Pensieve header: Exp relative to am, bm, cm, dm.

Follows code in Projects/SL2Portfolio/SL2PortfolioProgram.nb.

Startup

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio2"];
(*Once[<< KnotTheory`];*)
Once[<< "../Profile/Profile.m";];
<< "Engine-Speedy.m";
<< "Objects.m";
$k = 3;
HL[ℰ_] := Style[ℰ, Background → Green];
```

```
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
This version: April 2020. Original version: July 1994.
```

» Warning: On Sep 4 2019 I swapped the operations ϵ and η. Some incompatibilities may arise in older notebooks.

Exponentials as needed.

Task. Define $\text{Exp}_{m,i,k}[P]$ to compute $e^{\mathbb{O}(P)}$ to ϵ^k in the using the $\text{mm}_{i,i\to i}$ multiplication, where P is an ϵ-dependent near-docile element, giving the answer in $\mathbb{E}$-form.

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\lambda\,\mathbb{O}(P)} = \mathbb{O}\left(e^{\lambda P_0} F(\lambda)\right)$, then $F(\lambda=0)=1$ and we have:

$\mathbb{O}\left(e^{\lambda P_0}(P_0\,F(\lambda) + \partial_\lambda F)\right) = \mathbb{O}\left(\partial_\lambda e^{\lambda P_0} F(\lambda)\right) = \partial_\lambda \mathbb{O}\left(e^{\lambda P_0} F(\lambda)\right) = \partial_\lambda e^{\lambda\,\mathbb{O}(P)} = e^{\lambda\,\mathbb{O}(P)}\,\mathbb{O}(P) = \mathbb{O}\left(e^{\lambda P_0} F(\lambda)\right)\mathbb{O}(P)$.

This is a linear ODE for F. Setting inductively $F_k = F_{k-1} + \epsilon^k\,\varphi$ we find that $F_0 = 1$ and solve for φ.

In[∘]:=
```
(* Bug: The first line is valid only if 𝕆(𝕖^P0)==𝕖^𝕆(P0). *)
Exp_{mm_,i_,0}[P_] := Module[{LQ = Normal@P /. ϵ → 0},
   𝔼[LQ /. (x | y)_i → 0, LQ /. (b | a | t)_i → 0, 1] ];
Exp_{mm_,i_,k_}[P_] := Block[{$k = k},
  Module[{P0, λ, φ, F, j, rhs, err = 0, pows, at0, atλ},
   P0 = Normal@P /. ϵ → 0;
   F = Normal@Last@Exp_{mm,i,k-1}[λ P];
   (*Unary*)While[
    If[err =!= 0,
     pows = Echo[First /@ CoefficientRules[err, {y_i, b_i, a_i, x_i}]];
     F += Sum[ϵ^k φ_js[λ] Times @@ {y_i, b_i, a_i, x_i}^js, {js, pows}];
     rhs = Normal@Last@mm_{i,j→i}[
        𝔼_{{}→{i}}[λ P0 /. (x | y)_i → 0, λ P0 /. (b | a | t)_i → 0, F]_k sσ_{i→j}@𝔼_{{}→{i}}[0, 0, P]_k];
     err = CF[(∂_λ F) + P0 F - rhs];
     at0 = Table[φ_js[0] == 0, {js, pows}];
     atλ = (# == 0) & /@ (pows /. CoefficientRules[err, {y_i, b_i, a_i, x_i}]);
     F = F /. DSolve[And @@ (at0 ⋃ atλ), Table[φ_js[λ], {js, pows}], λ][[1]]
    ];
    rhs = Normal@Last@mm_{i,j→i}[
       𝔼_{{}→{i}}[λ P0 /. (x | y)_i → 0, λ P0 /. (b | a | t)_i → 0, F]_k sσ_{i→j}@𝔼_{{}→{i}}[0, 0, P]_k];
    err = CF[(∂_λ F) + P0 F - rhs];
    err =!= 0
   ];
   𝔼_{{}→{i}}[P0 /. (x | y)_i → 0, P0 /. (b | a | t)_i → 0, F + O[ϵ]^(k+1) /. λ → 1]]
 ]
```

In[∘]:= $\text{Exp}_{dm,1,2}[\xi\,(x_1+\epsilon\,y_1)]$

» {{1, 0, 0, 0}, {0, 0, 0, 0}}

» {{2, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}, {0, 0, 0, 0}}

Out[∘]=
$$\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ \xi\,x_1,\ 1+\left(-\frac{\xi^2\,(-1+B_1)}{2\,\hbar}+\xi\,y_1\right)\epsilon+\left(\frac{\xi^4\,(-1+B_1)^2}{8\,\hbar^2}+\frac{1}{2}\,\xi^2\,a_1\,B_1-\frac{1}{6}\,\gamma\,\xi^3\,(-1+3\,B_1)\,x_1-\frac{\xi^3\,(-1+B_1)\,y_1}{2\,\hbar}+\frac{1}{2}\,\gamma\,\xi^2\,\hbar\,x_1\,y_1+\frac{1}{2}\,\xi^2\,y_1^2\right)\epsilon^2+O[\epsilon]^3\Big]$$

In[∘]:= $dS_1[\mathbb{E}_{\{\}\to\{1\}}[0,0,\#]]\ \&\ /@\ \{y_1,x_1\}$

Out[∘]=
$$\Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ -\frac{y_1}{B_1}+\frac{\gamma\,\hbar\,y_1\,\epsilon}{B_1}-\frac{(\gamma^2\,\hbar^2\,y_1)\,\epsilon^2}{2\,B_1}+\frac{\gamma^3\,\hbar^3\,y_1\,\epsilon^3}{6\,B_1}+O[\epsilon]^4\Big],$$
$$\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ -x_1-\hbar\,a_1\,x_1\,\epsilon-\frac{1}{2}\,(\hbar^2\,a_1^2\,x_1)\,\epsilon^2-\frac{1}{6}\,(\hbar^3\,a_1^3\,x_1)\,\epsilon^3+O[\epsilon]^4\Big]\Big\}$$

In[∘]:=
```
Timing@{lhs = 𝔼{1}→{1}[0, ξ1 x1, 1] // dS1,
  rhs = Expdm,1,$k[ξ1 Last@dS1[𝔼{}→{1}[0, 0, x1]]] /. {{} → {1}}; HL[lhs ≡ rhs]}
```

» {{0, 0, 1, 1}, {0, 0, 0, 2}}

» {{0, 0, 2, 2}, {0, 0, 2, 1}, {0, 0, 1, 3}, {0, 0, 1, 2}, {0, 0, 0, 4}, {0, 0, 0, 3}, {0, 0, 0, 2}}

» {{0, 0, 3, 3}, {0, 0, 3, 2}, {0, 0, 3, 1}, {0, 0, 2, 4}, {0, 0, 2, 3},
{0, 0, 2, 2}, {0, 0, 1, 5}, {0, 0, 1, 4}, {0, 0, 1, 3}, {0, 0, 1, 2},
{0, 0, 0, 6}, {0, 0, 0, 5}, {0, 0, 0, 4}, {0, 0, 0, 3}, {0, 0, 0, 2}}

Out[∘]=

$$\Big\{3.53125, \Big\{\mathbb{E}_{\{1\}\to\{1\}}\Big[0, -x_1\,\xi_1, 1 + \Big(-\hbar\, a_1\, x_1\,\xi_1 - \frac{1}{2}\gamma\,\hbar\, x_1^2\,\xi_1^2\Big)\epsilon + \Big(-\frac{1}{2}\hbar^2 a_1^2 x_1\,\xi_1 + \frac{1}{4}\gamma^2\hbar^2 x_1^2\,\xi_1^2 - \gamma\,\hbar^2 a_1\, x_1^2\,\xi_1^2 + \frac{1}{2}\hbar^2 a_1^2 x_1^2\,\xi_1^2 - \frac{1}{2}\gamma^2\hbar^2 x_1^3\,\xi_1^3 + \frac{1}{2}\gamma\,\hbar^2 a_1\, x_1^3\,\xi_1^3 + \frac{1}{8}\gamma^2\hbar^2 x_1^4\,\xi_1^4\Big)\epsilon^2 +$$

$$\Big(-\frac{1}{6}\hbar^3 a_1^3 x_1\,\xi_1 - \frac{1}{12}\gamma^3\hbar^3 x_1^2\,\xi_1^2 + \frac{1}{2}\gamma^2\hbar^3 a_1\, x_1^2\,\xi_1^2 - \gamma\,\hbar^3 a_1^2 x_1^2\,\xi_1^2 + \frac{1}{2}\hbar^3 a_1^3 x_1^2\,\xi_1^2 + \frac{2}{3}\gamma^3\hbar^3 x_1^3\,\xi_1^3 - \frac{7}{4}\gamma^2\hbar^3 a_1\, x_1^3\,\xi_1^3 + \frac{5}{4}\gamma\,\hbar^3 a_1^2 x_1^3\,\xi_1^3 - \frac{1}{6}\hbar^3 a_1^3 x_1^3\,\xi_1^3 - \frac{19}{24}\gamma^3\hbar^3 x_1^4\,\xi_1^4 + \gamma^2\hbar^3 a_1\, x_1^4\,\xi_1^4 - \frac{1}{4}\gamma\,\hbar^3 a_1^2 x_1^4\,\xi_1^4 + \frac{1}{4}\gamma^3\hbar^3 x_1^5\,\xi_1^5 - \frac{1}{8}\gamma^2\hbar^3 a_1\, x_1^5\,\xi_1^5 - \frac{1}{48}\gamma^3\hbar^3 x_1^6\,\xi_1^6\Big)\epsilon^3 + O[\epsilon]^4\Big], \text{True}\Big\}\Big\}$$

In[∘]:=
```
Timing@{lhs = 𝔼{1}→{1}[0, η1 y1, 1] // dS1,
  rhs = Expdm,1,$k[η Last@dS1[𝔼{}→{1}[0, 0, y1]]] /. {η → η1, {} → {1}}, HL[lhs ≡ rhs]}
```

» {{2, 0, 0, 0}, {1, 0, 0, 0}}

» {{4, 0, 0, 0}, {3, 0, 0, 0}, {2, 0, 0, 0}, {1, 0, 0, 0}}

» {{6, 0, 0, 0}, {5, 0, 0, 0}, {4, 0, 0, 0}, {3, 0, 0, 0}, {2, 0, 0, 0}, {1, 0, 0, 0}}

Out[∘]=

$$\Big\{3.29688, \Big\{\mathbb{E}_{\{1\}\to\{1\}}\Big[0, -\frac{y_1\,\eta_1}{B_1}, 1 + \Big(\frac{\gamma\,\hbar\, y_1\,\eta_1}{B_1} - \frac{\gamma\,\hbar\, y_1^2\,\eta_1^2}{2 B_1^2}\Big)\epsilon + \Big(-\frac{\gamma^2\hbar^2 y_1\,\eta_1}{2 B_1} + \frac{7\gamma^2\hbar^2 y_1^2\,\eta_1^2}{4 B_1^2} - \frac{\gamma^2\hbar^2 y_1^3\,\eta_1^3}{B_1^3} + \frac{\gamma^2\hbar^2 y_1^4\,\eta_1^4}{8 B_1^4}\Big)\epsilon^2 +$$

$$\Big(\frac{\gamma^3\hbar^3 y_1\,\eta_1}{6 B_1} - \frac{25\gamma^3\hbar^3 y_1^2\,\eta_1^2}{12 B_1^2} + \frac{23\gamma^3\hbar^3 y_1^3\,\eta_1^3}{6 B_1^3} - \frac{49\gamma^3\hbar^3 y_1^4\,\eta_1^4}{24 B_1^4} + \frac{3\gamma^3\hbar^3 y_1^5\,\eta_1^5}{8 B_1^5} - \frac{\gamma^3\hbar^3 y_1^6\,\eta_1^6}{48 B_1^6}\Big)\epsilon^3 + O[\epsilon]^4\Big], \mathbb{E}_{\{1\}\to\{1\}}\Big[0, -\frac{y_1\,\eta_1}{B_1},$$

$$1 - \frac{\big(\gamma\,\hbar\,(-2 B_1 y_1\,\eta_1 + y_1^2\,\eta_1^2)\big)\epsilon}{2 B_1^2} + \frac{\gamma^2\hbar^2\,(-4 B_1^3 y_1\,\eta_1 + 14 B_1^2 y_1^2\,\eta_1^2 - 8 B_1 y_1^3\,\eta_1^3 + y_1^4\,\eta_1^4)\,\epsilon^2}{8 B_1^4} -$$

$$\frac{\big(\gamma^3\hbar^3\,(-8 B_1^5 y_1\,\eta_1 + 100 B_1^4 y_1^2\,\eta_1^2 - 184 B_1^3 y_1^3\,\eta_1^3 + 98 B_1^2 y_1^4\,\eta_1^4 - 18 B_1 y_1^5\,\eta_1^5 + y_1^6\,\eta_1^6)\big)\epsilon^3}{48 B_1^6} + O[\epsilon]^4\Big], \text{True}\Big\}\Big\}$$

```
In[•]:= Timing@{lhs = 𝔼_{{1}→{1}}[0, ξ₁ x₁, 1] // cS₁,
          rhs = Exp_{cm,1,$k}[ξ₁ Last@cS₁[𝔼_{{}→{1}}[0, 0, x₁]]] /. {} → {1}; HL[lhs ≡ rhs]}
Out[•]= {3.67188, {𝔼_{{1}→{1}}[0, -x₁ ξ₁, 1 + O[∈]^4], True}}

In[•]:= Timing@{lhs = 𝔼_{{1}→{1}}[0, η₁ y₁, 1] // cS₁,
          rhs = Exp_{cm,1,$k}[η₁ Last@cS₁[𝔼_{{}→{1}}[0, 0, y₁]]] /. {} → {1}; HL[lhs ≡ rhs]}
Out[•]= {0.296875, {𝔼_{{1}→{1}}[0, -y₁ η₁, 1 + O[∈]^4], True}}
```