Pensieve header: Benchmarking: QU testing, Knot[10,100] at k=1.

Header

Benchmarking in QU

Startup

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio2"];
<< KnotTheory`;
<< "../Profile/Profile.m";
<< "Engine-Speedy.m";
<< "Objects.m";
<< "KT.m";
BeginProfile[];
PopupWindow[Button["Show Profile Monitor"],
 Dynamic[PrintProfile[], UpdateInterval → 3, TrackedSymbols → {}]]
```

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, WikiLink, mathematica}].

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, QuantumGroups}].

```
Loading KnotTheory` version of January 20, 2015, 10:42:19.1122.
Read more at http://katlas.org/wiki/KnotTheory.
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
This version: June 2018. Original version: July 1994.
```

» Warning: On Sep 4 2019 I swapped the operations ϵ and η. Some incompatibilities may arise in older notebooks.

Out[∘]= Show Profile Monitor

In[∘]:=
```
$k = 2; (*ℏ=γ=1;*)
```

Utilities

In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Testing

In[∘]:=
```
$QZipFail = True;
```

In[∘]:=
```
Block[{$k = 1}, {
    am → am_{i,j→k}, bm → bm_{i,j→k}, dm → dm_{i,j→k}, R → R_{i,j}, R̄ → R̄_{i,j}, P → P_{i,j},
    aS → aS_i, aS̄ → aS̄_i, bS → bS_i, bS̄ → bS̄_i, dS → dS_i, aΔ → aΔ_{i→j,k}, bΔ → bΔ_{i→j,k},
    dΔ → dΔ_{i→j,k}, C → C_i, C̄ → C̄_i, Kink → Kink_i, Kink̄ → Kink̄_i, b2t → b2t_i, t2b → t2b_i
  }] //
 Column
```

Out[∘]=

$$\mathrm{am} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\,\alpha_i + a_k\,\alpha_j,\ \frac{x_k\,\xi_i}{\mathcal{A}_j} + x_k\,\xi_j,\ 1\right]$$

$$\mathrm{bm} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[b_k\,\beta_i + b_k\,\beta_j,\ y_k\,\eta_i + y_k\,\eta_j,\ 1 - y_k\,\beta_i\,\eta_j\,\epsilon + O[\epsilon]^2\right]$$

$$\mathrm{dm} \to \mathbb{E}_{\{i,j\}\to\{k\}}\Big[a_k\,\alpha_i + a_k\,\alpha_j + b_k\,\beta_i + b_k\,\beta_j,\ y_k\,\eta_i + \frac{y_k\,\eta_j}{\mathcal{A}_i} + \frac{x_k\,\xi_i}{\mathcal{A}_j} + \frac{(1-B_k)\,\eta_j\,\xi_i}{\hbar} + x_k\,\xi_j,$$
$$1 + \left(-\frac{y_k\,\beta_i\,\eta_j}{\mathcal{A}_i} - \frac{x_k\,\beta_j\,\xi_i}{\mathcal{A}_j} + a_k\,B_k\,\eta_j\,\xi_i + \frac{\gamma\,\hbar\,x_k\,y_k\,\eta_j\,\xi_i}{\mathcal{A}_i\,\mathcal{A}_j} + \frac{(\gamma - 3\,\gamma\,B_k)\,y_k\,\eta_j^2\,\xi_i}{2\,\mathcal{A}_i} + \frac{(\gamma - 3\,\gamma\,B_k)\,x_k\,\eta_j\,\xi_i^2}{2\,\mathcal{A}_j} + \frac{(\gamma - 4\,\gamma\,B_k + 3\,\gamma\,B_k^2)\,\eta_j^2\,\xi_i^2}{4\,\hbar}\right)\epsilon +$$
$$O[\epsilon]^2\Big]$$

$$\mathrm{R} \to \mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\,a_j\,b_i,\ \hbar\,x_j\,y_i,\ 1 - \frac{1}{4}\left(\gamma\,\hbar^3\,x_j^2\,y_i^2\right)\epsilon + O[\epsilon]^2\right]$$

$$\overline{\mathrm{R}} \to \mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\,a_j\,b_i,\ -\frac{\hbar\,x_j\,y_i}{B_i},\ 1 + \left(-\frac{\hbar^2\,a_j\,x_j\,y_i}{B_i} - \frac{3\,\gamma\,\hbar^3\,x_j^2\,y_i^2}{4\,B_i^2}\right)\epsilon + O[\epsilon]^2\right]$$

$$\mathrm{P} \to \mathbb{E}_{\{i,j\}\to\{\}}\left[\frac{\alpha_j\,\beta_i}{\hbar},\ \frac{\eta_i\,\xi_j}{\hbar},\ 1 + \frac{\gamma\,\eta_i^2\,\xi_j^2\,\epsilon}{4\,\hbar} + O[\epsilon]^2\right]$$

$$\mathrm{aS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i,\ -x_i\,\mathcal{A}_i\,\xi_i,\ 1 + \left(-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i - \frac{1}{2}\,\gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right)\epsilon + O[\epsilon]^2\right]$$

$$\overline{\mathrm{aS}} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i,\ -x_i\,\mathcal{A}_i\,\xi_i,\ 1 + \left(\gamma\,\hbar\,x_i\,\mathcal{A}_i\,\xi_i - \hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i - \frac{1}{2}\,\gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right)\epsilon + O[\epsilon]^2\right]$$

$$\mathrm{bS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i,\ -\frac{y_i\,\eta_i}{B_i},\ 1 + \left(-\frac{y_i\,\beta_i\,\eta_i}{B_i} - \frac{\gamma\,\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right)\epsilon + O[\epsilon]^2\right]$$

$$\overline{\mathrm{bS}} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i,\ -\frac{y_i\,\eta_i}{B_i},\ 1 + \left(\frac{\gamma\,\hbar\,y_i\,\eta_i}{B_i} - \frac{y_i\,\beta_i\,\eta_i}{B_i} - \frac{\gamma\,\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right)\epsilon + O[\epsilon]^2\right]$$

$$\mathrm{dS} \to \mathbb{E}_{\{i\}\to\{i\}}\Big[-a_i\,\alpha_i - b_i\,\beta_i,\ -\frac{y_i\,\mathcal{A}_i\,\eta_i}{B_i} - x_i\,\mathcal{A}_i\,\xi_i + \frac{(\mathcal{A}_i - B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{\hbar\,B_i},$$
$$1 + \Big(\frac{\gamma\,\hbar\,y_i\,\mathcal{A}_i\,\eta_i}{B_i} - \frac{y_i\,\mathcal{A}_i\,\beta_i\,\eta_i}{B_i} - \frac{\gamma\,\hbar\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2}{2\,B_i^2} - \hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i - x_i\,\mathcal{A}_i\,\beta_i\,\xi_i + \frac{a_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{B_i} -$$
$$\frac{\gamma\,\hbar\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i}{B_i} + \frac{(-\gamma\,\mathcal{A}_i + \gamma\,B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{B_i} + \frac{(\mathcal{A}_i - B_i\,\mathcal{A}_i)\,\beta_i\,\eta_i\,\xi_i}{\hbar\,B_i} + \frac{y_i\,(3\,\gamma\,\mathcal{A}_i^2 - \gamma\,B_i\,\mathcal{A}_i^2)\,\eta_i^2\,\xi_i}{2\,B_i^2} -$$
$$\frac{1}{2}\,\gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2 + \frac{x_i\,(3\,\gamma\,\mathcal{A}_i^2 - \gamma\,B_i\,\mathcal{A}_i^2)\,\eta_i\,\xi_i^2}{2\,B_i} + \frac{(-3\,\gamma\,\mathcal{A}_i^2 + 4\,\gamma\,B_i\,\mathcal{A}_i^2 - \gamma\,B_i^2\,\mathcal{A}_i^2)\,\eta_i^2\,\xi_i^2}{4\,\hbar\,B_i^2}\Big)\epsilon + O[\epsilon]^2\Big]$$

$$\mathrm{a}\triangle \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i + a_k\,\alpha_i,\ x_j\,\xi_i + x_k\,\xi_i,\ 1 + \left(-\hbar\,a_j\,x_k\,\xi_i + \frac{1}{2}\,\gamma\,\hbar\,x_j\,x_k\,\xi_i^2\right)\epsilon + O[\epsilon]^2\right]$$

$$\mathrm{b}\triangle \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[b_j\,\beta_i + b_k\,\beta_i,\ B_k\,y_j\,\eta_i + y_k\,\eta_i,\ 1 + \frac{1}{2}\,\gamma\,\hbar\,B_k\,y_j\,y_k\,\eta_i^2\,\epsilon + O[\epsilon]^2\right]$$

$$\mathrm{d}\triangle \to \mathbb{E}_{\{i\}\to\{j,k\}}\Big[a_j\,\alpha_i + a_k\,\alpha_i + b_j\,\beta_i + b_k\,\beta_i,\ y_j\,\eta_i + B_j\,y_k\,\eta_i + x_j\,\xi_i + x_k\,\xi_i,$$
$$1 + \left(\frac{1}{2}\,\gamma\,\hbar\,B_j\,y_j\,y_k\,\eta_i^2 - \hbar\,a_j\,x_k\,\xi_i + \frac{1}{2}\,\gamma\,\hbar\,x_j\,x_k\,\xi_i^2\right)\epsilon + O[\epsilon]^2\Big]$$

$$\mathrm{C} \to \mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \sqrt{B_i} - \frac{1}{2}\left(\hbar\,a_i\,\sqrt{B_i}\right)\epsilon + O[\epsilon]^2\right]$$

$$\overline{\mathrm{C}} \to \mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \frac{1}{\sqrt{B_i}} + \frac{\hbar\,a_i\,\epsilon}{2\,\sqrt{B_i}} + O[\epsilon]^2\right]$$

$$\mathrm{Kink} \to \mathbb{E}_{\{\}\to\{i\}}\left[\hbar\,a_i\,b_i,\ \hbar\,x_i\,y_i,\ \frac{1}{\sqrt{B_i}} + \left(\frac{\hbar\,a_i}{2\,\sqrt{B_i}} - \frac{\gamma\,\hbar^3\,x_i^2\,y_i^2}{4\,\sqrt{B_i}}\right)\epsilon + O[\epsilon]^2\right]$$

$$\overline{\mathrm{Kink}} \to \mathbb{E}_{\{\}\to\{i\}}\left[-\hbar\,a_i\,b_i,\ -\frac{\hbar\,x_i\,y_i}{B_i},\ \sqrt{B_i} + \left(-\frac{1}{2}\,\hbar\,a_i\,\sqrt{B_i} - \frac{\hbar^2\,a_i\,x_i\,y_i}{\sqrt{B_i}} - \frac{3\,\gamma\,\hbar^3\,x_i^2\,y_i^2}{4\,B_i^{3/2}}\right)\epsilon + O[\epsilon]^2\right]$$

$$\mathrm{b2t} \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i - \frac{t_i\,\beta_i}{\gamma},\ y_i\,\eta_i + x_i\,\xi_i,\ 1 + \frac{a_i\,\beta_i\,\epsilon}{\gamma} + O[\epsilon]^2\right]$$

$$\mathrm{t2b} \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i - \gamma\,b_i\,\tau_i,\ y_i\,\eta_i + x_i\,\xi_i,\ 1 + a_i\,\tau_i\,\epsilon + O[\epsilon]^2\right]$$

Check that on the generators this agrees with our conventions in the handout:

In[∘]:=
```
Timing@
 {{"[a,x]" → ((𝔼_{}→{1,2}[0, 0, a₂ x₁] // am_{1,2→1})[3] - (𝔼_{}→{1,2}[0, 0, a₁ x₂] // am_{1,2→1})[3]),
    "[b,y]" → ((𝔼_{}→{1,2}[0, 0, y₂ b₁] // bm_{1,2→1})[3] - (𝔼_{}→{1,2}[0, 0, y₁ b₂] // bm_{1,2→1})[3])} /.
   z_1 → z,
  {"Δ[y]" → Last[𝔼_{}→{1}[0, 0, y₁] ~B₁~ bΔ_{1→1,2}],
   "Δ[b]" → Last[𝔼_{}→{1}[0, 0, b₁] ~B₁~ bΔ_{1→1,2}],
   "Δ[a]" → Last[𝔼_{}→{1}[0, 0, a₁] ~B₁~ aΔ_{1→1,2}],
   "Δ[x]" → Last[𝔼_{}→{1}[0, 0, x₁] ~B₁~ aΔ_{1→1,2}]},
  {
    "S(a)" → ((𝔼_{}→{1}[0, 0, a₁] ~B₁~ aS₁)[3]),
    "S(x)" → ((𝔼_{}→{1}[0, 0, x₁] ~B₁~ aS₁)[3]),
    "S(b)" → ((𝔼_{}→{1}[0, 0, b₁] ~B₁~ bS₁)[3]),
    "S(y)" → ((𝔼_{}→{1}[0, 0, y₁] ~B₁~ bS₁)[3])
  } /. z_1 → z}
```

Out[∘]= $\{1., \{\{[a,x] \to -x\gamma,\ [b,y] \to -y\epsilon + O[\epsilon]^3\}, \{\Delta[y] \to (B_2 y_1 + y_2) + O[\epsilon]^3,\ \Delta[b] \to (b_1 + b_2) + O[\epsilon]^3,\ \Delta[a] \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta[x] \to (x_1 + x_2) - \hbar a_1 x_2 \epsilon + \frac{1}{2}\hbar^2 a_1^2 x_2 \epsilon^2 + O[\epsilon]^3\}, \{S(a) \to -a + O[\epsilon]^3,\ S(x) \to -x - a x \hbar \epsilon - \frac{1}{2}(a^2 x \hbar^2)\epsilon^2 + O[\epsilon]^3,\ S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + O[\epsilon]^3\}\}\}$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

In[∘]:=
```
Timing@Block[{$k = 3},
  HL /@ {(am_{1,2→1} // am_{1,3→1}) ≡ (am_{2,3→2} // am_{1,2→1}), (bm_{1,2→1} // bm_{1,3→1}) ≡ (bm_{2,3→2} // bm_{1,2→1})}
 ]
```

Out[∘]= {0.125, {True, True}}

R and *P* are inverses:

In[∘]:=
```
Timing@Block[{$k = 3}, {R_{i,j}, P_{i,k}, HL[(R_{i,j} // P_{i,k}) ≡ 𝔼_{k}→{j}[a_j α_k, x_j ξ_k, 1]]}]
```

Out[∘]= $\{0.28125, \{\mathbb{E}_{\{\}\to\{i,j\}}[\hbar a_j b_i, \hbar x_j y_i, 1 - \frac{1}{4}(\gamma \hbar^3 x_j^2 y_i^2)\epsilon + (\frac{1}{9}\gamma^2 \hbar^5 x_j^3 y_i^3 + \frac{1}{32}\gamma^2 \hbar^6 x_j^4 y_i^4)\epsilon^2 + (\frac{1}{48}\gamma^3 \hbar^5 x_j^2 y_i^2 - \frac{1}{16}\gamma^3 \hbar^7 x_j^4 y_i^4 - \frac{1}{36}\gamma^3 \hbar^8 x_j^5 y_i^5 - \frac{1}{384}\gamma^3 \hbar^9 x_j^6 y_i^6)\epsilon^3 + O[\epsilon]^4], \mathbb{E}_{\{i,k\}\to\{\}}[\frac{\alpha_k \beta_i}{\hbar}, \frac{\eta_i \xi_k}{\hbar}, 1 + \frac{\gamma \eta_i^2 \xi_k^2 \epsilon}{4\hbar} + \frac{(36\gamma^2\hbar^2\eta_i^2\xi_k^2 + 40\gamma^2\hbar\eta_i^3\xi_k^3 + 9\gamma^2\eta_i^4\xi_k^4)\epsilon^2}{288\hbar^2} + (\frac{1}{24}\gamma^3\hbar\eta_i^2\xi_k^2 + \frac{1}{6}\gamma^3\eta_i^3\xi_k^3 + \frac{13\gamma^3\eta_i^4\xi_k^4}{96\hbar} + \frac{5\gamma^3\eta_i^5\xi_k^5}{144\hbar^2} + \frac{\gamma^3\eta_i^6\xi_k^6}{384\hbar^3})\epsilon^3 + O[\epsilon]^4], \text{True}\}\}$

as and $\overline{\text{aS}}$ are inverses, bs and $\overline{\text{bS}}$ are inverses:

In[∘]:=
```
Timing[HL /@ {(aS̄₁ // aS₁) ≡ 𝔼_{1}→{1}[a₁ α₁, x₁ ξ₁, 1], (bS̄₁ // bS₁) ≡ 𝔼_{1}→{1}[b₁ β₁, y₁ η₁, 1]}]
```

Out[∘]= {0.28125, {True, True}}

(co)-associativity on both sides

```
In[•]:= Timing[
  HL /@ {(aΔ1→1,2 // aΔ2→2,3) ≡ (aΔ1→1,3 // aΔ1→1,2), (bΔ1→1,2 // bΔ2→2,3) ≡ (bΔ1→1,3 // bΔ1→1,2),
    (am1,2→1 // am1,3→1) ≡ (am2,3→2 // am1,2→1), (bm1,2→1 // bm1,3→1) ≡ (bm2,3→2 // bm1,2→1)}]
Out[•]= {0.265625, {True, True, True, True}}
```

Δ is an algebra morphism

```
In[•]:= Timing[HL /@ {(am1,2→1 // aΔ1→1,2) ≡ ((aΔ1→1,3 aΔ2→2,4) // (am3,4→2 am1,2→1)),
    (bm1,2→1 // bΔ1→1,2) ≡ ((bΔ1→1,3 bΔ2→2,4) // (bm3,4→2 bm1,2→1))}]
Out[•]= {0.359375, {True, True}}
```

An explicit formula for aS_i

```
In[•]:= Timing@Block[{$k = 4}, HL[aSi ≡ (𝔼{i}→{i,j}[-αi aj, -ξi xi,
        Sum[Expand[e^(ξi xi) (-ħ γ ϵ)^k / (2^k k!) Nest[Expand[xi^2 ∂{xi,2} #] &, e^(-ξi e^(ħ ϵ ai) xi), k]], {k, 0, $k}]]$k //
      ami,j→i)]]
Out[•]= {2.84375, True}
```

S is convolution inverse of id

```
In[•]:= Timing[HL[# ≡ 𝔼{1}→{1}[0, 0, 1]] & /@ {
    (aΔ1→1,2 ~ B1 ~ aS1) ~ B1,2 ~ am1,2→1, (aΔ1→1,2 ~ B2 ~ aS2) ~ B1,2 ~ am1,2→1,
    (bΔ1→1,2 ~ B1 ~ bS1) ~ B1,2 ~ bm1,2→1, (bΔ1→1,2 ~ B2 ~ bS2) ~ B1,2 ~ bm1,2→1}]
Out[•]= {0.390625, {True, True, True, True}}
```

But not with the opposite product:

```
In[•]:= Timing[Short[# ≡ 𝔼{1}→{1}[0, 0, 1]] & /@ {
    (aΔ1→1,2 ~ B1 ~ aS1) ~ B1,2 ~ am2,1→1, (aΔ1→1,2 ~ B2 ~ aS2) ~ B1,2 ~ am2,1→1,
    (bΔ1→1,2 ~ B1 ~ bS1) ~ B1,2 ~ bm2,1→1, (bΔ1→1,2 ~ B2 ~ bS2) ~ B1,2 ~ bm2,1→1}]
```

Out[•]= $\{0.515625, \{\frac{1}{2}\left(-2\gamma\epsilon\hbar x_1 \mathcal{A}_1 \xi_1 + \gamma^2\epsilon^2\hbar^2 \ll 1 \gg \mathcal{A}_1\xi_1 - \ll 1 \gg + 2\gamma^2\epsilon^2\hbar^2 x_1^2 \mathcal{A}_1^2 \xi_1^2\right) == 0,$

$\frac{1}{2}\left(-2\gamma\epsilon\hbar x_1\xi_1 - \gamma^2\epsilon^2\hbar^2 x_1\xi_1 + 2\gamma^2\epsilon^2\hbar^2 x_1^2\xi_1^2\right) == 0,$

$\frac{1}{2}\left(-2\gamma\epsilon\hbar y_1\eta_1 - \gamma^2\epsilon^2\hbar^2 y_1\eta_1 + 2\gamma^2\epsilon^2\hbar^2 y_1^2\eta_1^2\right) == 0,$

$\frac{-2\gamma\epsilon\hbar B_1 y_1\eta_1 + \ll 1 \gg - \ll 1 \gg + 2 \ll 4 \gg \eta_1^2}{2 B_1^2} == 0\}\}$

S is an algebra anti-(co)morphism

```
In[•]:= Timing[HL /@ {am1,2→1 ~ B1 ~ aS1 ≡ (aS1 aS2) ~ B1,2 ~ am2,1→1, bm1,2→1 ~ B1 ~ bS1 ≡ (bS1 bS2) ~ B1,2 ~ bm2,1→1,
    aS1 ~ B1 ~ aΔ1→1,2 ≡ aΔ1→2,1 ~ B1,2 ~ (aS1 aS2), bS1 ~ B1 ~ bΔ1→1,2 ≡ bΔ1→2,1 ~ B1,2 ~ (bS1 bS2)}]
Out[•]= {0.78125, {True, True, True, True}}
```

Pairing axioms

```
In[•]:= Timing[HL /@ {(bm1,2→1 𝔼{3}→{3}[α3 a3, ξ3 x3, 1]) ~ B1,3 ~ P1,3 ≡
          (𝔼{1}→{1}[β1 b1, η1 y1, 1] 𝔼{2}→{2}[β2 b2, η2 y2, 1] aΔ3→4,5) ~ B1,4 ~ P1,4 ~ B2,5 ~ P2,5,
        (bΔ1→1,2 𝔼{3}→{3}[α3 a3, ξ3 x3, 1] 𝔼{4}→{4}[α4 a4, ξ4 x4, 1]) ~ B1,3 ~ P1,3 ~ B2,4 ~ P2,4 ≡
          (𝔼{1}→{1}[β1 b1, η1 y1, 1] am3,4→3) ~ B1,3 ~ P1,3 }]
Out[•]= {0.3125, {True, True}}
```

```
In[•]:= Timing[HL /@ {((bS1 𝔼{2}→{2}[α2 a2, ξ2 x2, 1]) // P1,2) ≡ ((𝔼{1}→{1}[β1 b1, η1 y1, 1] aS2) // P1,2),
        (bS̄1 𝔼{2}→{2}[α2 a2, ξ2 x2, 1]) ~ B1,2 ~ P1,2 ≡ (𝔼{1}→{1}[β1 b1, η1 y1, 1] aS̄2) ~ B1,2 ~ P1,2}]
Out[•]= {0.15625, {True, True}}
```

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

```
In[•]:= Timing@{{
          "[a,y]" →
           ((𝔼{}→{1,2}[0, 0, y2 a1] ~ B1,2 ~ dm1,2→1)[3] - (𝔼{}→{1,2}[0, 0, y1 a2] ~ B1,2 ~ dm1,2→1)[3]),
          "[b,x]"  → ((𝔼{}→{1,2}[0, 0, x2 b1] ~ B1,2 ~ dm1,2→1)[3] -
             (𝔼{}→{1,2}[0, 0, x1 b2] ~ B1,2 ~ dm1,2→1)[3]),
          "xy-qyx" → ((𝔼{}→{1,2}[0, 0, x1 y2] ~ B1,2 ~ dm1,2→1)[3] -
             (1 + ϵ) (𝔼{}→{1,2}[0, 0, y1 x2] ~ B1,2 ~ dm1,2→1)[3])
         } /. {z_1 → z} // Expand // Factor,
      {
        "Δ(a)" → ((𝔼{}→{1}[0, 0, a1] ~ B1 ~ dΔ1→1,2)[3]),
        "Δ(x)" → ((𝔼{}→{1}[0, 0, x1] ~ B1 ~ dΔ1→1,2)[3]),
        "Δ(b)" → ((𝔼{}→{1}[0, 0, b1] ~ B1 ~ dΔ1→1,2)[3]),
        "Δ(y)" → ((𝔼{}→{1}[0, 0, y1] ~ B1 ~ dΔ1→1,2)[3])
       } // Simplify,
      {
         "S(a)" → ((𝔼{}→{1}[0, 0, a1] ~ B1 ~ dS1)[3]),
         "S(x)" → ((𝔼{}→{1}[0, 0, x1] ~ B1 ~ dS1)[3]),
         "S(b)" → ((𝔼{}→{1}[0, 0, b1] ~ B1 ~ dS1)[3]),
         "S(y)" → ((𝔼{}→{1}[0, 0, y1] ~ B1 ~ dS1)[3])
       } /. {z_1 → z} // Simplify
     }
```

Out[•]= $\{2.48438, \{\{[a,y] \to -y\,\gamma + O[\epsilon]^3,\ [b,x] \to x\,\epsilon + O[\epsilon]^3,$

$\text{xy-qyx} \to \frac{1-B}{\hbar} + (a\,B - x\,y + x\,y\,\gamma\,\hbar)\,\epsilon + \left(-\frac{1}{2} a^2\,B\,\hbar + \frac{1}{2} x\,y\,\gamma^2\,\hbar^2\right)\epsilon^2 + O[\epsilon]^3\},$

$\{\Delta(a) \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta(x) \to (x_1 + x_2) - \hbar\,a_1\,x_2\,\epsilon + \frac{1}{2}\hbar^2\,a_1^2\,x_2\,\epsilon^2 + O[\epsilon]^3,$

$\Delta(b) \to (b_1 + b_2) + O[\epsilon]^3,\ \Delta(y) \to (y_1 + B_1\,y_2) + O[\epsilon]^3\},$

$\{S(a) \to -a + O[\epsilon]^3,\ S(x) \to -x - a\,x\,\hbar\,\epsilon - \frac{1}{2}(a^2\,x\,\hbar^2)\,\epsilon^2 + O[\epsilon]^3,$

$S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + \frac{y\,\gamma\,\hbar\,\epsilon}{B} - \frac{(y\,\gamma^2\,\hbar^2)\,\epsilon^2}{2\,B} + O[\epsilon]^3\}\}\}$

(co)-associativity

In[∘]:= $\text{Timing}[\text{HL} /@ \{(d\Delta_{1\to1,2} \mathbin{//} d\Delta_{2\to2,3}) \equiv (d\Delta_{1\to1,3} \mathbin{//} d\Delta_{1\to1,2}),\ (dm_{1,2\to1} \mathbin{//} dm_{1,3\to1}) \equiv (dm_{2,3\to2} \mathbin{//} dm_{1,2\text{->}1})\}]$

Out[∘]= {1.29688, {True, True}}

Δ is an algebra morphism

In[∘]:= $\text{Timing}@\text{HL}[dm_{1,2\to1} \sim B_1 \sim d\Delta_{1\to1,2} \equiv (d\Delta_{1\to1,3}\ d\Delta_{2\to2,4}) \sim B_{1,2,3,4} \sim (dm_{3,4\to2}\ dm_{1,2\to1})]$

Out[∘]= {1.35938, True}

S_2 inverts R, but not S_1:

In[∘]:= $\text{Timing}@\{R_{1,2} \sim B_1 \sim dS_1 \equiv \overline{R}_{1,2},\ \text{HL}[R_{1,2} \sim B_2 \sim dS_2 \equiv \overline{R}_{1,2}]\}$

Out[∘]= $\{0.265625, \{\frac{1}{4 B_1^3} (4 \gamma \epsilon \hbar^2 B_1^2 x_2 y_1 - 2 \gamma^2 \epsilon^2 \hbar^3 B_1^2 x_2 y_1 + 4 \gamma \epsilon^2 \hbar^3 a_2 B_1^2 x_2 y_1 + 8 \gamma^2 \epsilon^2 \hbar^4 B_1 x_2^2 y_1^2 - 4 \gamma \epsilon^2 \hbar^4 a_2 B_1 x_2^2 y_1^2 - 3 \gamma^2 \epsilon^2 \hbar^5 x_2^3 y_1^3) == 0, \text{True}\}\}$

S is convolution inverse of id

In[∘]:= $\text{Timing}[\text{HL}[\# \equiv \mathbb{E}_{\{1\}\to\{1\}}[0, 0, 1]]\ \&\ /@ \{(d\Delta_{1\to1,2} \sim B_1 \sim dS_1) \sim B_{1,2} \sim dm_{1,2\to1},\ (d\Delta_{1\to1,2} \sim B_2 \sim dS_2) \mathbin{//} dm_{1,2\to1}\}]$

Out[∘]= {2.23438, {True, True}}

S is a (co)-algebra anti-morphism

In[∘]:= $\text{Timing}[\text{HL} /@ \text{Expand} /@ \{dm_{1,2\to1} \sim B_1 \sim dS_1 \equiv (dS_1\ dS_2) \sim B_{1,2} \sim dm_{2,1\to1},\ dS_1 \sim B_1 \sim d\Delta_{1\to1,2} \equiv d\Delta_{1\to2,1} \sim B_{1,2} \sim (dS_1\ dS_2)\}]$

Out[∘]= {4.90625, {True, True}}

Quasi-triangular axiom 1:

In[∘]:= $\text{Timing}@\text{HL}[R_{1,2} \sim B_1 \sim d\Delta_{1\text{->}1,3} \equiv (R_{1,4}\ R_{3,2}) \sim B_{2,4} \sim dm_{2,4\to2}]$

Out[∘]= {0.125, True}

Quasi-triangular axiom 2:

In[∘]:= $\text{Timing}@\text{HL}[((d\Delta_{1\text{->}1,2}\ R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3\to1}\ dm_{2,4\to2})) \equiv ((d\Delta_{1\to2,1}\ R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{3,1\to1}\ dm_{4,2\to2}))]$

Out[∘]= {1.01563, True}

The Drinfel'd element inverse property, $(u_1\, \overline{u}_2) \sim B_{1,2} \sim dm_{1,2\to1} \equiv \mathbb{E}[0, 0, 1]$:

In[∘]:= $\text{Timing}@\text{HL}[((R_{1,2} \sim B_1 \sim dS_1 \sim B_{1,2} \sim dm_{2,1\to i})\ (R_{1,2} \sim B_2 \sim dS_2 \sim B_2 \sim dS_2 \sim B_{1,2} \sim dm_{2,1\to j})) \sim B_{i,j} \sim dm_{i,j\to i} \equiv \mathbb{E}_{\{\}\to\{i\}}[0, 0, 1]]$

Out[∘]= {0.90625, True}

The ribbon element v satisfies $v^2 = S(u)\, u$. The spinner C=uv^{-1}. It is convenient to compute $z = S(u)\, u^{-1}$ which is something easy.

In[∘]:= **Timing@Block**[{$k = 2},
((($R_{1,2}$ ~ B_1 ~ dS_1 ~ $B_{1,2}$ ~ $dm_{2,1\to i}$) ~ B_i ~ dS_i) ($R_{1,2}$ ~ B_2 ~ dS_2 ~ B_2 ~ dS_2 ~ $B_{1,2}$ ~ $dm_{2,1\to j}$)) ~ $B_{i,j}$ ~ $dm_{i,j\to i}$]

Out[∘]= $\left\{1.25, \mathbb{E}_{\{\}\to\{i\}}\left[0, 0, \frac{1}{B_i} + \frac{\hbar a_i \epsilon}{B_i} + \frac{\hbar^2 a_i^2 \epsilon^2}{2 B_i} + O[\epsilon]^3\right]\right\}$

In[∘]:= **Timing@Block**[{$k = 2}, HL /@ {($C_i \bar{C}_j$) ~ $B_{i,j}$ ~ $dm_{i,j\to i}$ ≡ $\mathbb{E}_{\{\}\to\{i\}}$[0, 0, 1], ($\bar{C}_i \bar{C}_j$) ~ $B_{i,j}$ ~ $dm_{i,j\to i}$ ≡
((($R_{1,2}$ ~ B_1 ~ dS_1 ~ $B_{1,2}$ ~ $dm_{2,1\to i}$) ~ B_i ~ dS_i) ($R_{1,2}$ ~ B_2 ~ dS_2 ~ B_2 ~ dS_2 ~ $B_{1,2}$ ~ $dm_{2,1\to j}$)) ~ $B_{i,j}$ ~ $dm_{i,j\to i}$}]

Out[∘]= {1.45313, {True, True}}

Reidemeister 2:

In[∘]:= **Timing**[HL[# ≡ $\mathbb{E}_{\{\}\to\{1,2\}}$[0, 0, 1]] & /@
{($\bar{R}_{1,2} R_{3,4}$) ~ $B_{1,2,3,4}$ ~ ($dm_{1,3\to 1}$ $dm_{2,4\to 2}$), ($R_{1,2} \bar{R}_{3,4}$) ~ $B_{1,2,3,4}$ ~ ($dm_{1,3\to 1}$ $dm_{2,4\to 2}$)}]

Out[∘]= {0.859375, {True, True}}

Cyclic Reidemeister 2:

In[∘]:= **Timing@HL**[($R_{1,4} \bar{R}_{5,2} \bar{C}_3$) ~ $B_{2,4}$ ~ $dm_{2,4\to 2}$ ~ $B_{1,3}$ ~ $dm_{1,3\to 1}$ ~ $B_{1,5}$ ~ $dm_{1,5\to 1}$ ≡ $\bar{C}_1$ $\mathbb{E}_{\{\}\to\{2\}}$[0, 0, 1]]

Out[∘]= {0.609375, True}

Reidemeister 3:

In[∘]:= **Timing@HL**[(($R_{1,2} R_{4,3} R_{5,6}$) ~ $B_{1,4}$ ~ $dm_{1,4\to 1}$ ~ $B_{2,5}$ ~ $dm_{2,5\to 2}$ ~ $B_{3,6}$ ~ $dm_{3,6\to 3}$) ≡
(($R_{1,6} R_{2,3} R_{4,5}$) ~ $B_{1,4}$ ~ $dm_{1,4\to 1}$ ~ $B_{2,5}$ ~ $dm_{2,5\to 2}$ ~ $B_{3,6}$ ~ $dm_{3,6\to 3}$)]

Out[∘]= {0.921875, True}

Relations between the four kinks:

In[∘]:= **Timing**[HL /@ {$Kink_i$ ≡ ($R_{3,1} C_2$) ~ $B_{1,2}$ ~ $dm_{1,2\to 1}$ ~ $B_{1,3}$ ~ $dm_{1,3\to i}$,
$\overline{Kink}_j$ ≡ ($\bar{R}_{3,1} \bar{C}_2$) ~ $B_{1,2}$ ~ $dm_{1,2\to 1}$ ~ $B_{1,3}$ ~ $dm_{1,3\to j}$, ($Kink_i \overline{Kink}_j$) ~ $B_{i,j}$ ~ $dm_{i,j\to 1}$ ≡ $\mathbb{E}_{\{\}\to\{1\}}$[0, 0, 1]}]

Out[∘]= {1.70313, {True, True, True}}

The Trefoil

In[∘]:= **Timing@Block**[{$k = 1},
Z31 = $R_{1,5} R_{6,2} R_{3,7} \bar{C}_4 \overline{Kink}_8 \overline{Kink}_9 \overline{Kink}_{10}$;
Do[Z31 = Z31 ~ $B_{1,r}$ ~ $dm_{1,r\to 1}$, {r, 2, 10}];
{**Simplify** /@ Z31, **Simplify** /@ (Z31 ~ B_1 ~ $b2t_1$ /. T_1 → T)}]

Out[∘]= $\left\{1.4375, \left\{\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, \frac{B_1}{1 - B_1 + B_1^2} - \frac{\hbar B_1 \left(-a_1 \left(-1 + B_1 - B_1^3 + B_1^4\right) + \gamma \left(B_1 - 2 B_1^2 - 2 B_1^4 + 2 \hbar x_1 y_1 + B_1^3 \left(3 + 2 \hbar x_1 y_1\right)\right)\right) \epsilon}{\left(1 - B_1 + B_1^2\right)^3} + O[\epsilon]^2\right], \mathbb{E}_{\{\}\to\{1\}}\left[0, 0, \frac{T}{1 - T + T^2} + \frac{T \hbar \left(T \left(-1 + 2 T - 3 T^2 + 2 T^3\right) \gamma + 2 \left(-1 + T - T^3 + T^4\right) a_1 - 2 \left(1 + T^3\right) \gamma \hbar x_1 y_1\right) \epsilon}{\left(1 - T + T^2\right)^3} + O[\epsilon]^2\right]\right\}\right\}$

In[ ]:=
```
Timing@Block[{$k = 1},
  Z31 = kR1,5 kR6,2 kR3,7 kC4 kKink8 kKink9 kKink10;
  Do[Z31 = Z31 ~ B1,r ~ km1,r→1, {r, 2, 10}];
  Simplify /@ Z31]
```

Out[ ]= $\{0.9375, \mathbb{E}_{\{\}\to\{1\}}[0, 0, \frac{T}{1-T+T^2} + \frac{T\hbar\left(T\left(-1+2T-3T^2+2T^3\right)\gamma+2\left(-1+T-T^3+T^4\right)a_1-2\left(1+T^3\right)\gamma\hbar x_1 y_1\right)\epsilon}{\left(1-T+T^2\right)^3} + O[\epsilon]^2]\}$

Knot

In[ ]:=
```
$k = 1; Timing@Z@Knot[10, 100]
```

Knot

Get: ParentDirectory[File] in $Path is not a string.

Knot

KnotTheory: Loading precomputed data in PD4Knots`.

Knot

Out[ ]= $\{37.6563, \mathbb{E}_{\{\}\to\{0\}}[0, 0, \frac{T^4}{1-4T+9T^2-12T^3+13T^4-12T^5+9T^6-4T^7+T^8} + \left(\left(a\left(-8T^4\hbar+24T^5\hbar-36T^6\hbar+24T^7\hbar-24T^9\hbar+36T^{10}\hbar-24T^{11}\hbar+8T^{12}\hbar\right)\right)/\left(1-8T+34T^2-96T^3+203T^4-344T^5+492T^6-608T^7+653T^8-608T^9+492T^{10}-344T^{11}+203T^{12}-96T^{13}+34T^{14}-8T^{15}+T^{16}\right)+\left(-6T^4\gamma\hbar+44T^5\gamma\hbar-167T^6\gamma\hbar+410T^7\gamma\hbar-733T^8\gamma\hbar+1016T^9\gamma\hbar-1140T^{10}\gamma\hbar+1048T^{11}\gamma\hbar-776T^{12}\gamma\hbar+440T^{13}\gamma\hbar-156T^{14}\gamma\hbar-16T^{15}\gamma\hbar+79T^{16}\gamma\hbar-70T^{17}\gamma\hbar+37T^{18}\gamma\hbar-12T^{19}\gamma\hbar+2T^{20}\gamma\hbar\right)/\left(1-12T+75T^2-316T^3+1002T^4-2544T^5+5394T^6-9840T^7+15771T^8-22512T^9+28866T^{10}-33432T^{11}+35095T^{12}-33432T^{13}+28866T^{14}-22512T^{15}+15771T^{16}-9840T^{17}+5394T^{18}-2544T^{19}+1002T^{20}-316T^{21}+75T^{22}-12T^{23}+T^{24}\right)+\left(x\,y\left(-8T^4\gamma\hbar^2+16T^5\gamma\hbar^2-20T^6\gamma\hbar^2+4T^7\gamma\hbar^2+4T^8\gamma\hbar^2-20T^9\gamma\hbar^2+16T^{10}\gamma\hbar^2-8T^{11}\gamma\hbar^2\right)\right)/\left(1-8T+34T^2-96T^3+203T^4-344T^5+492T^6-608T^7+653T^8-608T^9+492T^{10}-344T^{11}+203T^{12}-96T^{13}+34T^{14}-8T^{15}+T^{16}\right)\right)\epsilon + O[\epsilon]^2]\}$

In[ ]:=
```
EndProfile[];
```

Profile

In[ ]:=
```
PrintProfile[]
```

Profile

Out[ ]=
```
ProfileRoot is root. Profiled time: 69.58
  (     1)   0.078/ 37.641 above Z
  (   157)   0.533/ 24.254 above B
  (    37)   0.265/  7.561 above Boot
  (   147)   0.093/  0.124 above CF
  (     1)       0/      0 above RVK
CF: called 12484 times, time in 23.078/55.581
  (  1047)   0.390/  0.970 under EEQ
  (     4)       0/  0.016 under Z
  (   122)   0.046/  0.093 under Boot
  (  1365)   4.967/ 15.337 under LZip
  (   147)   0.093/  0.124 under ProfileRoot
  (  9799)  17.582/ 39.041 under QZip
  ( 36494)   9.010/ 32.503 above CCF
Together: called 36494 times, time in 18.622/23.493
  ( 36494)  18.622/ 23.493 under CCF
```

```
  ( 36494)   4.871/  4.871 above Exp
CCF: called 36494 times, time in 9.01/32.503
  ( 36494)   9.010/ 32.503 under CF
  ( 36494)  18.622/ 23.493 above Together
Zip: called 2730 times, time in 7.529/33.188
  (   298)   0.802/  5.146 under LZip
  (   298)   0.783/  3.721 under QZip
  (  2134)   5.944/ 24.321 under Zip
  (  2730)   1.338/  1.338 above Collect
  (  2134)   5.944/ 24.321 above Zip
Exp: called 36494 times, time in 4.871/4.871
  ( 36494)   4.871/  4.871 under Together
LZip: called 298 times, time in 1.612/23.567
  (   298)   1.612/ 23.567 under B
  (  1047)   0.502/  1.472 above EEQ
  (  1365)   4.967/ 15.337 above CF
  (   298)   0.802/  5.146 above Zip
Collect: called 2730 times, time in 1.338/1.338
  (  2730)   1.338/  1.338 under Zip
Boot: called 64 times, time in 1.122/12.501
  (     3)      0/  0.062 under Z
  (    24)   0.857/  4.878 under Boot
  (    37)   0.265/  7.561 under ProfileRoot
  (    69)   0.108/  6.408 above B
  (    24)   0.857/  4.878 above Boot
  (   122)   0.046/  0.093 above CF
QZip: called 298 times, time in 1.021/43.783
  (   298)   1.021/ 43.783 under B
  (  9799)  17.582/ 39.041 above CF
  (   298)   0.783/  3.721 above Zip
B: called 298 times, time in 0.797/68.147
  (    72)   0.156/ 37.485 under Z
  (    69)   0.108/  6.408 under Boot
  (   157)   0.533/ 24.254 under ProfileRoot
  (   298)   1.612/ 23.567 above LZip
  (   298)   1.021/ 43.783 above QZip
EEQ: called 1047 times, time in 0.502/1.472
  (  1047)   0.502/  1.472 under LZip
  (  1047)   0.390/  0.970 above CF
Z: called 1 times, time in 0.078/37.641
  (     1)   0.078/ 37.641 under ProfileRoot
  (    72)   0.156/ 37.485 above B
  (     3)      0/  0.062 above Boot
  (     4)      0/  0.016 above CF
RVK: called 1 times, time in 0./0.
  (     1)      0/     0 under ProfileRoot
```