma\,a_3\,y_3\,\eta_2^3\,\xi_1^2}{\mathcal{A}_1}+\frac{\gamma^2\,b_3\,y_3\,\beta_1\,\eta_2^3\,\xi_1^2}{2\,\mathcal{A}_1}+\frac{\gamma^2\,y_3^2\,\eta_2^4\,\xi_1^2}{2\,\mathcal{A}_1^2}+\frac{\gamma\,x_3^2\,\beta_2\,\eta_2\,\xi_1^3}{\mathcal{A}_2^2}+$$
$$\frac{\gamma^2\,x_3\,\eta_2^2\,\xi_1^3}{\mathcal{A}_2}-\frac{\gamma\,a_3\,x_3\,\eta_2^2\,\xi_1^3}{\mathcal{A}_2}+\frac{\gamma^2\,b_3\,x_3\,\beta_2\,\eta_2^2\,\xi_1^3}{2\,\mathcal{A}_2}+\frac{1}{3}\gamma^3\,b_3\,\eta_2^3\,\xi_1^3-\frac{1}{2}\gamma^2\,a_3\,b_3\,\eta_2^3\,\xi_1^3+$$
$$\frac{\gamma^2\,x_3\,y_3\,\eta_2^3\,\xi_1^3}{\mathcal{A}_1\,\mathcal{A}_2}+\frac{\gamma^3\,b_3\,y_3\,\eta_2^4\,\xi_1^3}{2\,\mathcal{A}_1}+\frac{\gamma^2\,x_3^2\,\eta_2^2\,\xi_1^4}{2\,\mathcal{A}_2^2}+\frac{\gamma^3\,b_3\,x_3\,\eta_2^3\,\xi_1^4}{2\,\mathcal{A}_2}+\frac{1}{8}\gamma^4\,b_3^2\,\eta_2^4\,\xi_1^4\Bigg)\epsilon^2+O[\epsilon]^3\Big]$$

In[∘]:= $\text{Timing}@\text{Block}[\{\$k=4\},\ \text{HL}[(\text{cm}_{1,2\to1}\,//\,\text{cm}_{1,3\to1})\equiv(\text{cm}_{2,3\to2}\,//\,\text{cm}_{1,2\to1})]]$

Out[∘]= {53.4688, True}

In[∘]:= $\text{Define}\left[\text{c}\Delta_{i\to j,k}=\mathbb{E}_{\{i\}\to\{j,k\}}\left[(a_j+a_k)\,\alpha_i+\left(b_j+b_k\right)\beta_i,\ (y_j+y_k)\,\eta_i+(x_j+x_k)\,\xi_i,\ 1\right]_{\$k}\right]$

Associativity, co-associativity, and Δ is an algebra morphism:

In[∘]:= $\text{Timing}@\text{Block}[\{\$k=3\},\ \text{HL}\ /@\ \{(\text{cm}_{1,2\to1}\,//\,\text{cm}_{1,3\to1})\equiv(\text{cm}_{2,3\to2}\,//\,\text{cm}_{1,2\to1}),$
$(\text{c}\Delta_{1\to1,2}\,//\,\text{c}\Delta_{2\to2,3})\equiv(\text{c}\Delta_{1\to1,3}\,//\,\text{c}\Delta_{1\to1,2}),$
$(\text{cm}_{1,2\to1}\,//\,\text{c}\Delta_{1\to1,2})\equiv((\text{c}\Delta_{1\to1,3}\,\text{c}\Delta_{2\to2,4})\,//\,(\text{cm}_{3,4\to2}\,\text{cm}_{1,2\to1}))\}]$

Out[∘]= {13.4531, {True, True, True}}

In[∘]:= $\text{Define}[\text{cS}_i=\mathbb{E}_{\{i\}\to\{1,2,3,4\}}[-\beta_i\,b_2-\alpha_i\,a_3,\ -\eta_i\,y_1-\xi_i\,x_4,\ 1]\,//\,\text{cm}_{4,3\to i}\,//\,\text{cm}_{i,2\to i}\,//\,\text{cm}_{i,1\to i}]$

S is convolution inverse of id:

In[∘]:= $\text{Timing}@\text{Block}\Big[\{\$k=3\},\ \text{HL}[\#\equiv\mathbb{E}_{\{1\}\to\{1\}}[0,0,1]]\ \&\ /@\ \Big\{$
$\left(\text{c}\Delta_{1\to1,2}\sim B_1\sim\text{cS}_1\right)\sim B_{1,2}\sim\text{cm}_{1,2\to1},\ \left(\text{c}\Delta_{1\to1,2}\sim B_2\sim\text{cS}_2\right)\sim B_{1,2}\sim\text{cm}_{1,2\to1}\Big\}\Big]$

Out[∘]= {4.10938, {True, True}}

S is an algebra anti-(co)morphism

In[∘]:=
```
Timing@Block[{$k = 3},
  HL /@ {cm_{1,2→1} ~ B_1 ~ cS_1 ≡ (cS_1 cS_2) ~ B_{1,2} ~ cm_{2,1→1}, cS_1 ~ B_1 ~ cΔ_{1→1,2} ≡ cΔ_{1→2,1} ~ B_{1,2} ~ (cS_1 cS_2)}]
```

Out[∘]=
```
{19., {True, True}}
```

Classical is the $\hbar \to 0$ limit of quantum:

In[∘]:=
```
Timing[HL /@ {
   (cm_{1,2→3}) ≡ ((Limit[# /. U2l, ℏ → 0] //. l2U) & /@ dm_{1,2→3}),
   cΔ_{1→2,3} ≡ ((Limit[# /. U2l, ℏ → 0] //. l2U) & /@ dΔ_{1→2,3}),
   cS_1 ≡ ((Limit[# /. U2l, ℏ → 0] //. l2U) & /@ dS_1)
  }]
```

Out[∘]=
```
{35.7031, {True, True, True}}
```