Pensieve header: Exp relative to am, bm, cm, dm.

Follows code in Projects/SL2Portfolio/SL2PortfolioProgram.nb.

Startup

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio2"];
(*Once[<< KnotTheory`];*)
Once[<< "../Profile/Profile.m";];
<< "Engine-Speedy.m";
<< "Objects.m";
$k = 3;
HL[ℰ_] := Style[ℰ, Background → Green];
```

```
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
This version: June 2018. Original version: July 1994.
```

Exponentials as needed.

Task. Define $\text{Exp}_{m,i,k}[P]$ to compute $e^{\mathbb{O}(P)}$ to ϵ^k in the using the $m_{i,i\to i}$ multiplication, where P is an ϵ-dependent near-docile element, giving the answer in $\mathbb{E}$-form.

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\lambda\,\mathbb{O}(P)} = \mathbb{O}\left(e^{\lambda P_0} F(\lambda)\right)$, then $F(\lambda = 0) = 1$ and we have:

$$\mathbb{O}\left(e^{\lambda P_0}(P_0\,F(\lambda) + \partial_\lambda F)\right) = \mathbb{O}\left(\partial_\lambda e^{\lambda P_0} F(\lambda)\right) = \partial_\lambda\,\mathbb{O}\left(e^{\lambda P_0} F(\lambda)\right) = \partial_\lambda e^{\lambda\,\mathbb{O}(P)} = e^{\lambda\,\mathbb{O}(P)}\,\mathbb{O}(P) = \mathbb{O}\left(e^{\lambda P_0} F(\lambda)\right)\mathbb{O}(P).$$

This is a linear ODE for F. Setting inductively $F_k = F_{k-1} + \epsilon^k \varphi$ we find that $F_0 = 1$ and solve for φ.

In[∘]:=
```
(* Bug: The first line is valid only if 𝕆(e^P0)==e^𝕆(P0). *)
Exp_{m_,i_,0}[P_] := Module[{LQ = Normal@P /. ϵ → 0},
   𝔼[LQ /. (x | y)_i → 0, LQ /. (b | a | t)_i → 0, 1] ];
Exp_{m_,i_,k_}[P_] := Block[{$k = k},
  Module[{P0, λ, φ, φs, F, j, rhs, eqn, pows, at0, atλ},
   P0 = Normal@P /. ϵ → 0;
   F = Normal@Last@Exp_{m,i,k-1}[λ P];
   While[
    rhs = Normal@Last@m_{i,j→i}[
        𝔼_{{}→{i}}[λ P0 /. (x | y)_i → 0, λ P0 /. (b | a | t)_i → 0, F]_k sσ_{i→j}@𝔼_{{}→{i}}[0, 0, P]_k];
    eqn = CF[(∂_λ F) + P0 F - rhs];
    eqn =!= 0,
    (*do*) pows = Echo[First /@ CoefficientRules[eqn, {y_i, b_i, a_i, x_i}]];
    F += Sum[ϵ^k φ_js[λ] Times @@ {y_i, b_i, a_i, x_i}^js, {js, pows}];
    rhs = Normal@Last@m_{i,j→i}[
        𝔼_{{}→{i}}[λ P0 /. (x | y)_i → 0, λ P0 /. (b | a | t)_i → 0, F]_k sσ_{i→j}@𝔼_{{}→{i}}[0, 0, P]_k];
    eqn = CF[(∂_λ F) + P0 F - rhs];
    φs = Table[φ_js[λ], {js, pows}];
    at0 = Table[φ_js[0] == 0, {js, pows}];
    atλ = (# == 0) & /@ (pows /. CoefficientRules[eqn, {y_i, b_i, a_i, x_i}]);
    F = F /. DSolve[And @@ (at0 ⋃ atλ), φs, λ] [[1]]
   ];
   𝔼_{{}→{i}}[P0 /. (x | y)_i → 0, P0 /. (b | a | t)_i → 0, F + O[ϵ]^{k+1} /. λ → 1]]
 ]
```

In[∘]:= `Exp_{dm,1,2}[ξ (x_1 + ϵ y_1)]`

```
{{1, 0, 0, 0}, {0, 0, 0, 0}}
{{2, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}, {0, 0, 0, 0}}
```

Out[∘]=
$$\mathbb{E}_{\{\}\to\{1\}}\left[0,\ \xi\, x_1,\ 1+\left(-\frac{\xi^2\left(-1+B_1\right)}{2\hbar}+\xi\, y_1\right)\epsilon+\left(\frac{\xi^4\left(-1+B_1\right)^2}{8\hbar^2}+\frac{1}{2}\xi^2 a_1 B_1-\frac{1}{6}\gamma\,\xi^3\left(-1+3B_1\right)x_1-\frac{\xi^3\left(-1+B_1\right)y_1}{2\hbar}+\frac{1}{2}\gamma\,\xi^2\hbar\, x_1 y_1+\frac{1}{2}\xi^2 y_1^2\right)\epsilon^2+O[\epsilon]^3\right]$$

In[∘]:= `dS_1[𝔼_{{}→{1}}[0, 0, #]] & /@ {y_1, x_1}`

Out[∘]=
$$\left\{\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -\frac{y_1}{B_1}+\frac{\gamma\hbar\, y_1\,\epsilon}{B_1}-\frac{\left(\gamma^2\hbar^2 y_1\right)\epsilon^2}{2B_1}+\frac{\gamma^3\hbar^3 y_1\,\epsilon^3}{6B_1}+O[\epsilon]^4\right],\ \mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -x_1-\hbar\, a_1 x_1\,\epsilon-\frac{1}{2}\left(\hbar^2 a_1^2 x_1\right)\epsilon^2-\frac{1}{6}\left(\hbar^3 a_1^3 x_1\right)\epsilon^3+O[\epsilon]^4\right]\right\}$$

In[∘]:=
```
Timing@{lhs = 𝔼_{{1}→{1}}[0, ξ_1 x_1, 1] // dS_1,
  rhs = Exp_{dm,1,$k}[ξ_1 Last@dS_1[𝔼_{{}→{1}}[0, 0, x_1]]] /. {{} → {1}}; HL[lhs ≡ rhs]}
```

```
{{0, 0, 1, 1}, {0, 0, 0, 2}}
{{0, 0, 2, 2}, {0, 0, 2, 1}, {0, 0, 1, 3}, {0, 0, 1, 2}, {0, 0, 0, 4}, {0, 0, 0, 3}, {0, 0, 0, 2}}
{{0, 0, 3, 3}, {0, 0, 3, 2}, {0, 0, 3, 1}, {0, 0, 2, 4}, {0, 0, 2, 3},
 {0, 0, 2, 2}, {0, 0, 1, 5}, {0, 0, 1, 4}, {0, 0, 1, 3}, {0, 0, 1, 2},
 {0, 0, 0, 6}, {0, 0, 0, 5}, {0, 0, 0, 4}, {0, 0, 0, 3}, {0, 0, 0, 2}}
```

Out[∘]=
$$\Big\{3.53125, \Big\{\mathbb{E}_{\{1\}\to\{1\}}\Big[0, -x_1\,\xi_1, 1+\Big(-\hbar\,a_1\,x_1\,\xi_1-\frac{1}{2}\gamma\,\hbar\,x_1^2\,\xi_1^2\Big)\epsilon+\Big(-\frac{1}{2}\hbar^2\,a_1^2\,x_1\,\xi_1+\frac{1}{4}\gamma^2\,\hbar^2\,x_1^2\,\xi_1^2-\gamma\,\hbar^2\,a_1\,x_1^2\,\xi_1^2+\frac{1}{2}\hbar^2\,a_1^2\,x_1^2\,\xi_1^2-\frac{1}{2}\gamma^2\,\hbar^2\,x_1^3\,\xi_1^3+\frac{1}{2}\gamma\,\hbar^2\,a_1\,x_1^3\,\xi_1^3+\frac{1}{8}\gamma^2\,\hbar^2\,x_1^4\,\xi_1^4\Big)\epsilon^2+\Big(-\frac{1}{6}\hbar^3\,a_1^3\,x_1\,\xi_1-\frac{1}{12}\gamma^3\,\hbar^3\,x_1^2\,\xi_1^2+\frac{1}{2}\gamma^2\,\hbar^3\,a_1\,x_1^2\,\xi_1^2-\gamma\,\hbar^3\,a_1^2\,x_1^2\,\xi_1^2+\frac{1}{2}\hbar^3\,a_1^3\,x_1^2\,\xi_1^2+\frac{2}{3}\gamma^3\,\hbar^3\,x_1^3\,\xi_1^3-\frac{7}{4}\gamma^2\,\hbar^3\,a_1\,x_1^3\,\xi_1^3+\frac{5}{4}\gamma\,\hbar^3\,a_1^2\,x_1^3\,\xi_1^3-\frac{1}{6}\hbar^3\,a_1^3\,x_1^3\,\xi_1^3-\frac{19}{24}\gamma^3\,\hbar^3\,x_1^4\,\xi_1^4+\gamma^2\,\hbar^3\,a_1\,x_1^4\,\xi_1^4-\frac{1}{4}\gamma\,\hbar^3\,a_1^2\,x_1^4\,\xi_1^4+\frac{1}{4}\gamma^3\,\hbar^3\,x_1^5\,\xi_1^5-\frac{1}{8}\gamma^2\,\hbar^3\,a_1\,x_1^5\,\xi_1^5-\frac{1}{48}\gamma^3\,\hbar^3\,x_1^6\,\xi_1^6\Big)\epsilon^3+O[\epsilon]^4\Big], \text{True}\Big\}\Big\}$$

In[∘]:=
```
Timing@{lhs = 𝔼_{{1}→{1}}[0, η₁ y₁, 1] // dS₁,
  rhs = Exp_{dm,1,$k}[η Last@dS₁[𝔼_{{}→{1}}[0, 0, y₁]]] /. {η → η₁, {} → {1}}, HL[lhs ≡ rhs]}
```

```
{{2, 0, 0, 0}, {1, 0, 0, 0}}
{{4, 0, 0, 0}, {3, 0, 0, 0}, {2, 0, 0, 0}, {1, 0, 0, 0}}
{{6, 0, 0, 0}, {5, 0, 0, 0}, {4, 0, 0, 0}, {3, 0, 0, 0}, {2, 0, 0, 0}, {1, 0, 0, 0}}
```

Out[∘]=
$$\Big\{3.29688, \Big\{\mathbb{E}_{\{1\}\to\{1\}}\Big[0, -\frac{y_1\,\eta_1}{B_1}, 1+\Big(\frac{\gamma\,\hbar\,y_1\,\eta_1}{B_1}-\frac{\gamma\,\hbar\,y_1^2\,\eta_1^2}{2\,B_1^2}\Big)\epsilon+\Big(-\frac{\gamma^2\,\hbar^2\,y_1\,\eta_1}{2\,B_1}+\frac{7\,\gamma^2\,\hbar^2\,y_1^2\,\eta_1^2}{4\,B_1^2}-\frac{\gamma^2\,\hbar^2\,y_1^3\,\eta_1^3}{B_1^3}+\frac{\gamma^2\,\hbar^2\,y_1^4\,\eta_1^4}{8\,B_1^4}\Big)\epsilon^2+\Big(\frac{\gamma^3\,\hbar^3\,y_1\,\eta_1}{6\,B_1}-\frac{25\,\gamma^3\,\hbar^3\,y_1^2\,\eta_1^2}{12\,B_1^2}+\frac{23\,\gamma^3\,\hbar^3\,y_1^3\,\eta_1^3}{6\,B_1^3}-\frac{49\,\gamma^3\,\hbar^3\,y_1^4\,\eta_1^4}{24\,B_1^4}+\frac{3\,\gamma^3\,\hbar^3\,y_1^5\,\eta_1^5}{8\,B_1^5}-\frac{\gamma^3\,\hbar^3\,y_1^6\,\eta_1^6}{48\,B_1^6}\Big)\epsilon^3+O[\epsilon]^4\Big], \mathbb{E}_{\{1\}\to\{1\}}\Big[0, -\frac{y_1\,\eta_1}{B_1}, 1-\frac{\big(\gamma\,\hbar\,\big(-2\,B_1\,y_1\,\eta_1+y_1^2\,\eta_1^2\big)\big)\,\epsilon}{2\,B_1^2}+\frac{1}{8\,B_1^4}\gamma^2\,\hbar^2\,\big(-4\,B_1^3\,y_1\,\eta_1+14\,B_1^2\,y_1^2\,\eta_1^2-8\,B_1\,y_1^3\,\eta_1^3+y_1^4\,\eta_1^4\big)\,\epsilon^2-\frac{1}{48\,B_1^6}\big(\gamma^3\,\hbar^3\,\big(-8\,B_1^5\,y_1\,\eta_1+100\,B_1^4\,y_1^2\,\eta_1^2-184\,B_1^3\,y_1^3\,\eta_1^3+98\,B_1^2\,y_1^4\,\eta_1^4-18\,B_1\,y_1^5\,\eta_1^5+y_1^6\,\eta_1^6\big)\big)\,\epsilon^3+O[\epsilon]^4\Big], \text{True}\Big\}\Big\}$$

In[∘]:=
```
Timing@{lhs = 𝔼_{{1}→{1}}[0, ξ₁ x₁, 1] // cS₁,
  rhs = Exp_{cm,1,$k}[ξ₁ Last@cS₁[𝔼_{{}→{1}}[0, 0, x₁]]] /.  {} → {1}; HL[lhs ≡ rhs]}
```

Out[∘]= $\Big\{3.67188, \Big\{\mathbb{E}_{\{1\}\to\{1\}}\Big[0, -x_1\,\xi_1, 1+O[\epsilon]^4\Big], \text{True}\Big\}\Big\}$

In[∘]:=
```
Timing@{lhs = 𝔼_{{1}→{1}}[0, η₁ y₁, 1] // cS₁,
  rhs = Exp_{cm,1,$k}[η₁ Last@cS₁[𝔼_{{}→{1}}[0, 0, y₁]]] /. {} → {1}; HL[lhs ≡ rhs]}
```

Out[∘]= $\Big\{0.296875, \Big\{\mathbb{E}_{\{1\}\to\{1\}}\Big[0, -y_1\,\eta_1, 1+O[\epsilon]^4\Big], \text{True}\Big\}\Big\}$