Pensieve header: Exp relative to am, bm, cm, dm.

Follows code in Projects/SL2Portfolio/SL2PortfolioProgram.nb.

Startup

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio2"];
(*Once[<< KnotTheory`];*)
Once[<< "../Profile/Profile.m";];
<< "Engine-Speedy.m";
<< "Objects.m";
$k = 3;
HL[ℰ_] := Style[ℰ, Background → Green];
```

```
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
This version: June 2018. Original version: July 1994.
```

Exponentials as needed.

Task. Define $\text{Exp}_{m,i,k}[\lambda, P]$ which computes $e^{\lambda\,\mathbb{O}(P)}$ to ϵ^k in the using the $m_{i,i\to i}$ multiplication, where λ is a scalar and P is an ϵ-dependent near-docile element, giving the answer in $\mathbb{E}$-form.
Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\lambda\,\mathbb{O}(P)} = \mathbb{O}\left(e^{\lambda P_0} F(\lambda)\right)$, then $F(\lambda = 0) = 1$ and we have:
$\mathbb{O}\left(e^{\lambda P_0}(P_0 F(\lambda) + \partial_\lambda F)\right) = \mathbb{O}\left(\partial_\lambda e^{\lambda P_0} F(\lambda)\right) = \partial_\lambda \mathbb{O}\left(e^{\lambda P_0} F(\lambda)\right) = \partial_\lambda e^{\lambda\,\mathbb{O}(P)} = e^{\lambda\,\mathbb{O}(P)}\,\mathbb{O}(P) = \mathbb{O}\left(e^{\lambda P_0} F(\lambda)\right)\mathbb{O}(P)$.
This is a linear ODE for F. Setting inductively $F_k = F_{k-1} + \epsilon^k \varphi$ we find that $F_0 = 1$ and solve for φ.

In[∘]:=
```
(* Bug: The first line is valid only if 𝕆(𝕖^P0)==𝕖^𝕆(P0). *)
(* Bug: λ must be a symbol. *)
Exp_{m_,i_,0}[λ_, P_] := Module[{LQ = Normal@P /. ϵ → 0},
   𝔼[λ LQ /. (x | y)_i → 0, λ LQ /. (b | a | t)_i → 0, 1] ];
Exp_{m_,i_,k_}[λ_, P_] := Block[{$k = k},
  Module[{P0, φ, φs, F, j, rhs, at0, atλ},
   P0 = Normal@P /. ϵ → 0;
   F = Normal@Last@Exp_{m,i,k-1}[λ, P];
   While[
    rhs = Normal@Last@m_{i,j→i}[𝔼_{{}→{i}}[λ P0 /. (x | y)_i → 0, λ P0 /. (b | a | t)_i → 0, F + O[ϵ]^(k+1)]
          𝔼_{{}→{j}}[0, 0, P + O[ϵ]^(k+1) /. u_i :→ u_j]];
    eqn = CF[(∂_λ F) + P0 F - rhs];
    eqn =!= 0,
    (*do*) pows = First /@ CoefficientRules[eqn, {y_i, b_i, a_i, x_i}];
    F += Sum[ϵ^k φ_js[λ] Times @@ {y_i, b_i, a_i, x_i}^js, {js, pows}];
    rhs = Normal@Last@m_{i,j→i}[𝔼_{{}→{i}}[λ P0 /. (x | y)_i → 0,
          λ P0 /. (b | a | t)_i → 0, F + O[ϵ]^(k+1)] 𝔼_{{}→{j}}[0, 0, P + O[ϵ]^(k+1) /. u_i :→ u_j]];
    eqn = CF[(∂_λ F) + P0 F - rhs];
    φs = Table[φ_js[λ], {js, pows}];
    at0 = Table[φ_js[0] == 0, {js, pows}];
    atλ = (# == 0) & /@ (pows /. CoefficientRules[eqn, {y_i, b_i, a_i, x_i}]);
    F = F /. DSolve[And @@ (at0 ⋃ atλ), φs, λ] [[1]]
   ];
   𝔼_{{}→{i}}[λ P0 /. (x | y)_i → 0, λ P0 /. (b | a | t)_i → 0, F + O[ϵ]^(k+1)]
  ]
```

In[∘]:= `Exp_{dm,1,2}[ξ, x_1 + ϵ y_1]`

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ \xi x_1,\ 1 + \left(\frac{\xi^2 - \xi^2 B_1}{2\hbar} + \xi y_1\right)\epsilon + \left(\frac{\xi^4 (-1+B_1)^2}{8\hbar^2} + \frac{1}{2}\xi^2 a_1 B_1 - \frac{1}{6}\gamma\xi^3(-1+3B_1)x_1 - \frac{\xi^3(-1+B_1)y_1}{2\hbar} + \frac{1}{2}\gamma\xi^2\hbar x_1 y_1 + \frac{1}{2}\xi^2 y_1^2\right)\epsilon^2 + O[\epsilon]^3\Big]$

In[∘]:= `dS_1[𝔼_{{}→{1}}[0, 0, #]] & /@ {y_1, x_1}`

Out[∘]= $\Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ -\frac{y_1}{B_1} + \frac{\gamma\hbar y_1\epsilon}{B_1} - \frac{(\gamma^2\hbar^2 y_1)\epsilon^2}{2B_1} + \frac{\gamma^3\hbar^3 y_1\epsilon^3}{6B_1} + O[\epsilon]^4\Big],\ \mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ -x_1 - \hbar a_1 x_1\epsilon - \frac{1}{2}(\hbar^2 a_1^2 x_1)\epsilon^2 - \frac{1}{6}(\hbar^3 a_1^3 x_1)\epsilon^3 + O[\epsilon]^4\Big]\Big\}$

In[∘]:= **Timing@{lhs = $\mathbb{E}_{\{1\}\to\{1\}}$[0, ξ_1 x_1, 1] // dS_1,
rhs = $\text{Exp}_{dm,1,\$k}$[$\xi$, Last@$dS_1$[$\mathbb{E}_{\{\}\to\{1\}}$[0, 0, x_1]]] /. {$\xi \to \xi_1$, {} → {1}}; HL[lhs ≡ rhs]}**

Out[∘]= $\{4.375, \{\mathbb{E}_{\{1\}\to\{1\}}[0, -x_1\,\xi_1, 1 + \left(-\hbar\, a_1\, x_1\, \xi_1 - \frac{1}{2}\gamma\,\hbar\, x_1^2\,\xi_1^2\right)\epsilon + \left(-\frac{1}{2}\hbar^2\, a_1^2\, x_1\,\xi_1 + \frac{1}{4}\gamma^2\,\hbar^2\, x_1^2\,\xi_1^2 - \gamma\,\hbar^2\, a_1\, x_1^2\,\xi_1^2 + \frac{1}{2}\hbar^2\, a_1^2\, x_1^2\,\xi_1^2 - \frac{1}{2}\gamma^2\,\hbar^2\, x_1^3\,\xi_1^3 + \frac{1}{2}\gamma\,\hbar^2\, a_1\, x_1^3\,\xi_1^3 + \frac{1}{8}\gamma^2\,\hbar^2\, x_1^4\,\xi_1^4\right)\epsilon^2 + \left(-\frac{1}{6}\hbar^3\, a_1^3\, x_1\,\xi_1 - \frac{1}{12}\gamma^3\,\hbar^3\, x_1^2\,\xi_1^2 + \frac{1}{2}\gamma^2\,\hbar^3\, a_1\, x_1^2\,\xi_1^2 - \gamma\,\hbar^3\, a_1^2\, x_1^2\,\xi_1^2 + \frac{1}{2}\hbar^3\, a_1^3\, x_1^2\,\xi_1^2 + \frac{2}{3}\gamma^3\,\hbar^3\, x_1^3\,\xi_1^3 - \frac{7}{4}\gamma^2\,\hbar^3\, a_1\, x_1^3\,\xi_1^3 + \frac{5}{4}\gamma\,\hbar^3\, a_1^2\, x_1^3\,\xi_1^3 - \frac{1}{6}\hbar^3\, a_1^3\, x_1^3\,\xi_1^3 - \frac{19}{24}\gamma^3\,\hbar^3\, x_1^4\,\xi_1^4 + \gamma^2\,\hbar^3\, a_1\, x_1^4\,\xi_1^4 - \frac{1}{4}\gamma\,\hbar^3\, a_1^2\, x_1^4\,\xi_1^4 + \frac{1}{4}\gamma^3\,\hbar^3\, x_1^5\,\xi_1^5 - \frac{1}{8}\gamma^2\,\hbar^3\, a_1\, x_1^5\,\xi_1^5 - \frac{1}{48}\gamma^3\,\hbar^3\, x_1^6\,\xi_1^6\right)\epsilon^3 + O[\epsilon]^4], \text{True}\}\}$

In[∘]:= **Timing@{lhs = $\mathbb{E}_{\{1\}\to\{1\}}$[0, η_1 y_1, 1] // dS_1,
rhs = $\text{Exp}_{dm,1,\$k}$[$\eta$, Last@$dS_1$[$\mathbb{E}_{\{\}\to\{1\}}$[0, 0, y_1]]] /. {$\eta \to \eta_1$, {} → {1}}; HL[lhs ≡ rhs]}**

Out[∘]= $\{4.21875, \{\mathbb{E}_{\{1\}\to\{1\}}[0, -\frac{y_1\,\eta_1}{B_1}, 1 + \left(\frac{\gamma\,\hbar\, y_1\,\eta_1}{B_1} - \frac{\gamma\,\hbar\, y_1^2\,\eta_1^2}{2\,B_1^2}\right)\epsilon + \left(-\frac{\gamma^2\,\hbar^2\, y_1\,\eta_1}{2\,B_1} + \frac{7\,\gamma^2\,\hbar^2\, y_1^2\,\eta_1^2}{4\,B_1^2} - \frac{\gamma^2\,\hbar^2\, y_1^3\,\eta_1^3}{B_1^3} + \frac{\gamma^2\,\hbar^2\, y_1^4\,\eta_1^4}{8\,B_1^4}\right)\epsilon^2 + \left(\frac{\gamma^3\,\hbar^3\, y_1\,\eta_1}{6\,B_1} - \frac{25\,\gamma^3\,\hbar^3\, y_1^2\,\eta_1^2}{12\,B_1^2} + \frac{23\,\gamma^3\,\hbar^3\, y_1^3\,\eta_1^3}{6\,B_1^3} - \frac{49\,\gamma^3\,\hbar^3\, y_1^4\,\eta_1^4}{24\,B_1^4} + \frac{3\,\gamma^3\,\hbar^3\, y_1^5\,\eta_1^5}{8\,B_1^5} - \frac{\gamma^3\,\hbar^3\, y_1^6\,\eta_1^6}{48\,B_1^6}\right)\epsilon^3 + O[\epsilon]^4], \text{True}\}\}$

In[∘]:= **Timing@{lhs = $\mathbb{E}_{\{1\}\to\{1\}}$[0, ξ_1 x_1, 1] // cS_1,
rhs = $\text{Exp}_{cm,1,\$k}$[$\xi$, Last@$cS_1$[$\mathbb{E}_{\{\}\to\{1\}}$[0, 0, x_1]]] /. {$\xi \to \xi_1$, {} → {1}}; HL[lhs ≡ rhs]}**

Out[∘]= $\{5.625, \{\mathbb{E}_{\{1\}\to\{1\}}[0, -x_1\,\xi_1, 1 + O[\epsilon]^4], \text{True}\}\}$

In[∘]:= **Timing@{lhs = $\mathbb{E}_{\{1\}\to\{1\}}$[0, η_1 y_1, 1] // cS_1,
rhs = $\text{Exp}_{cm,1,\$k}$[$\eta$, Last@$cS_1$[$\mathbb{E}_{\{\}\to\{1\}}$[0, 0, y_1]]] /. {$\eta \to \eta_1$, {} → {1}}; HL[lhs ≡ rhs]}**

Out[∘]= $\{0.59375, \{\mathbb{E}_{\{1\}\to\{1\}}[0, -y_1\,\eta_1, 1 + O[\epsilon]^4], \text{True}\}\}$